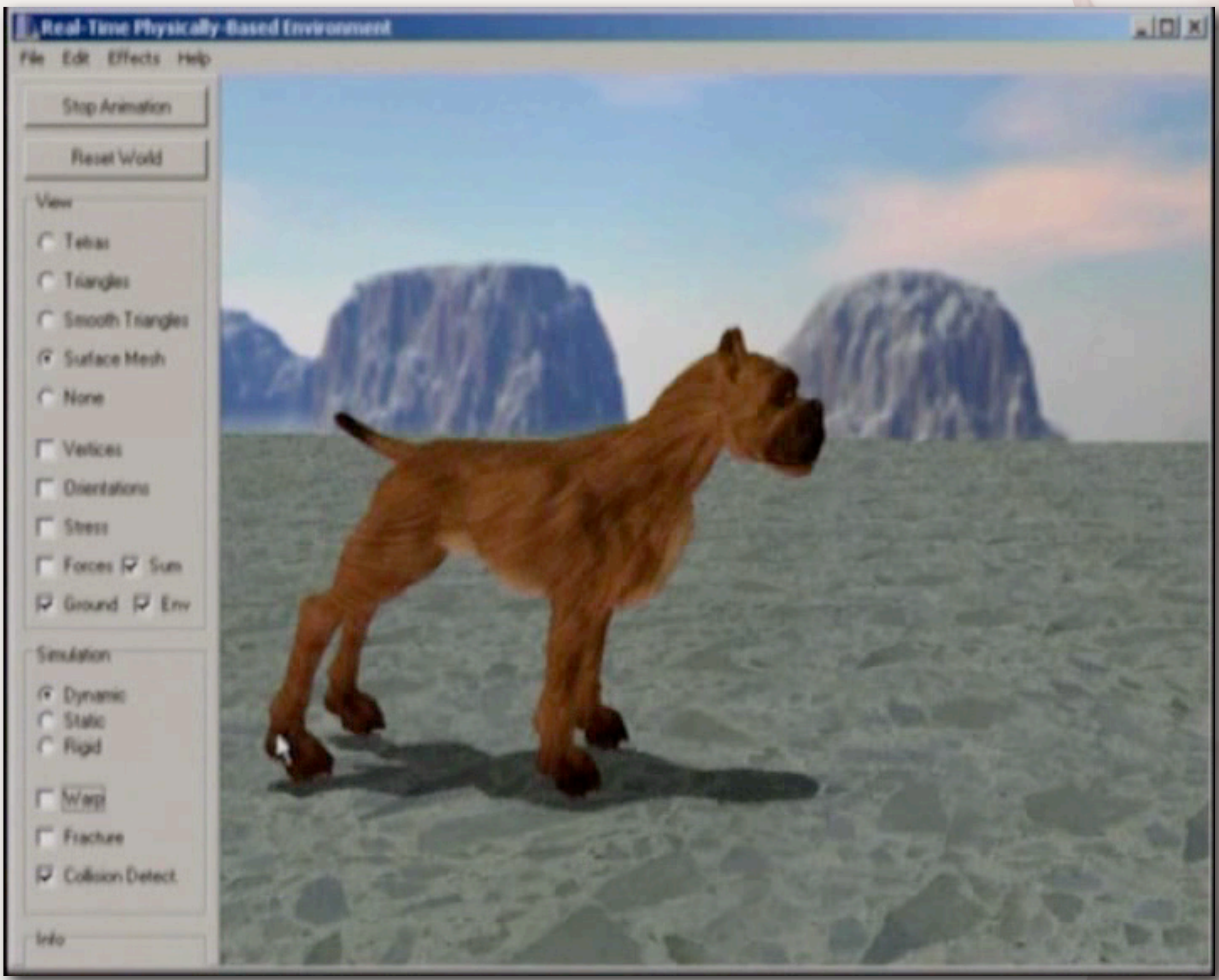
Deformable Materials 2

Adrien Treuille

source: Müller, Stam, James, Thürey. ***Real-Time Physics Class Notes*.**

Goal


Real-Time Physically-Based Environment
File Edit Effects Help
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Triangles
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Surface Mesh
None
Vertices
Orientations
Stress
Forces Sum
Ground Env
Simulation
Dynamic
Static
Rigid
Warp
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Collision Detect
Info

Overview

- Strain (Recap)
- Stress
- From Strain to Stress
- Discretization
- Simulation

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Deformations

Spring deformed by Δx:

$$\text{force} = -E\Delta x$$

stress: σ

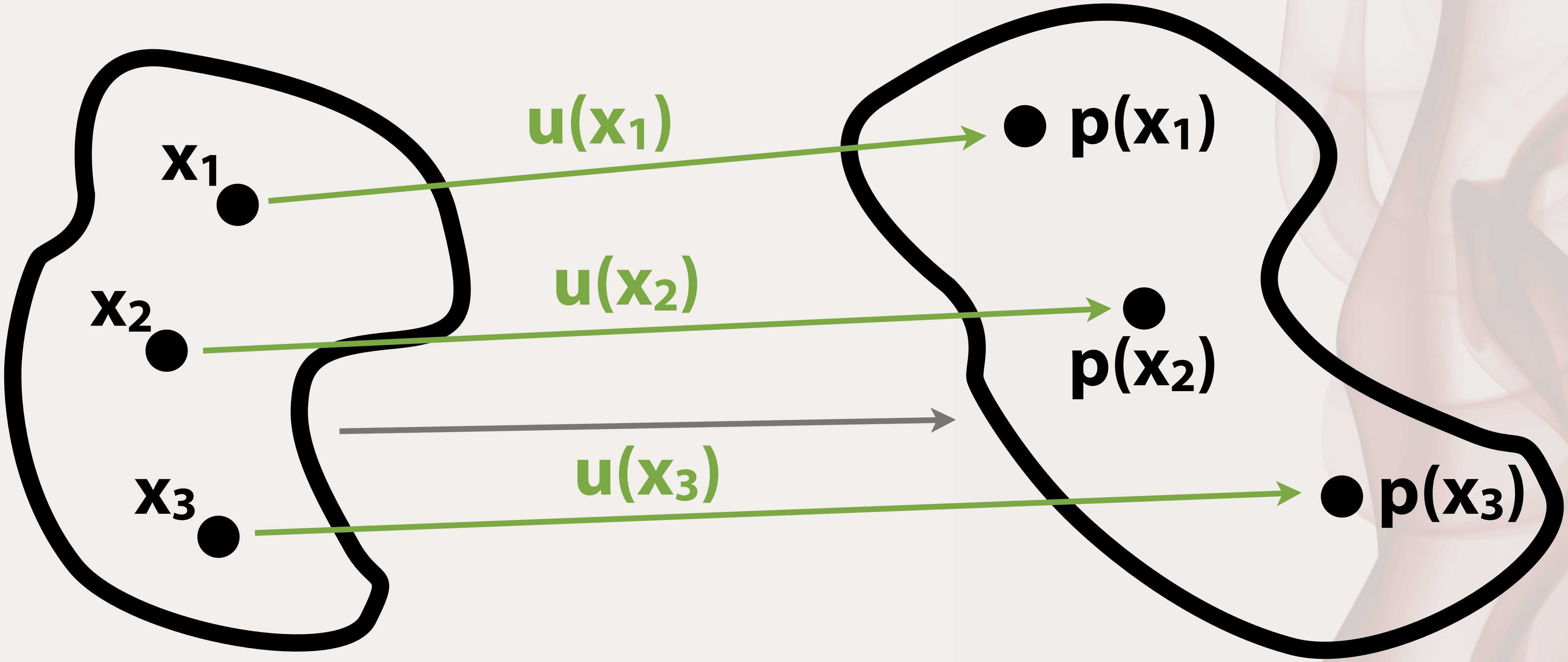
Young's modulus

strain: ε

Hooke's Law:

$$\sigma = E\epsilon$$

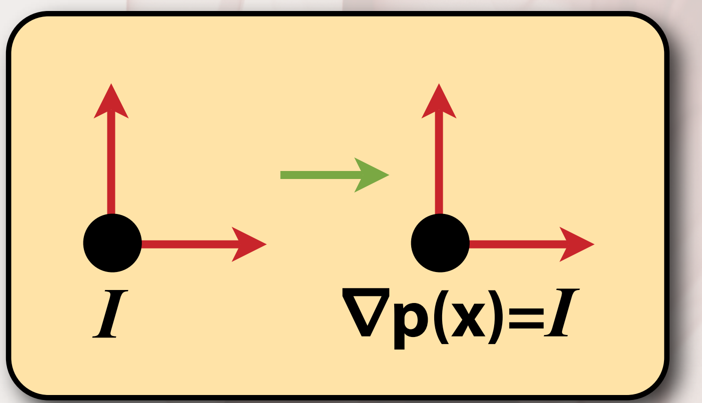
In 3D...



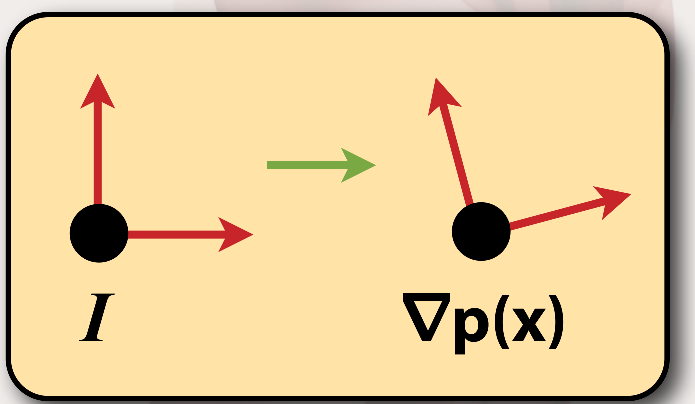
$$u(x) = p(x) - x$$

Defining Strain

- Strain is **invariant to translation**.
  - Ignore $p(x)$
  - Define in terms of local coordinate system transform: $\nabla p(x)$.



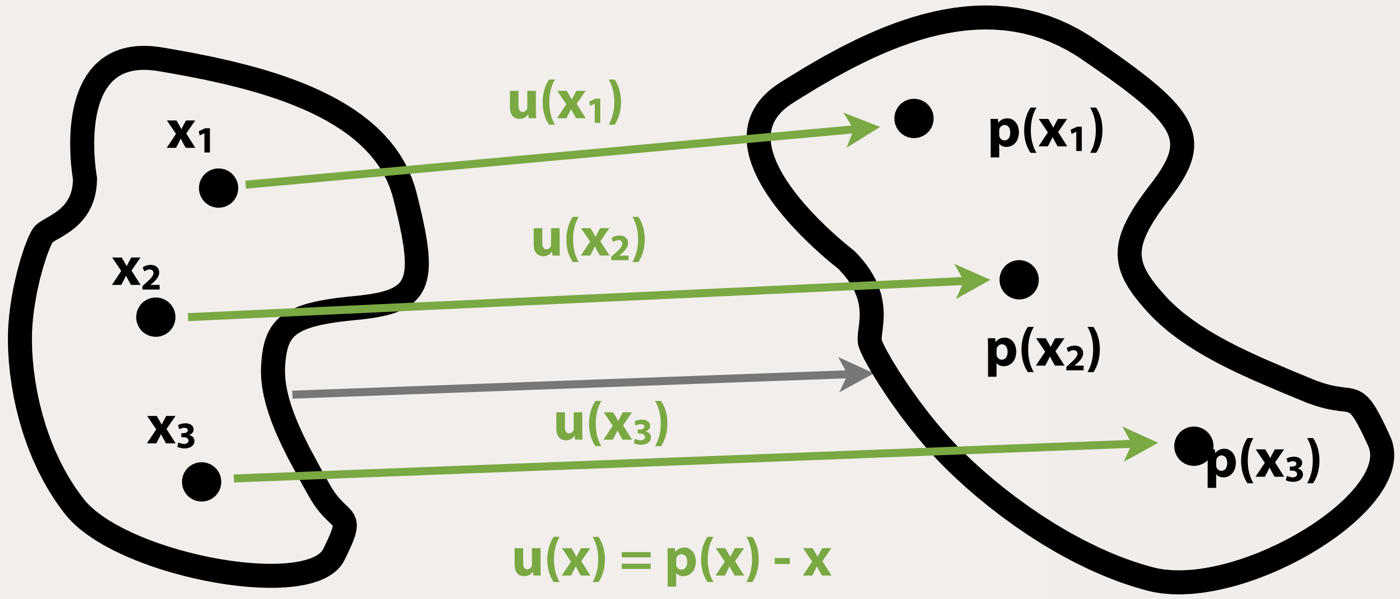
- Strain is **invariant to rotation**.
  - If $[\nabla p(x)]^T \nabla p(x) = I$,
  - Then $\varepsilon=0$



- Natural to define strain as:
  - $\varepsilon = ½([\nabla p(x)]^T \nabla p(x) - I)$
  - 6 DOFs

$$\epsilon = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$$

Green's Strain



$$\epsilon_G = \frac{1}{2}\left(\nabla u + [\nabla u]^T + [\nabla u]^T \nabla u\right)$$

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Stress

$$\sigma = E\epsilon$$

Stress

Direct Stress:

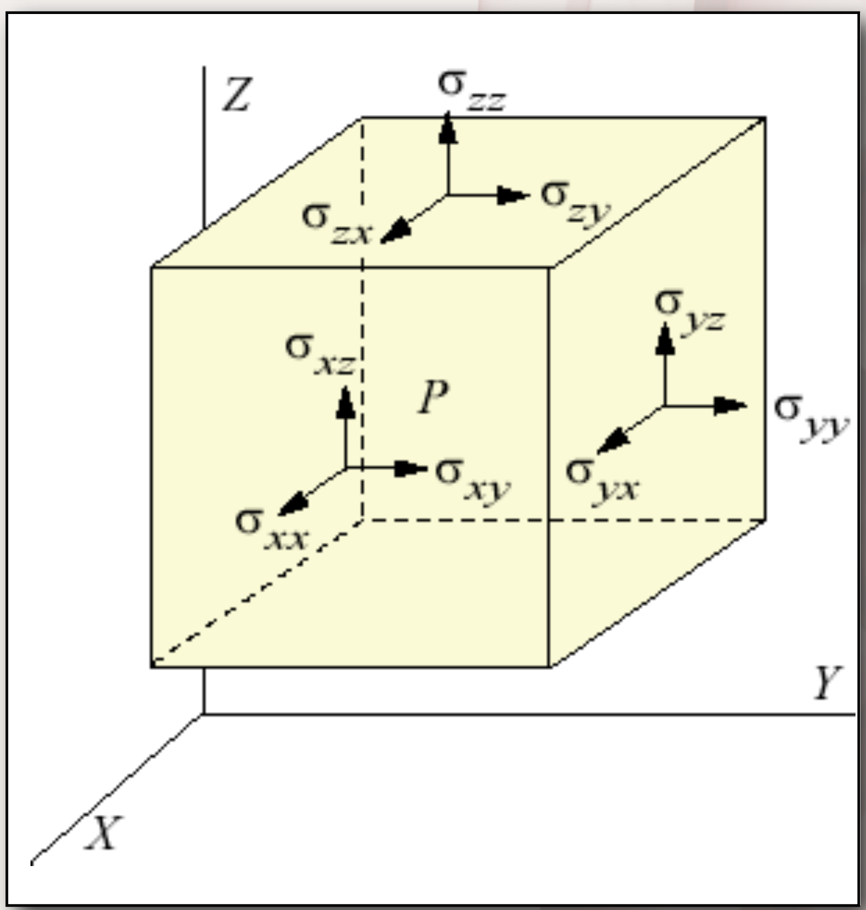
Direct stresses cause compression.

σ_{xx}, σ_{yy}, σ_{zz}

Shear Stress:

Shear stresses resist compression.

σ_{xy}, σ_{yz}, σ_{xz}

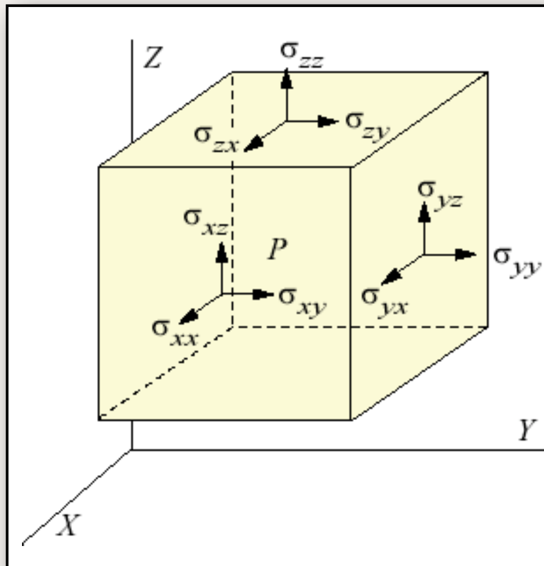


source: http://www.efunda.com/formulae/solid_mechanics/mat_mechanics/stress.cfm

Stress Tensor:

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

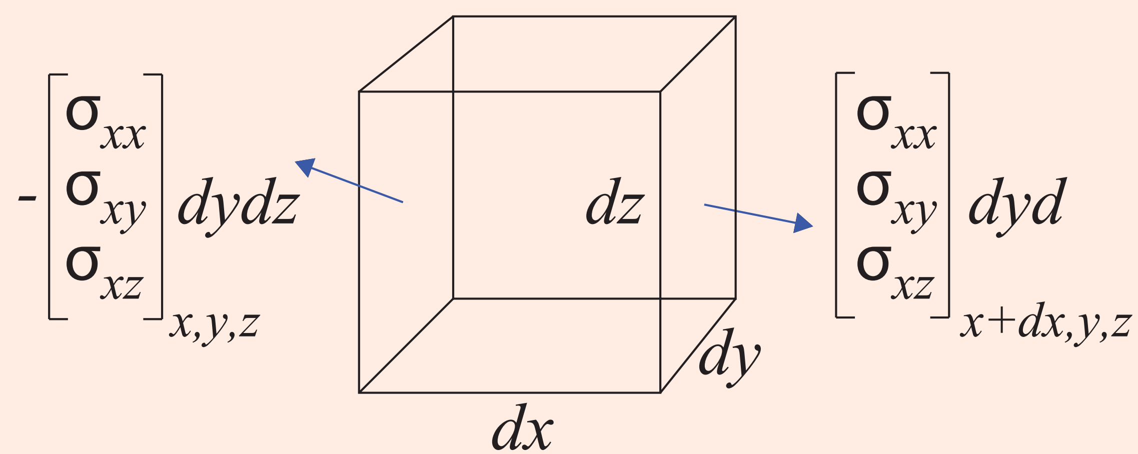
Stress Tensor Interpretation



$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

Stress measures the force on each face:

$$\frac{d\mathbf{f}}{dA} = \sigma \cdot \mathbf{n}.$$



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Young's Modulus

$$\sigma = E\epsilon$$

Voigt Notation

Stress

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

$$\{\sigma\} = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{xz}]^T \in \mathbf{R}^6$$

Strain

$$\epsilon = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$$

$$\{\epsilon\} = [\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{yz}, \epsilon_{xz}]^T \in \mathbf{R}^6$$

Isotropic Materials

$$\{\sigma\} = E\{\epsilon\} \qquad E \in \mathbf{R}^{6\times 6}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}$$

E **Elastic Stiffness**

How strongly the material resists deformation.

$v \in \left[0, \frac{1}{2}\right)$ **Poisson's Ratio**

How much volume is conserved.

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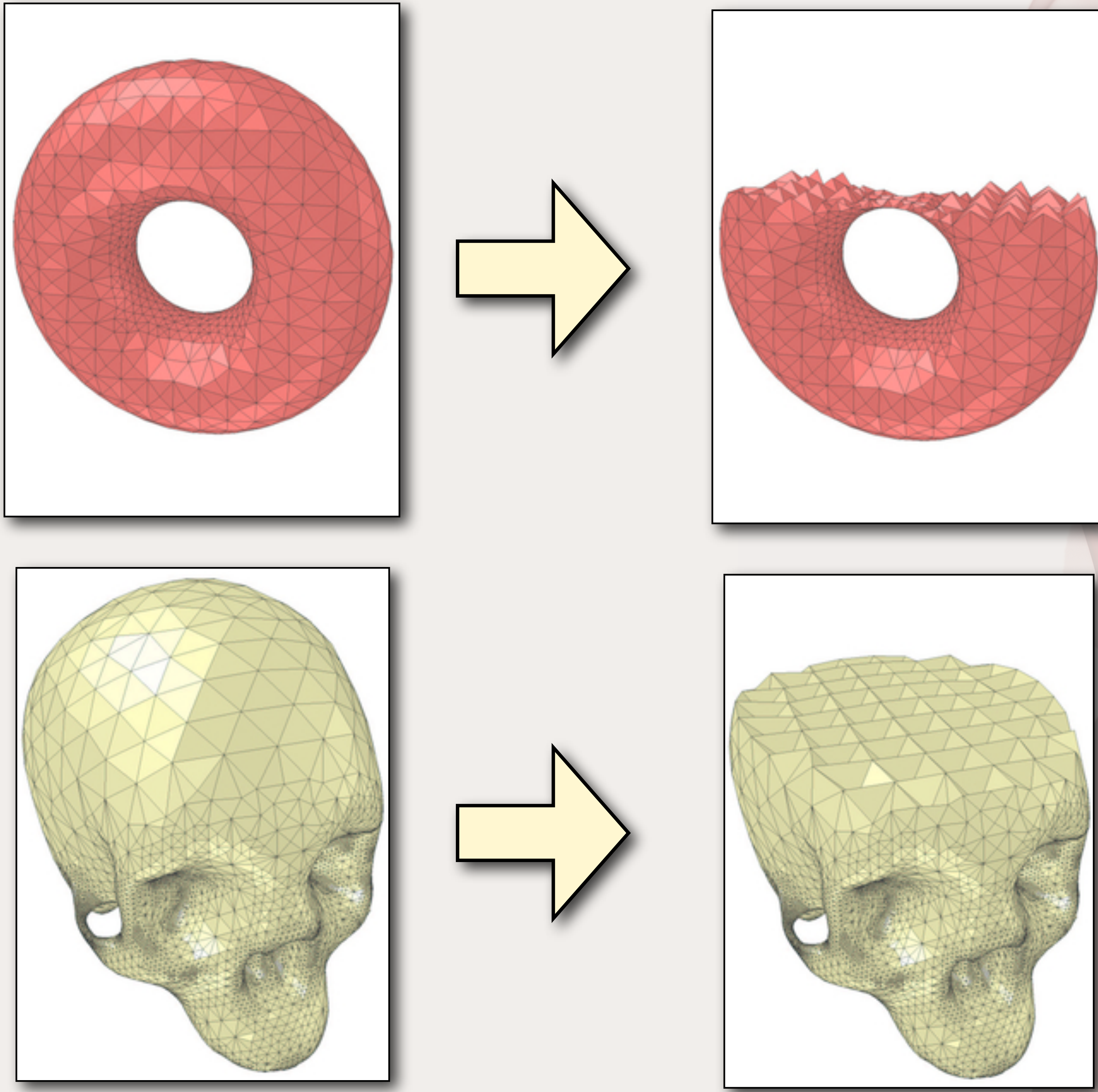
Discretization



source: Bridson, R., Teran, J., Molino, N. and Fedkiw, R., "Adaptive Physics Based Tetrahedral Mesh Generation Using Level Sets", Engineering with Computers 21, 2-18 (2005).

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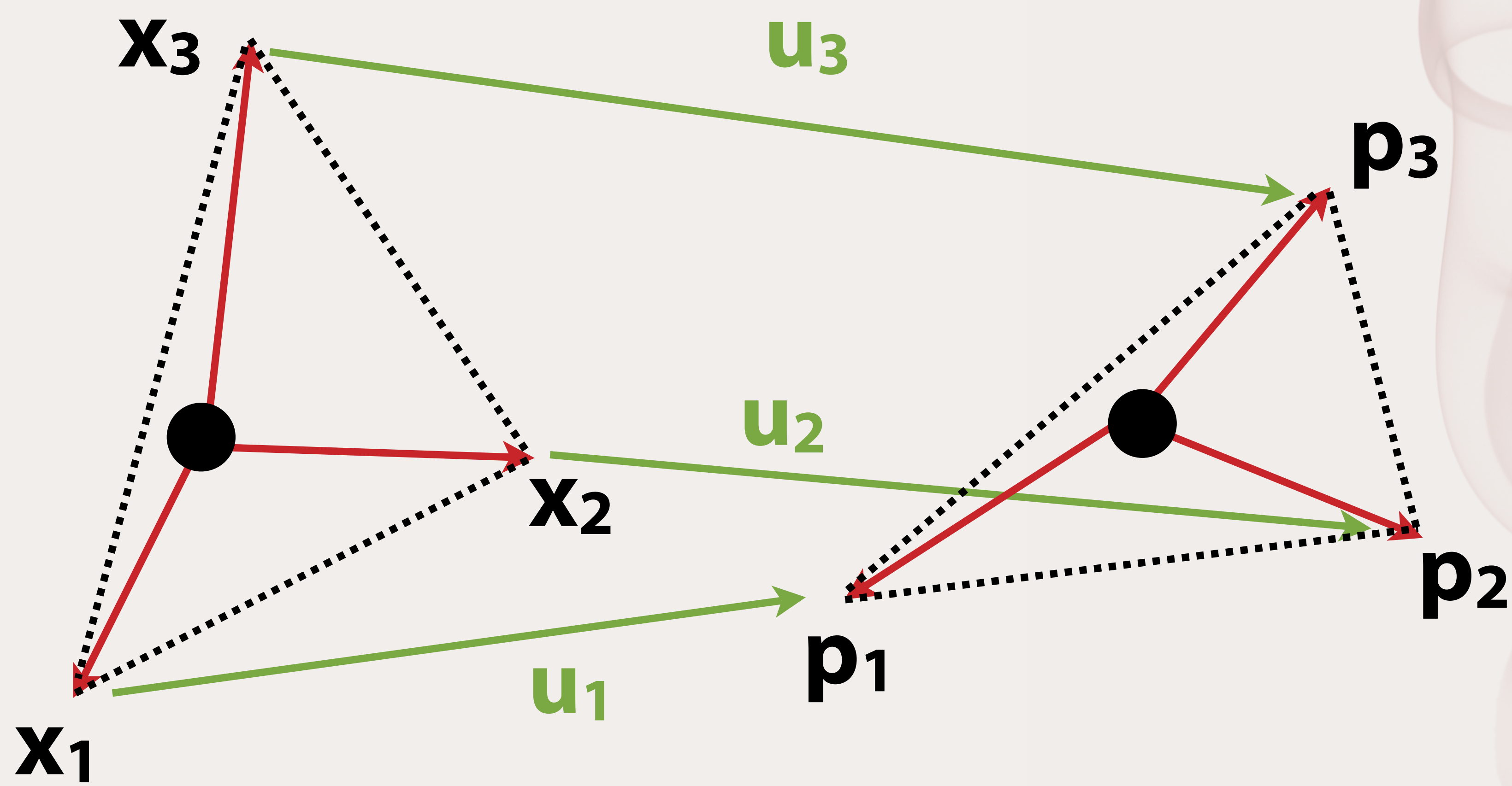
Discrete Strain



$$\mathbf{p}(\mathbf{x}) = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3]\,[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]^{-1}\,\mathbf{x} = \mathbf{P}\mathbf{x}$$

$$\nabla \mathbf{p} = \mathbf{P} \text{ and } \nabla \mathbf{u} = \mathbf{P} - \mathbf{I}$$

$$\varepsilon = \frac{1}{2}(\nabla \mathbf{u} + [\nabla \mathbf{u}]^T + [\nabla \mathbf{u}]^T \nabla \mathbf{u})$$

Simulation Loop

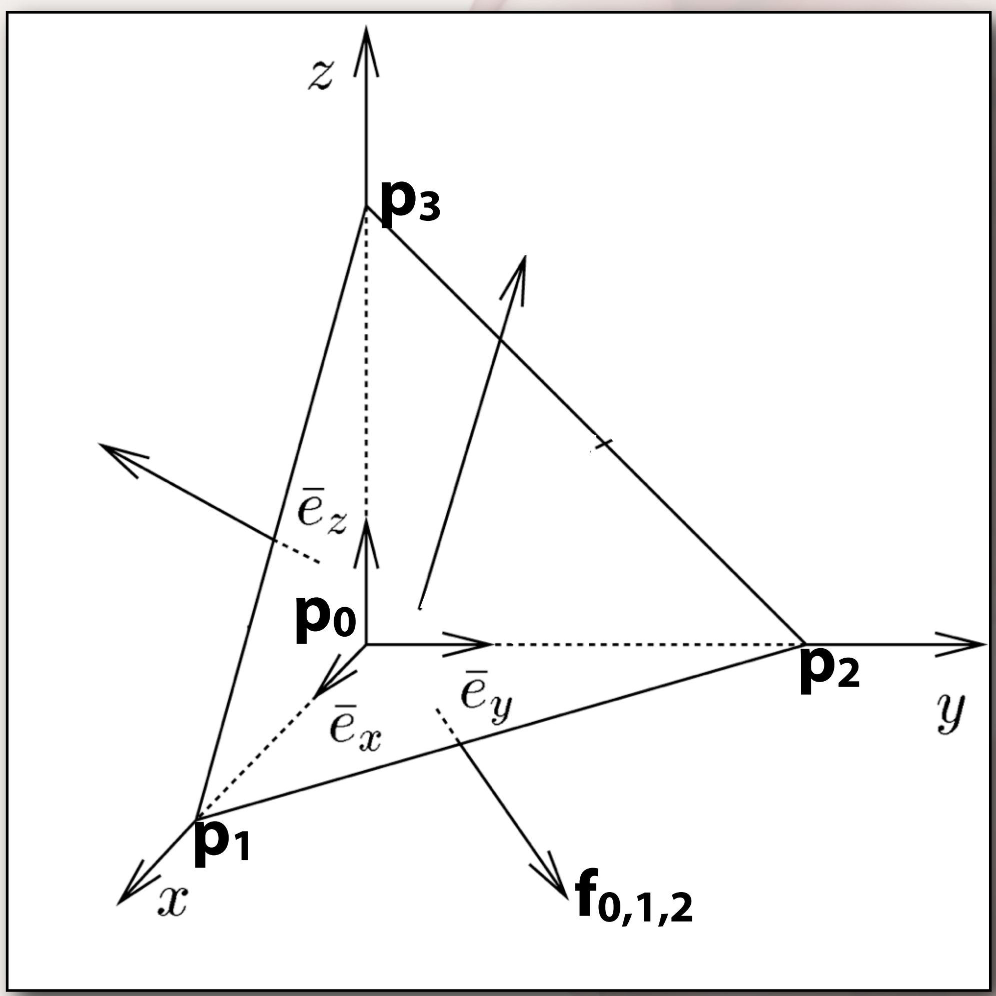
- **Compute Strain:**

$$\varepsilon = \frac{1}{2}(\nabla \mathbf{u} + [\nabla \mathbf{u}]^T + [\nabla \mathbf{u}]^T \nabla \mathbf{u})$$

- **Convert to Stress:**

$$\sigma = E\epsilon$$

$$\frac{d\mathbf{f}}{dA} = \sigma \cdot \mathbf{n}.$$



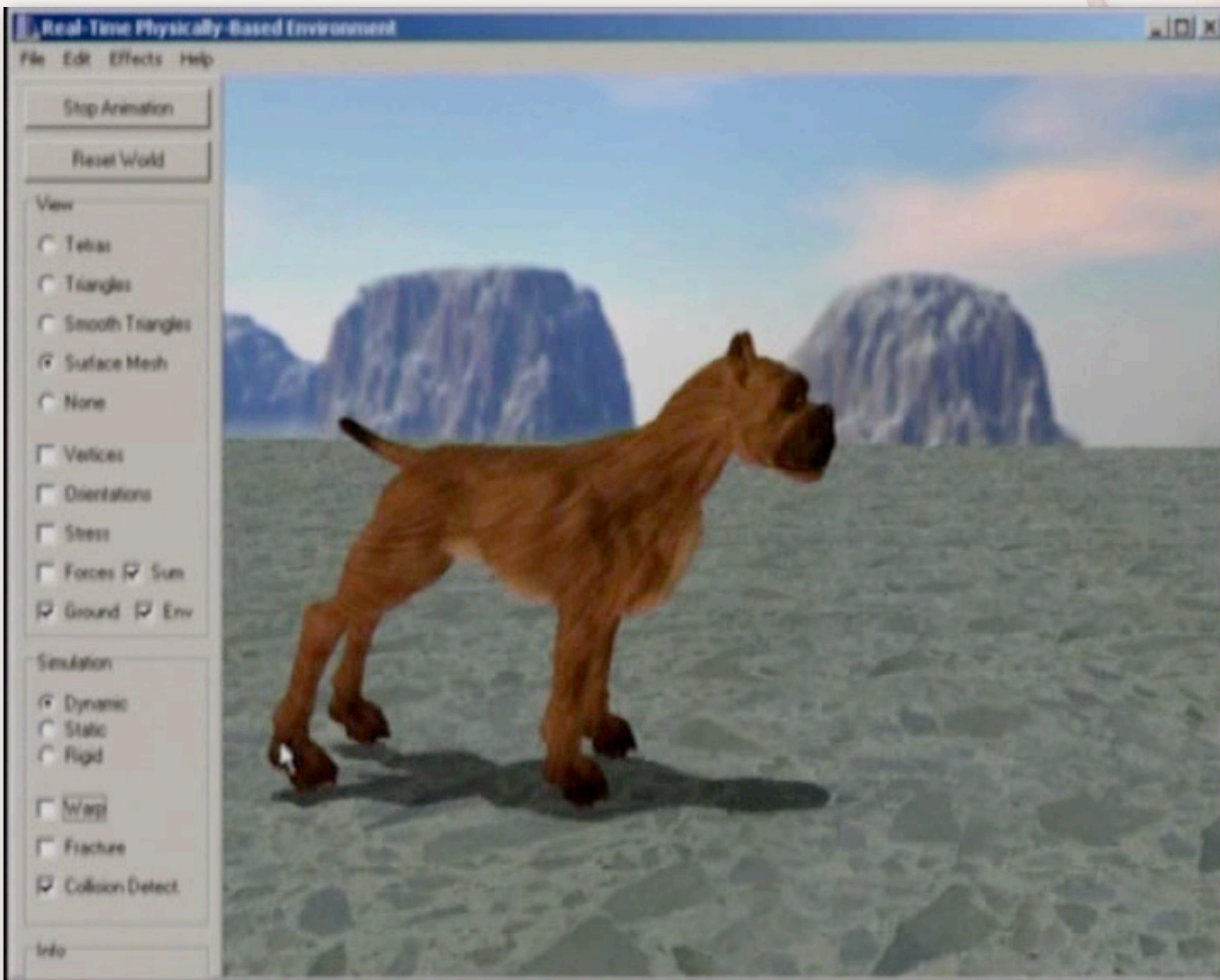
source: http://www.emeraldinsight.com/fig/1740200102032.png

- **Compute Face Forces:**

$$\begin{aligned}\mathbf{f}_{0,1,2} &= \boldsymbol{\sigma} \cdot \mathbf{n}_{0,1,2} \cdot A_{0,1,2} \\ &= \boldsymbol{\sigma}[(\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0)]\end{aligned}$$

- **Distribute to vertices.**
- **Integrate eqns of motion (e.g. 4th order RK).**

Examples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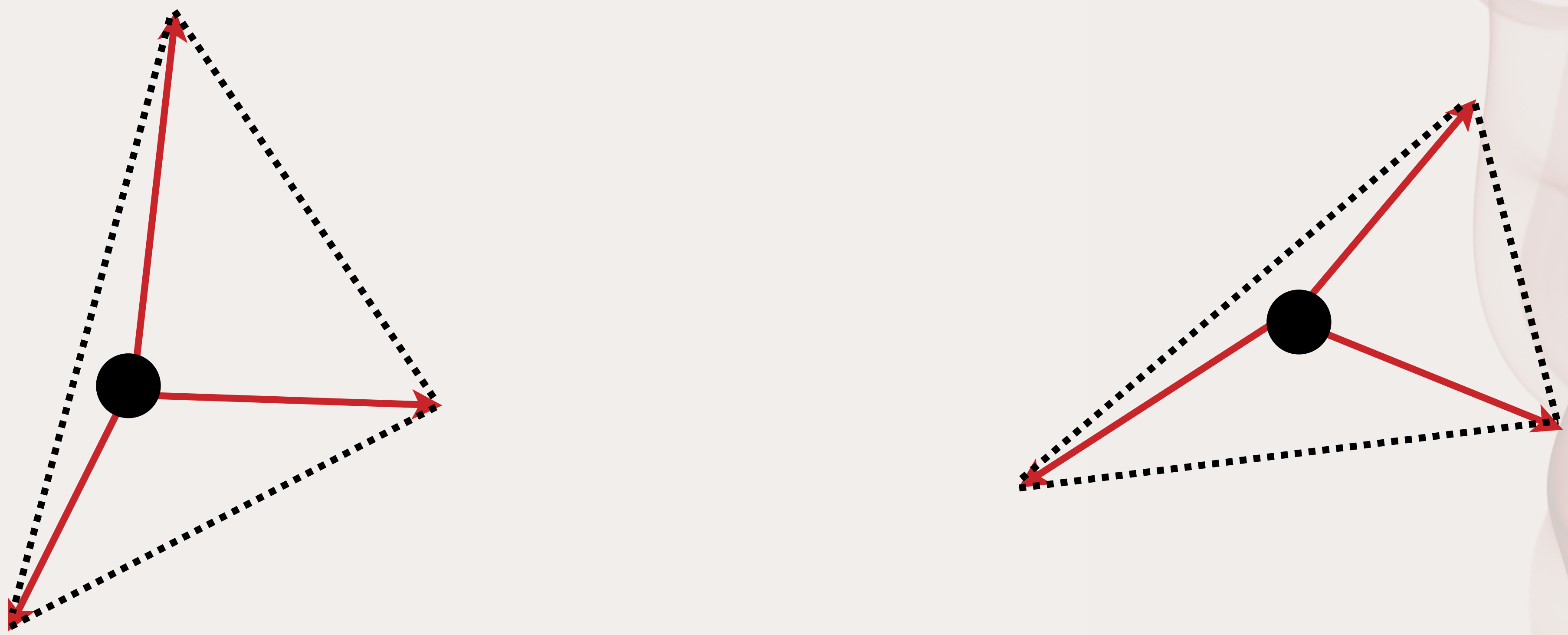
Question



Question

- How could we reduce the cost of simulation for a very finely discretized surface?
- Are there cheap ways of getting volumetric behavior without a full tetrahedralization?
- How can collision constraints be integrated?
- How to simulate plasticity?

Solutions

- bounding volume tree w/ tetrahedra at leaves
  - simulate parent nodes instead of leaves (if stresses are close)
- simulate on a simplified mesh (make details into bump maps)
- adaptive tetrahedralization based on force magnitudes
- come up with tetrahedralization that best captures the simulation based on precomputed simulations
- springs connected to a “skeleton”
- plasticity based on sparse springs connecting the surface mesh to itself