Recovering Transformations

What if we know $f$ and $g$ and want to recover the transform $T$?

- e.g. better align images from Project 1
- willing to let user provide correspondences
  - How many do we need?
Translation: # correspondences?

How many correspondences needed for translation?
How many Degrees of Freedom?
What is the transformation matrix?

\[
M = \begin{bmatrix}
1 & 0 & p'_x - p_x \\
0 & 1 & p'_y - p_y \\
0 & 0 & 1
\end{bmatrix}
\]
Euclidian: # correspondences?

How many correspondences needed for translation+rotation? How many DOF?
Affine: # correspondences?

How many correspondences needed for affine?
How many DOF?
Projective: # correspondences?

How many correspondences needed for projective?
How many DOF?
Example: warping triangles

Given two triangles: ABC and A’B’C’ in 2D (12 numbers)
Need to find transform T to transfer all pixels from one to the other.
What kind of transformation is T?
How can we compute the transformation matrix:

\[
\begin{bmatrix}
x' \\
y' \\
1
\end{bmatrix} = \begin{bmatrix}
a & b & c \\
d & e & f \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]
HINT: warping triangles

Don’t forget to move the origin too!

Very useful for Project 2…
Morphing = Object Averaging

The aim is to find “an average” between two objects
• Not an average of two images of objects…
• …but an image of the average object!
• How can we make a smooth transition in time?
  – Do a “weighted average” over time t

How do we know what the average object looks like?
• We haven’t a clue!
• But we can often fake something reasonable
  – Usually required user/artist input
What’s the average of P and Q?

Linear Interpolation (Affine Combination): New point $aP + bQ$, defined only when $a+b = 1$
So $aP+bQ = aP+(1-a)Q$

P and Q can be anything:
- points on a plane (2D) or in space (3D)
- Colors in RGB or HSV (3D)
- Whole images (m-by-n D)… etc.
Idea #1: Cross-Dissolve

Interpolate whole images:

\[ \text{Image}_{\text{halfway}} = (1-t)\text{Image}_1 + t\text{image}_2 \]

This is called **cross-dissolve** in film industry

But what is the images are not aligned?
Idea #2: Align, then cross-dissolve

Align first, then cross-dissolve

• Alignment using global warp – picture still valid
Dog Averaging

What to do?
- Cross-dissolve doesn’t work
- Global alignment doesn’t work
  - Cannot be done with a global transformation (e.g. affine)
- Any ideas?

Feature matching!
- Nose to nose, tail to tail, etc.
- This is a local (non-parametric) warp
Idea #3: Local warp, then cross-dissolve

Morphing procedure:

for every $t$,

1. Find the average shape (the “mean dog” 😊)
   - local warping
2. Find the average color
   - Cross-dissolve the warped images
Local (non-parametric) Image Warping

Need to specify a more detailed warp function

- Global warps were functions of a few (2,4,8) parameters
- Non-parametric warps $u(x,y)$ and $v(x,y)$ can be defined independently for every single location $x,y$!
- Once we know vector field $u,v$ we can easily warp each pixel (use backward warping with interpolation)
Image Warping – non-parametric

Move control points to specify a spline warp
Spline produces a smooth vector field
Warp specification - dense

How can we specify the warp?

Specify corresponding *spline control points*

- *interpolate* to a complete warping function

But we want to specify only a few points, not a grid
Warp specification - sparse

How can we specify the warp?

Specify corresponding *points*

- *interpolate* to a complete warping function
- How do we do it?

How do we go from feature points to pixels?
Triangular Mesh

1. Input correspondences at key feature points
2. Define a triangular mesh over the points
   • Same mesh in both images!
   • Now we have triangle-to-triangle correspondences
3. Warp each triangle separately from source to destination
   • How do we warp a triangle?
   • 3 points = affine warp!
   • Just like texture mapping
Triangulations

A *triangulation* of set of points in the plane is a *partition* of the convex hull to triangles whose vertices are the points, and do not contain other points.

There are an exponential number of triangulations of a point set.
An $O(n^3)$ Triangulation Algorithm

Repeat until impossible:

- Select two sites.
- If the edge connecting them does not intersect previous edges, keep it.
“Quality” Triangulations

Let $\alpha(T) = (\alpha_1, \alpha_2, \ldots, \alpha_{3t})$ be the vector of angles in the triangulation $T$ in increasing order.

A triangulation $T_1$ will be “better” than $T_2$ if $\alpha(T_1) > \alpha(T_2)$ lexicographically.

The Delaunay triangulation is the “best”
- Maximizes smallest angles

![good triangulation](image1)

![bad triangulation](image2)
Improving a Triangulation

In any convex quadrangle, an *edge flip* is possible. If this flip *improves* the triangulation locally, it also improves the global triangulation.

If an edge flip improves the triangulation, the first edge is called *illegal*.
**Illegal Edges**

**Lemma:** An edge $pq$ is illegal iff one of its opposite vertices is inside the circle defined by the other three vertices.

**Proof:** By Thales’ theorem.

![Diagram of illegal edges](image)

**Theorem:** A Delaunay triangulation does not contain illegal edges.

**Corollary:** A triangle is Delaunay iff the circle through its vertices is empty of other sites.

**Corollary:** The Delaunay triangulation is not unique if more than three sites are co-circular.
Naïve Delaunay Algorithm

Start with an arbitrary triangulation. Flip any illegal edge until no more exist. Could take a long time to terminate.
Delaunay Triangulation by Duality

General position assumption: There are no four co-circular points.

Draw the dual to the Voronoi diagram by connecting each two neighboring sites in the Voronoi diagram.

**Corollary:** The DT may be constructed in $O(n \log n)$ time.

This is what Matlab’s `delaunay` function uses.
Image Morphing

We know how to warp one image into the other, but how do we create a morphing sequence?

1. Create an intermediate shape (by interpolation)
2. Warp both images towards it
3. Cross-dissolve the colors in the newly warped images
Warp interpolation

How do we create an intermediate warp at time \( t \)?

- Assume \( t = [0,1] \)
- Simple linear interpolation of each feature pair
- \((1-t)p_1 + t p_0\) for corresponding features \( p_0 \) and \( p_1 \)
Other Issues

Beware of folding
  • You are probably trying to do something 3D-ish

Morphing can be generalized into 3D
  • If you have 3D data, that is!

Extrapolation can sometimes produce interesting effects
  • Caricatures
Dynamic Scene
Project #2: Due Tu, Sept 27

- Given two photos, produce a 60-frame morph animation
  - Use triangulation-based morphing (lots of helpful Matlab tools)
  - Need to write triangle-to-triangle warp (can’t use Matlab tools)
- We put all animations together into a movie!