Rigid Body Dynamics

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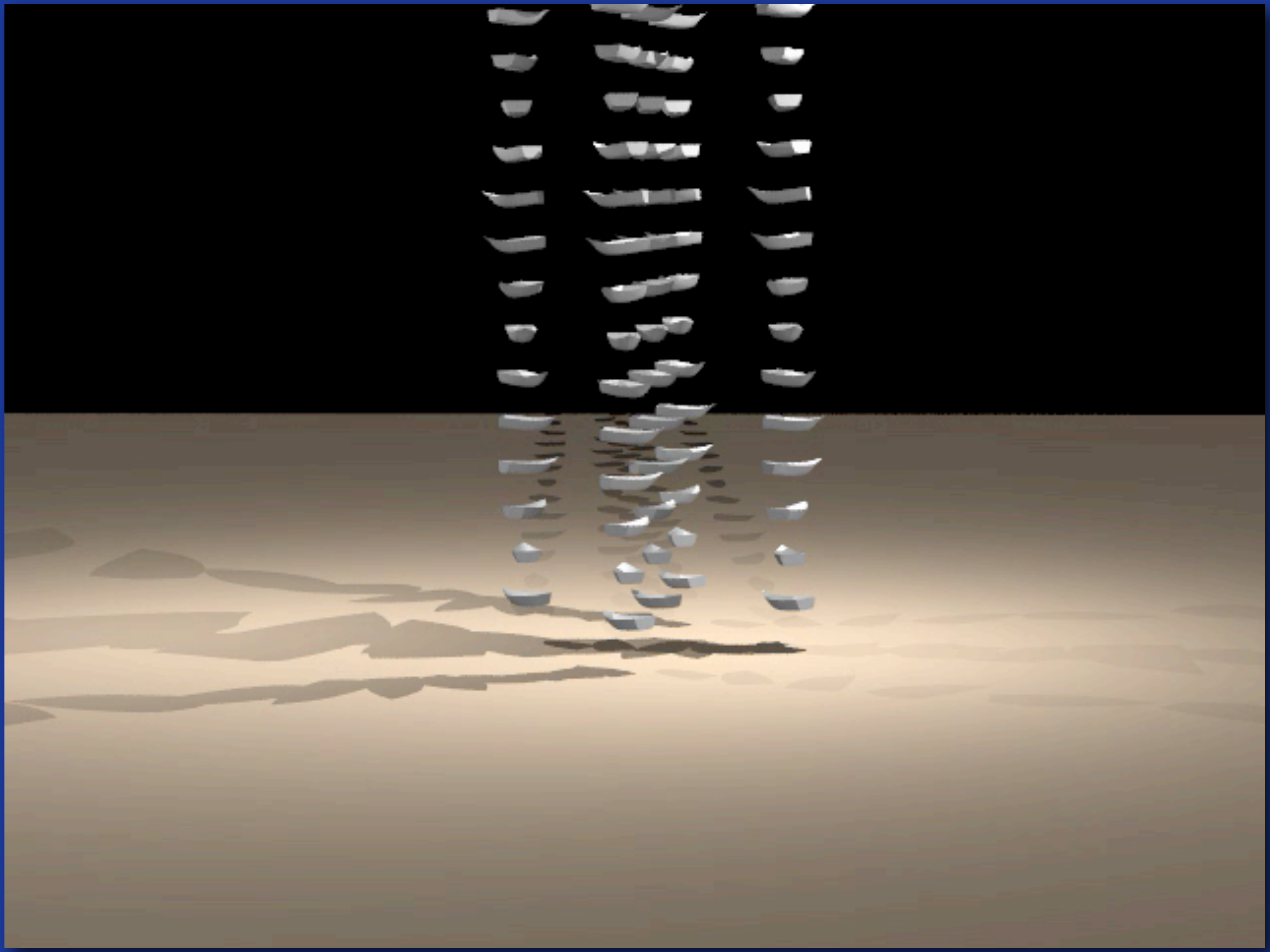
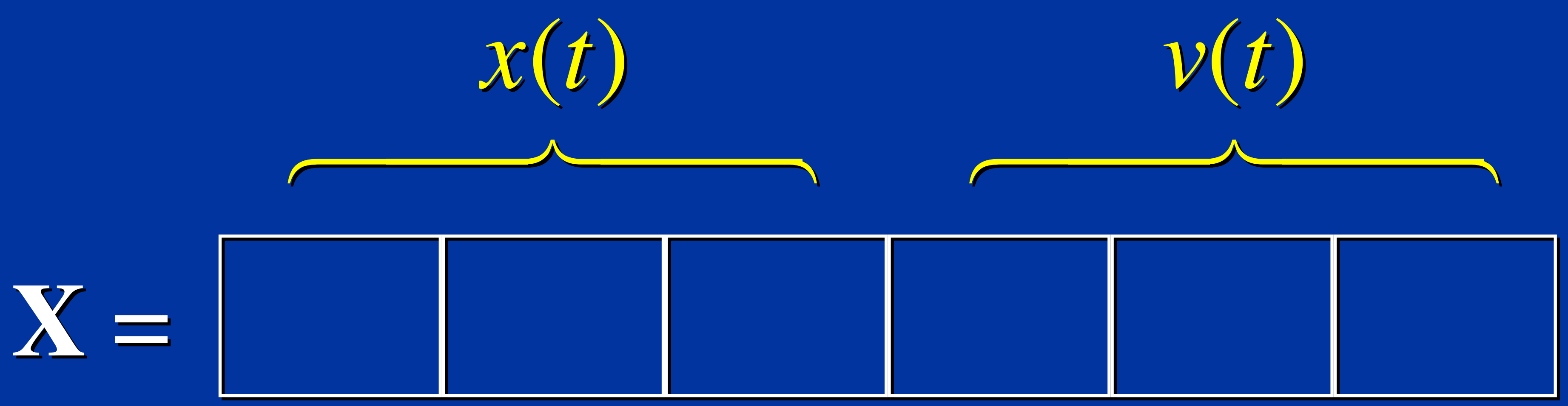
PIXAR
ANIMATION STUDIOS

What is a Rigid Body?

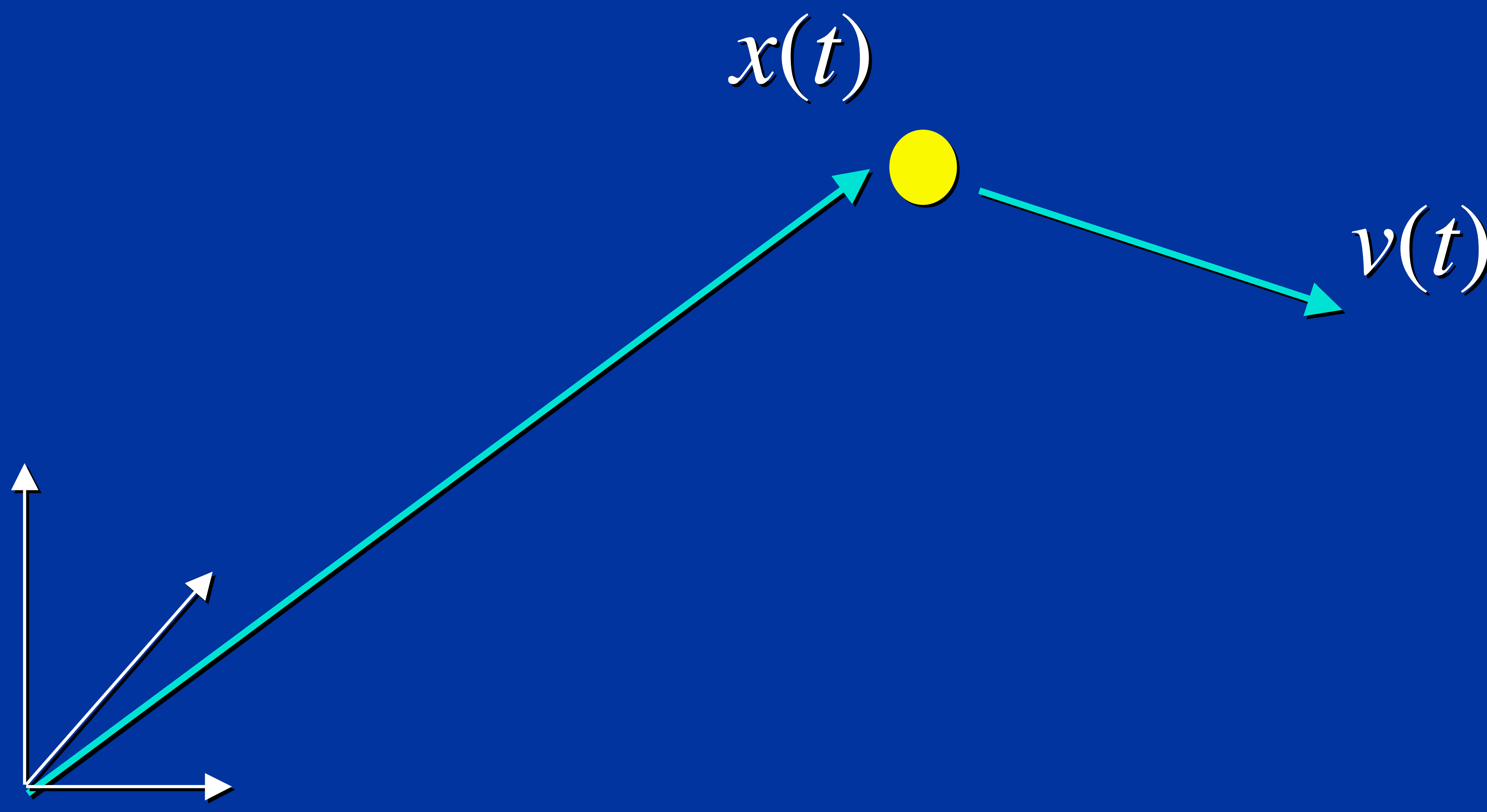


Particle State

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$$



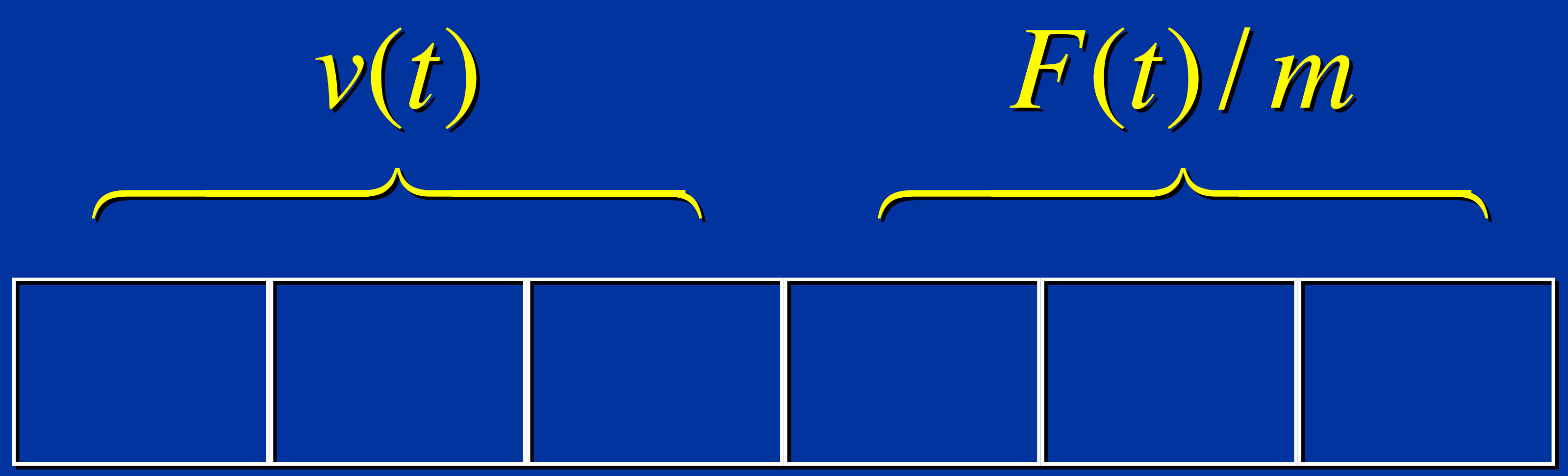
Particle Motion



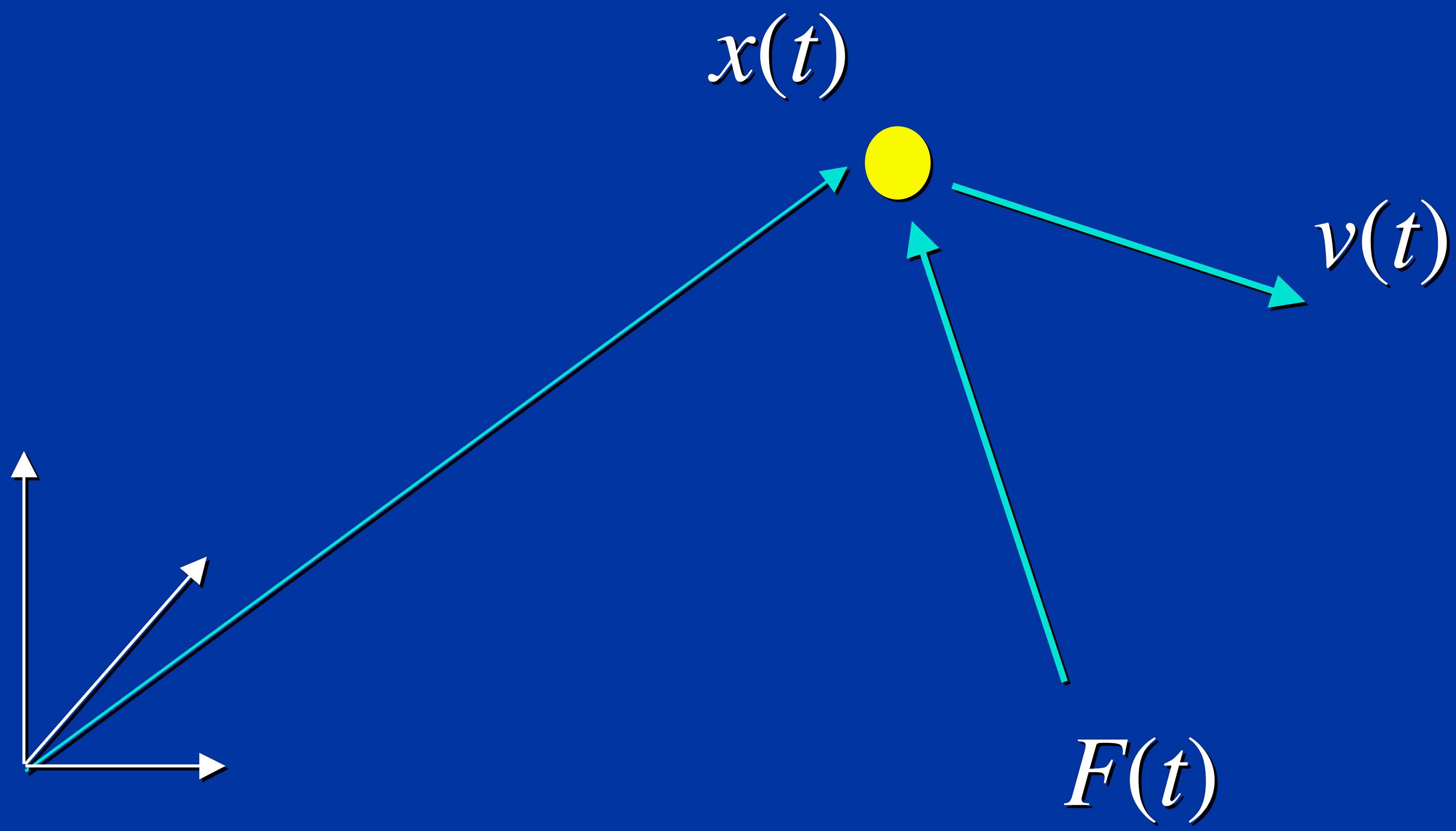
State Derivative

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ F(t)/m \end{pmatrix}$$

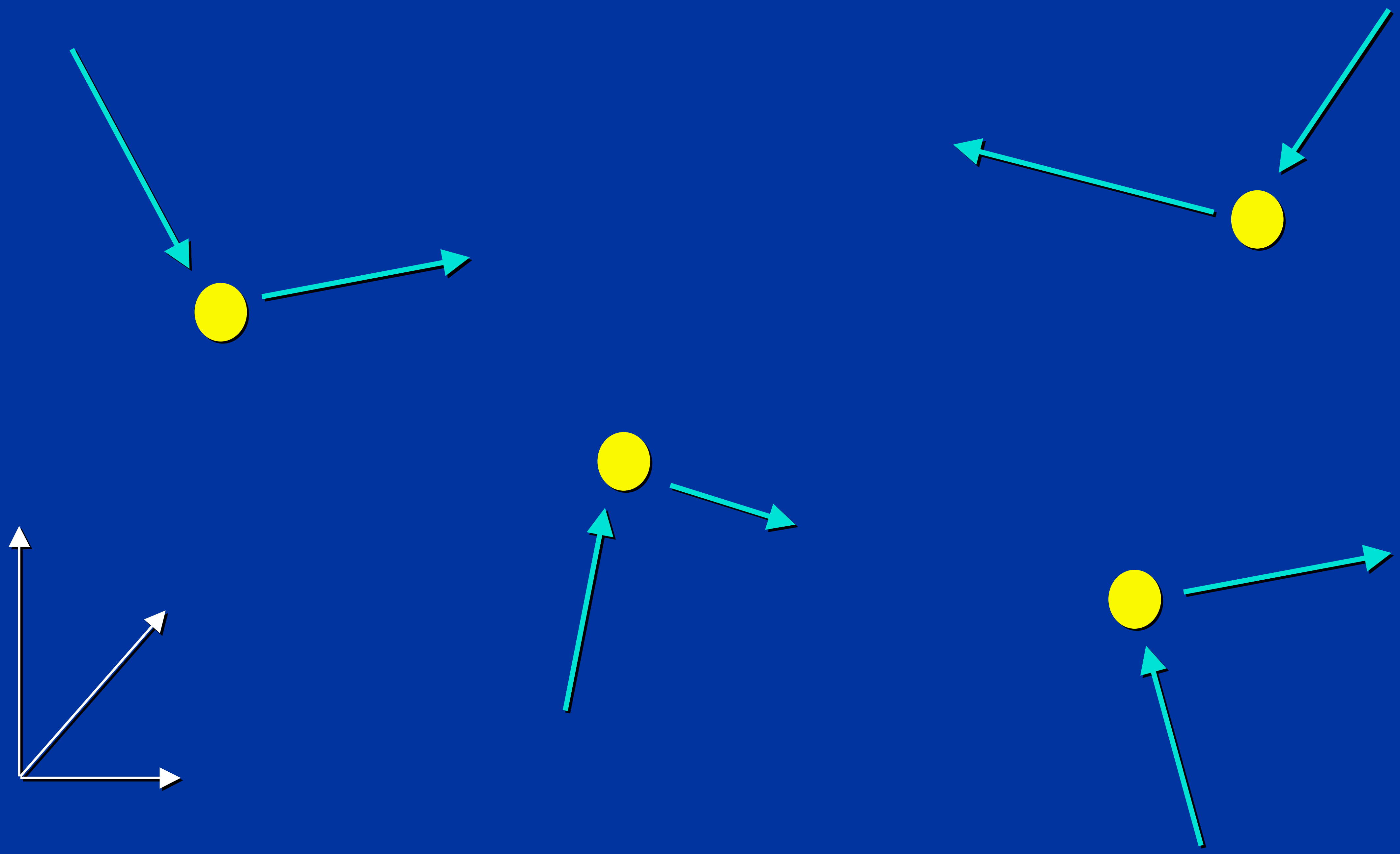
$$\frac{d}{dt}\mathbf{X} =$$



Particle Dynamics



Multiple Particles



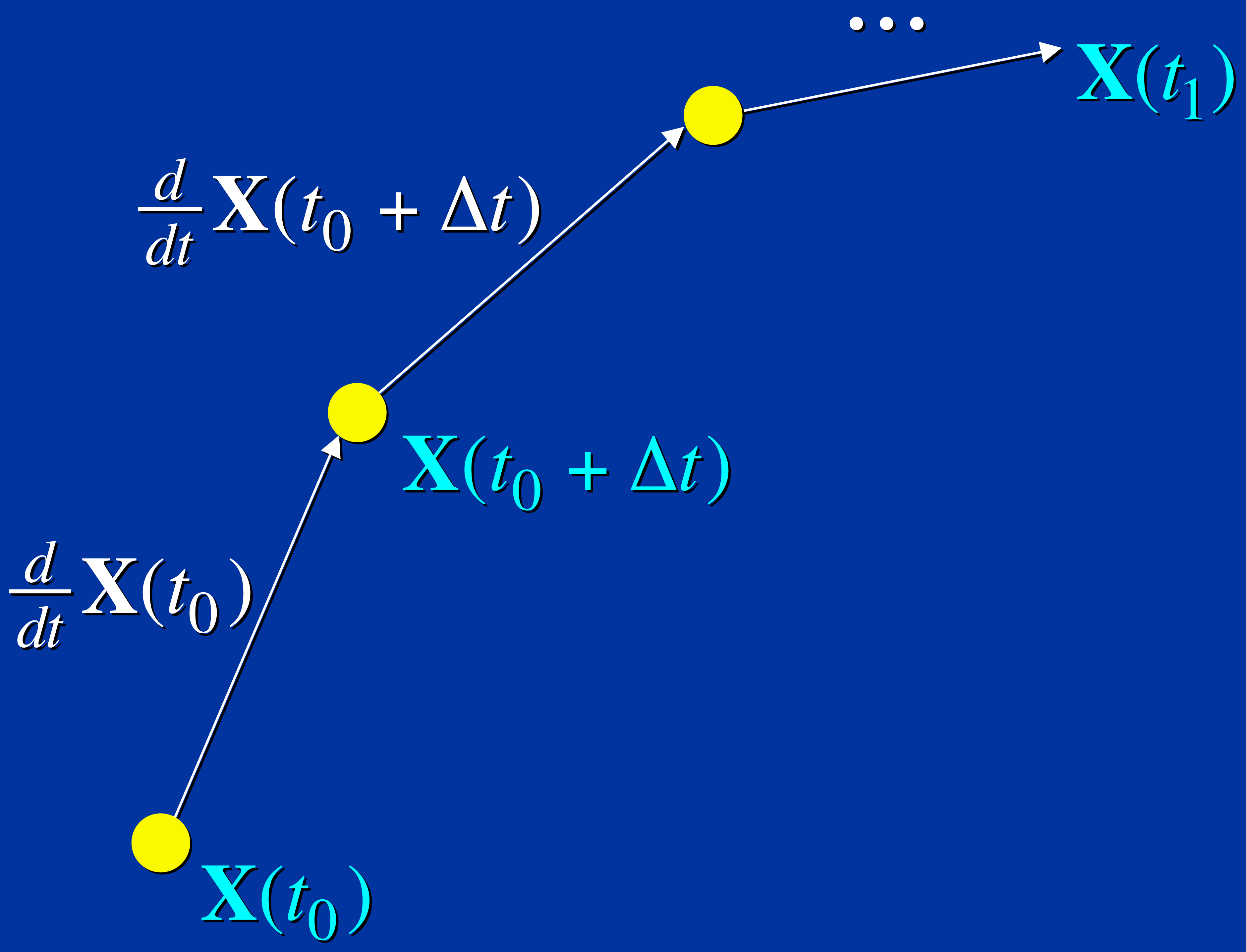
State Derivative

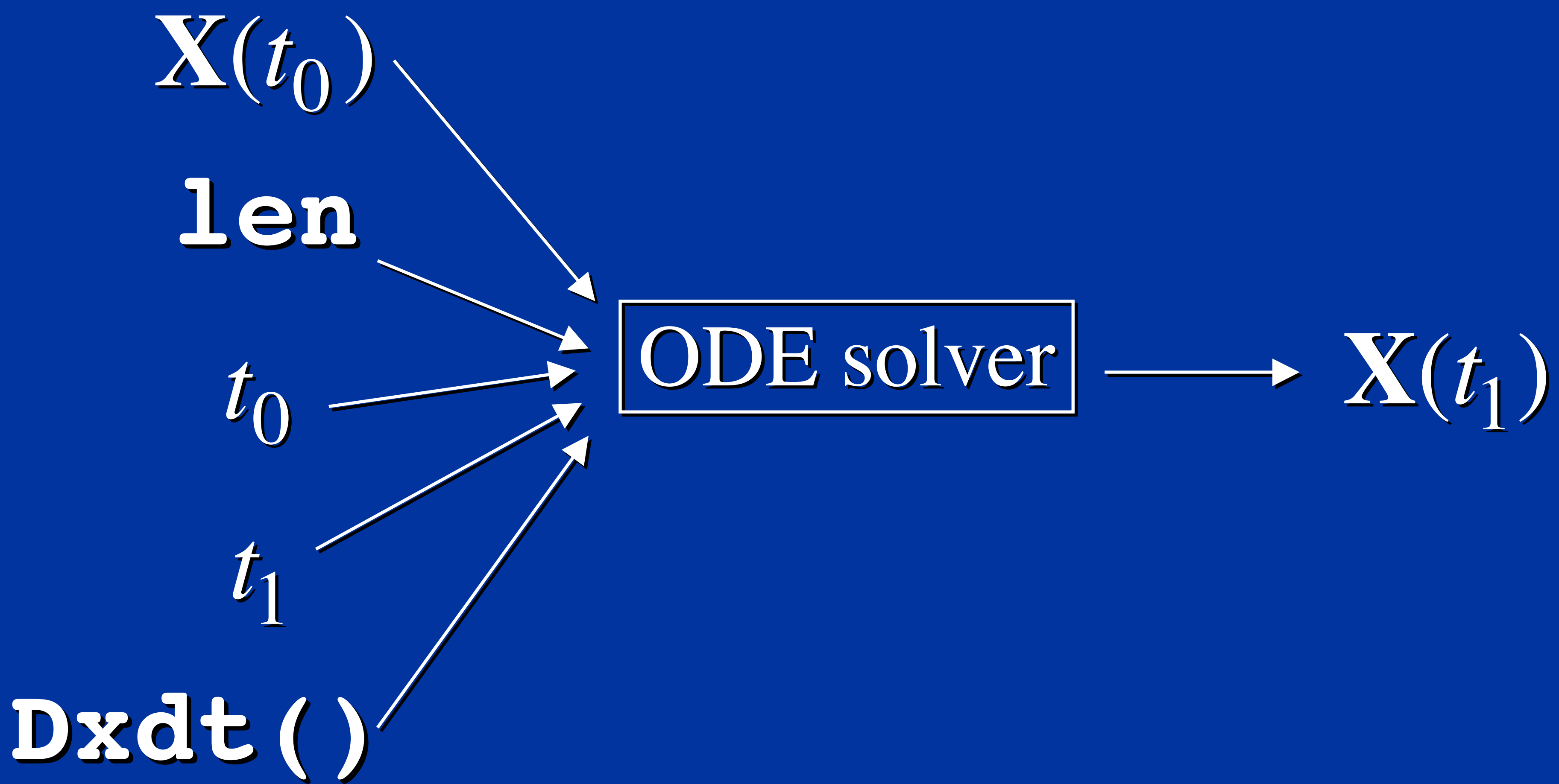
$$\frac{d}{dt}\mathbf{X} = \frac{d}{dt}\begin{pmatrix} x_1(t) \\ v_1(t) \\ \vdots \\ x_n(t) \\ v_n(t) \end{pmatrix} = \begin{pmatrix} v_1(t) \\ F_1(t)/m_1 \\ \vdots \\ v_n(t) \\ F_n(t)/m_n \end{pmatrix}$$

$$\frac{d}{dt}\mathbf{X} =$$

		··· $6n$ elements ···		

ODE solution





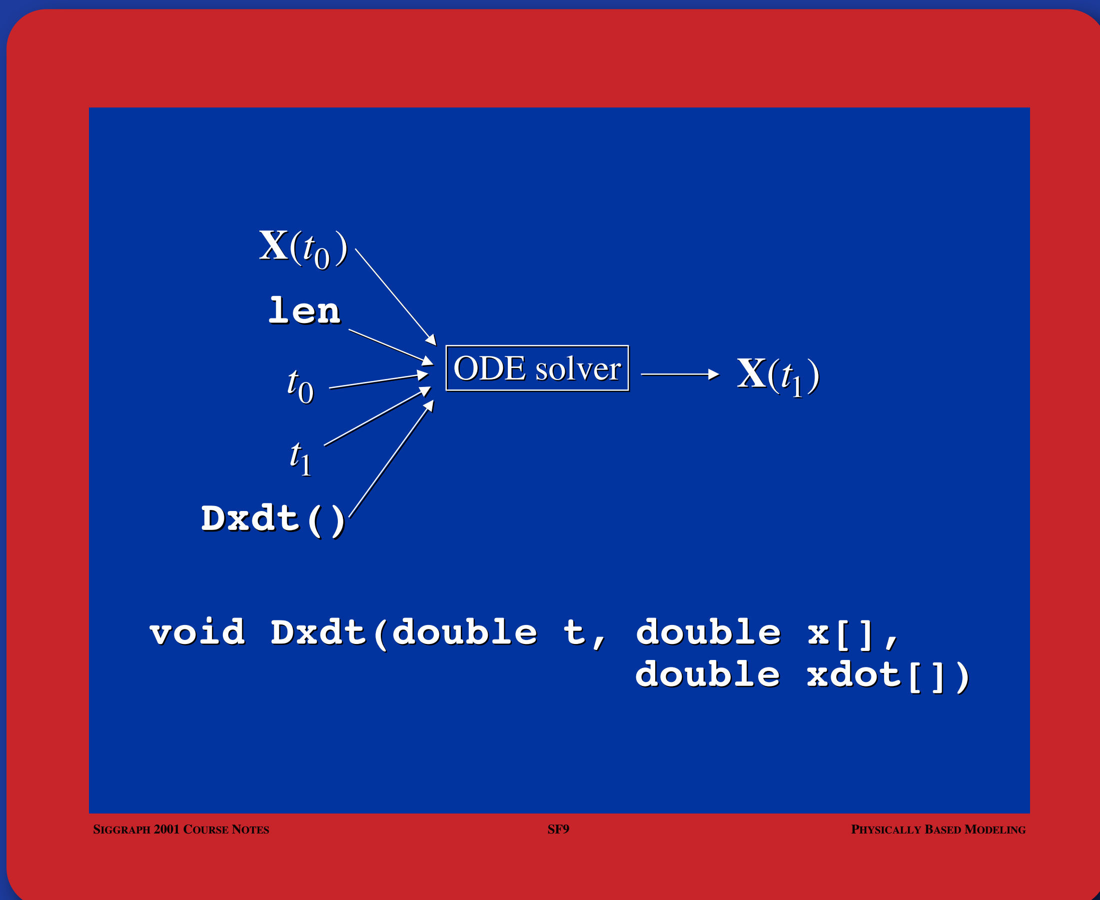

```
void Dxdt(double t, double x[],
                    double xdot[])
```

`Dxdt()`

$$\mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ v_1(t) \\ \vdots \\ x_n(t) \\ v_n(t) \end{bmatrix}$$

$$\frac{d}{dt}\mathbf{X}(t) = \begin{bmatrix} v_1(t) \\ F_1(t)/m_1 \\ \vdots \\ v_n(t) \\ F_n(t)/m_n \end{bmatrix}$$

What We Have



X(t0)
len
t0
t1
Dxdt()
ODE solver
X(t1)
void Dxdt(double t, double x[],
double xdot[])
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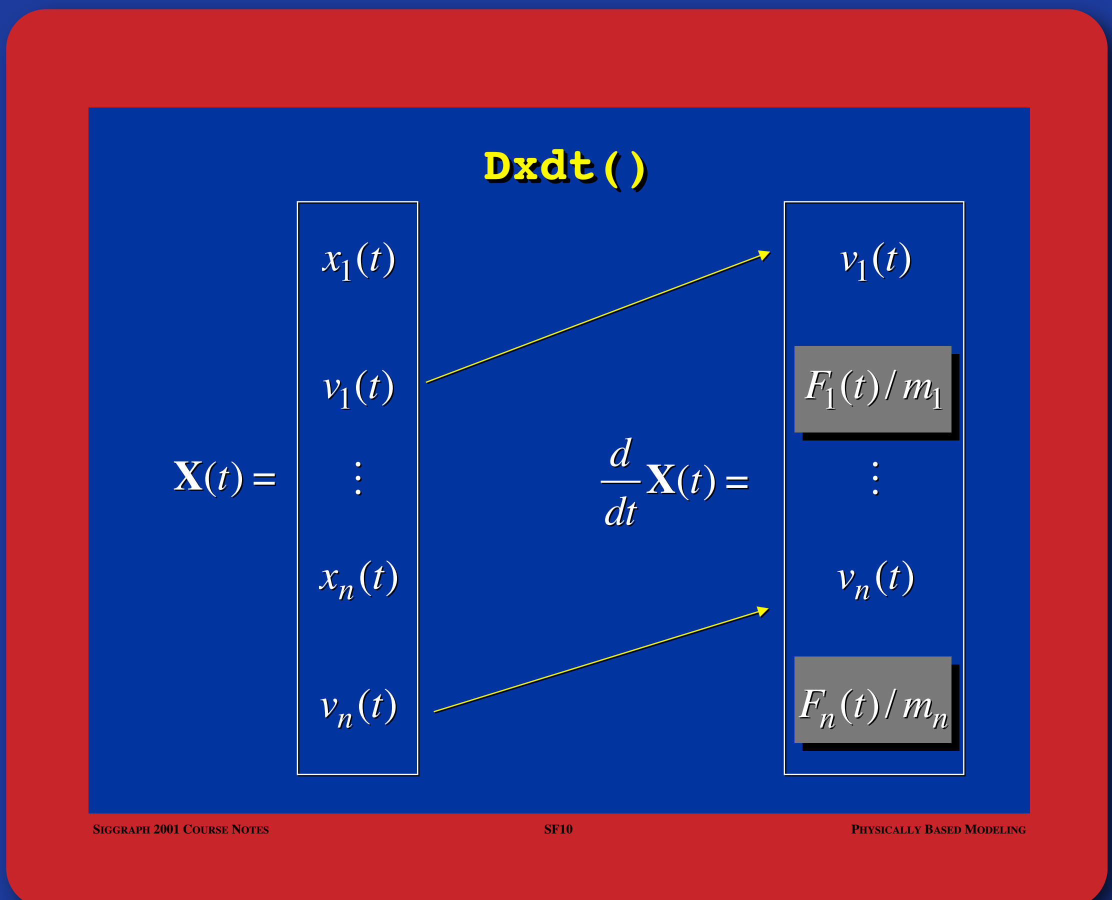
SF9
PHYSICALLY BASED MODELING



Dxdt()
X(t) =
x1(t)
v1(t)
xn(t)
vn(t)
d/dt X(t) =
v1(t)
F1(t)/m1
vn(t)
Fn(t)/mn
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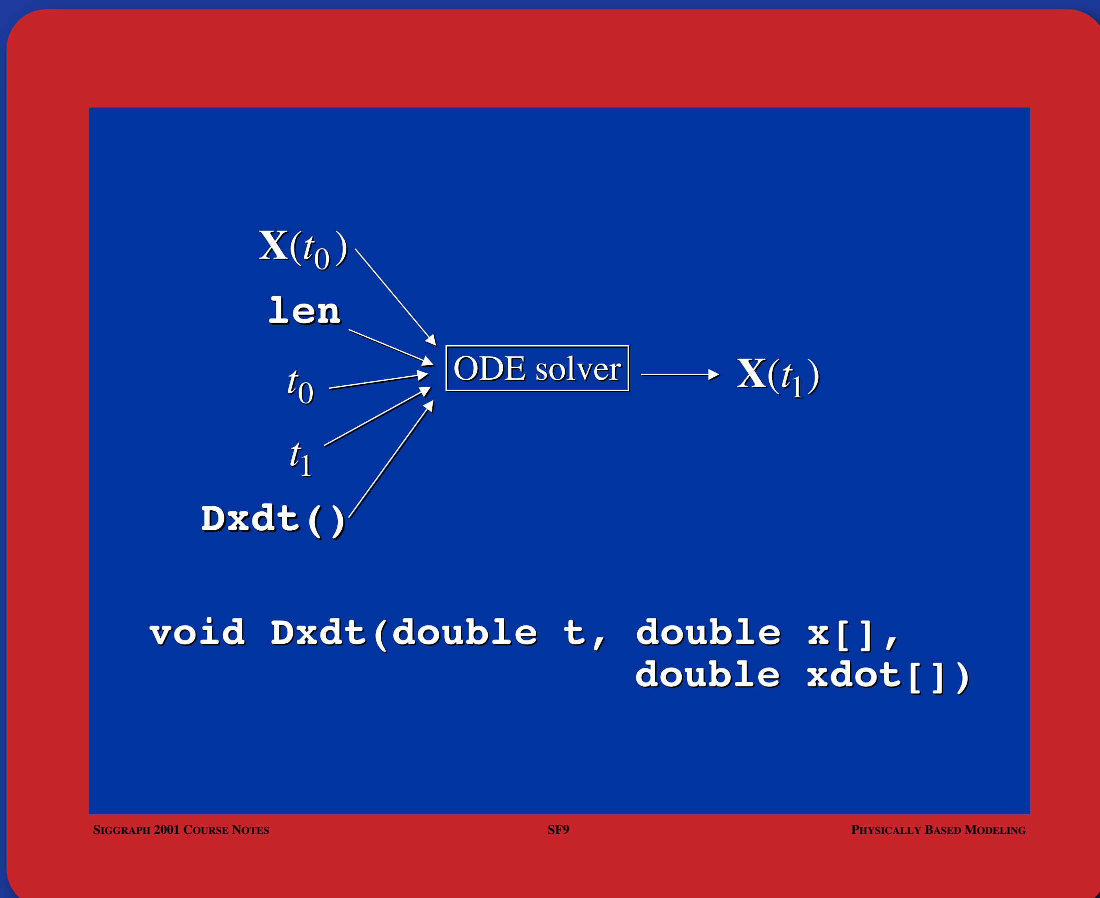
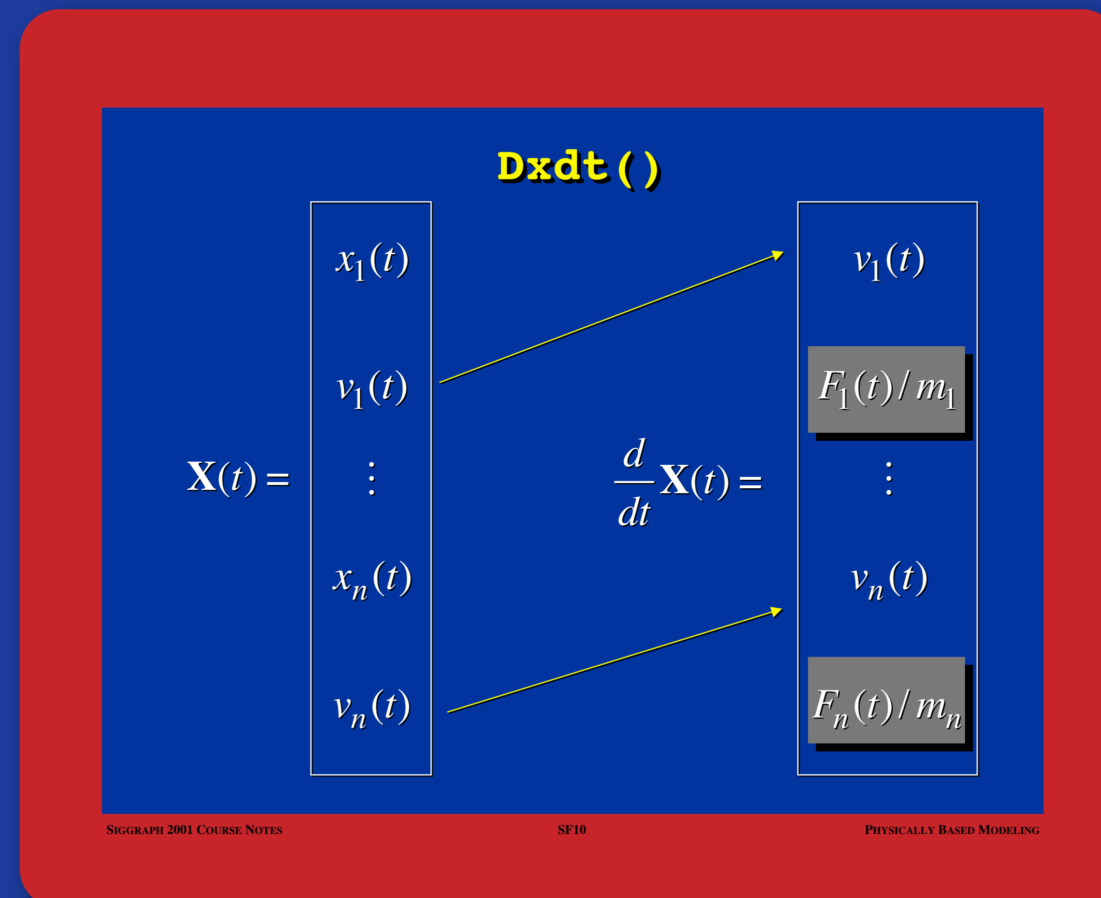
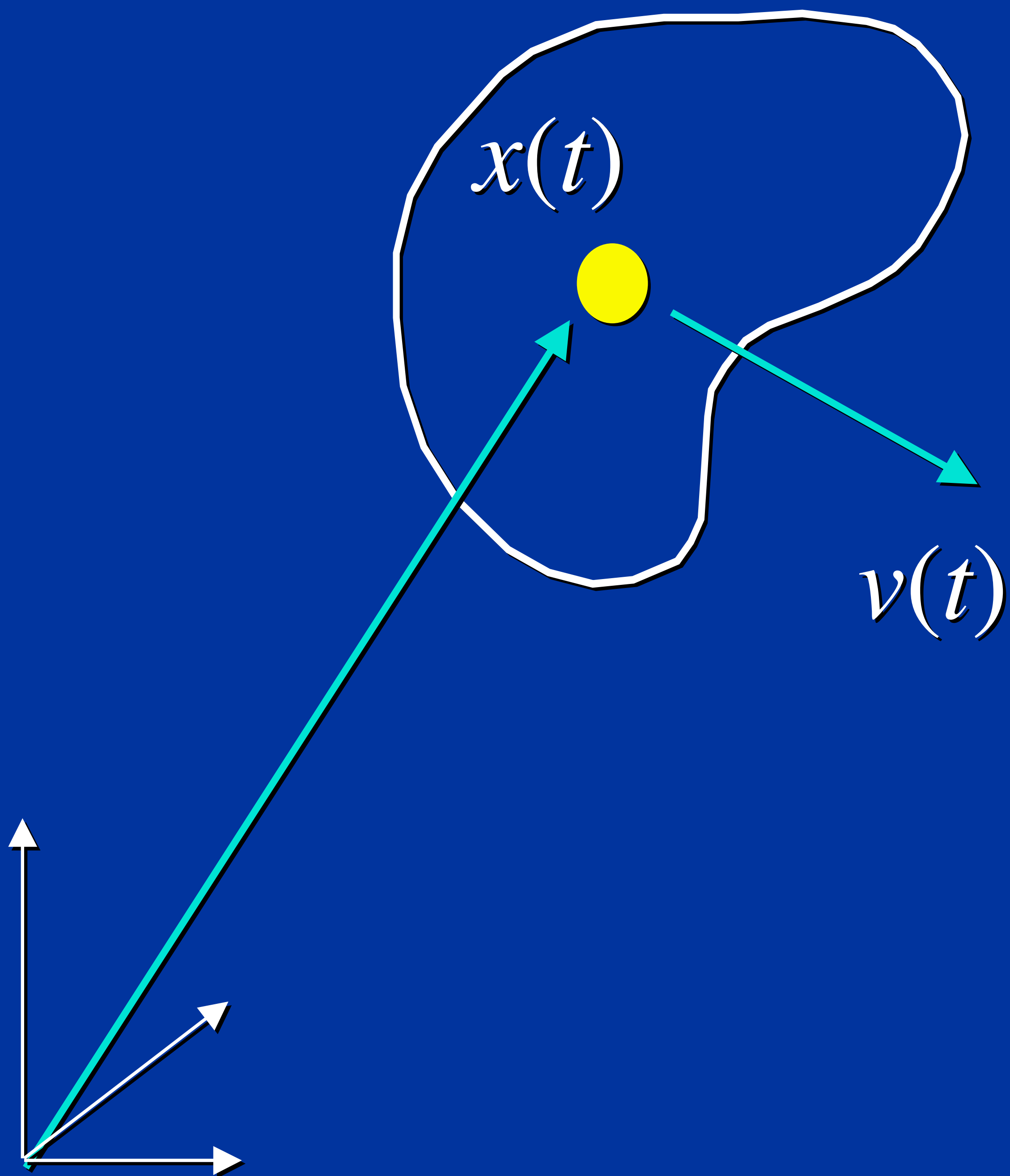
Our Goal





Replicate this approach
for rigid bodies.

Rigid Body State



$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ ? \\ v(t) \\ ? \end{pmatrix}$$

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ ? \\ Mv(t) \\ ? \end{pmatrix} = \begin{pmatrix} ? \end{pmatrix}$$

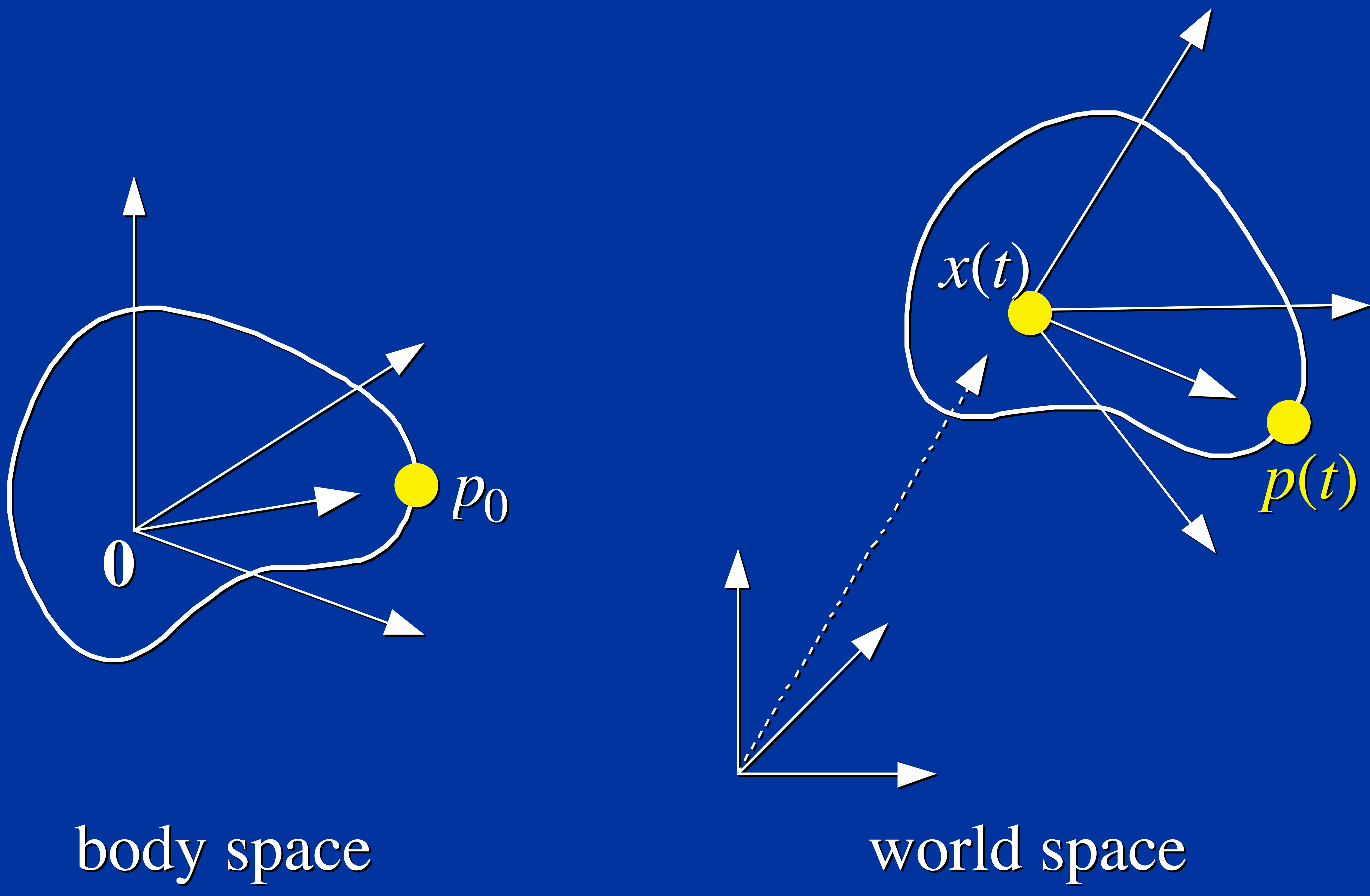
- Use Momentum P=Mv instead of just v.
- What is this?

Orientation

We represent orientation as a rotation matrix[†] $\mathbf{R}(t)$. Points are transformed from body-space to world-space as:

$$p(t) = \mathbf{R}(t)\, p_0 + x(t)$$

[†] He's lying. Actually, we use quaternions.

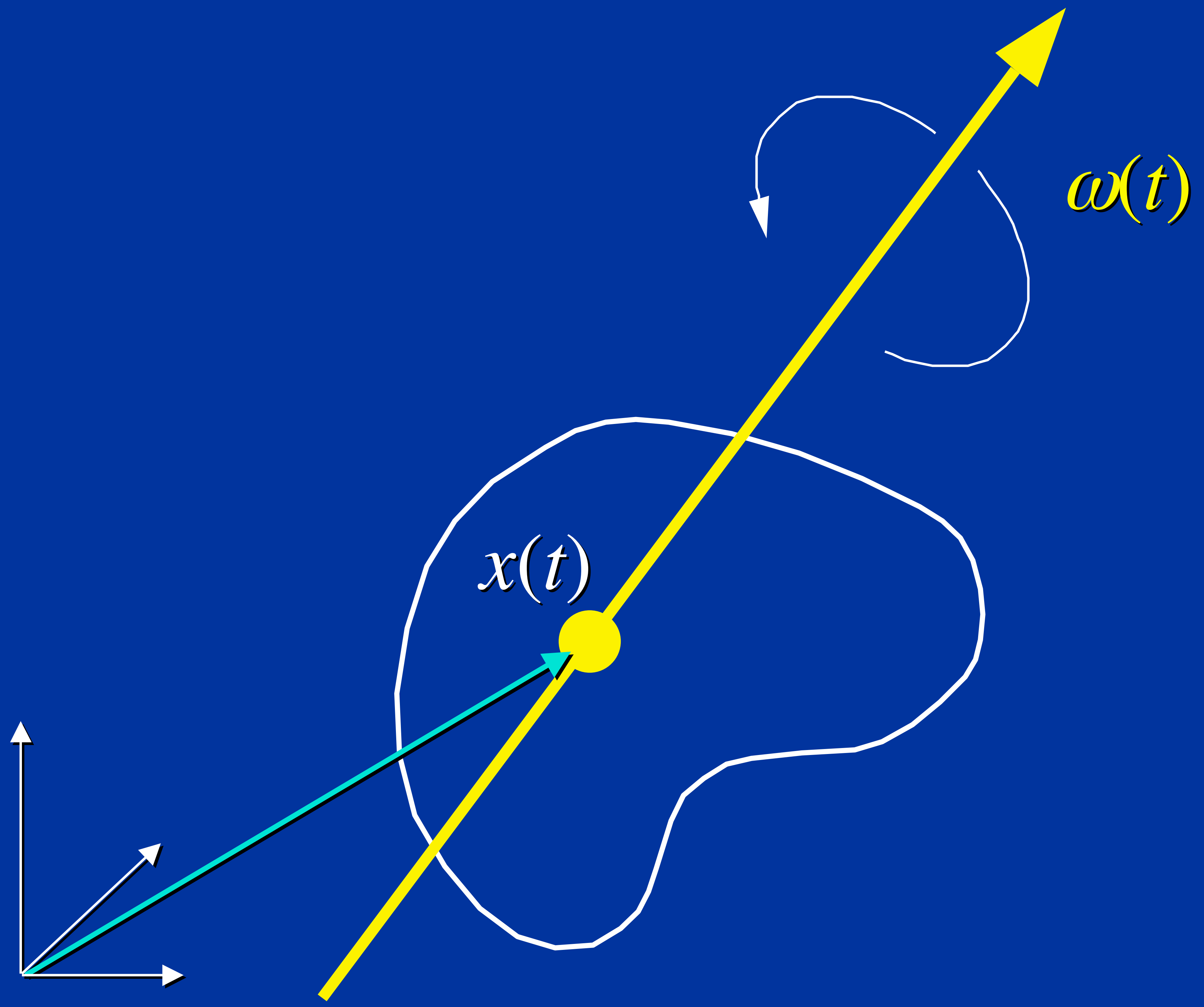

p_0
0
body space
$x(t)$
$p(t)$
world space

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ R(t) \\ Mv(t) \\ ? \end{pmatrix} = \begin{pmatrix} ? \end{pmatrix}$$

- What is this?

Angular Velocity Definition

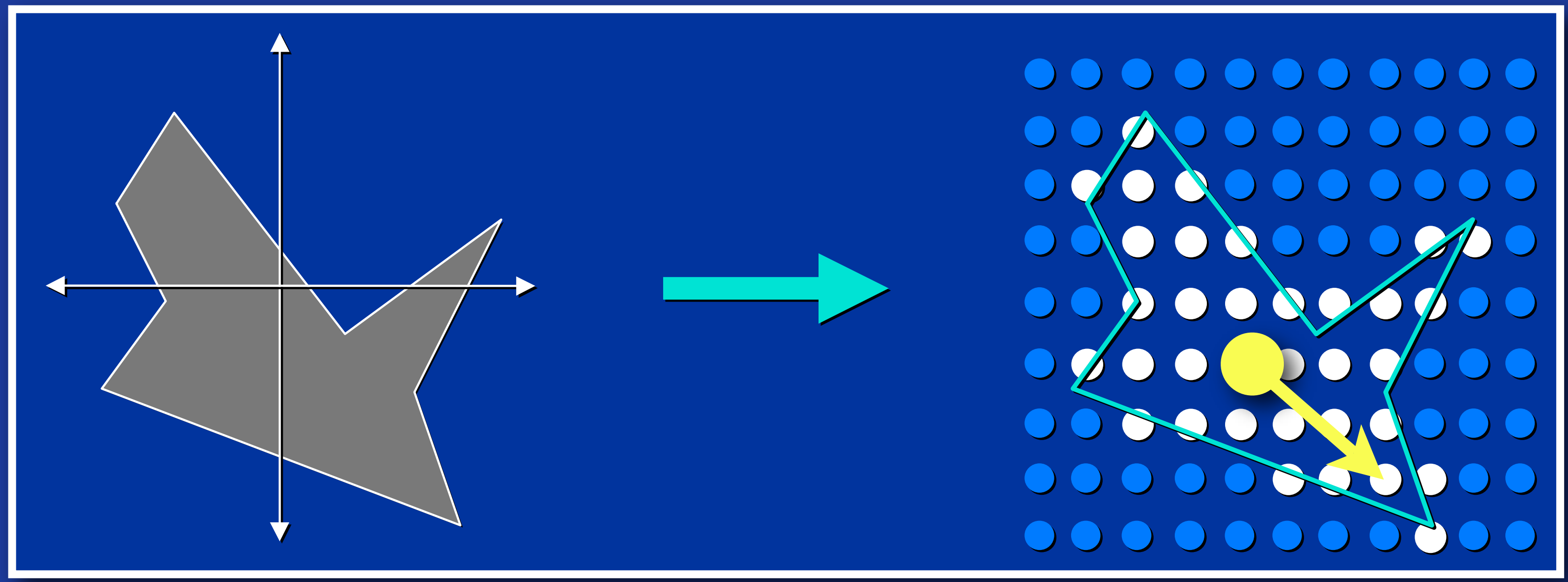


Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ R(t) \\ Mv(t) \\ \omega(t) \end{pmatrix} = \begin{pmatrix} ? \end{pmatrix}$$

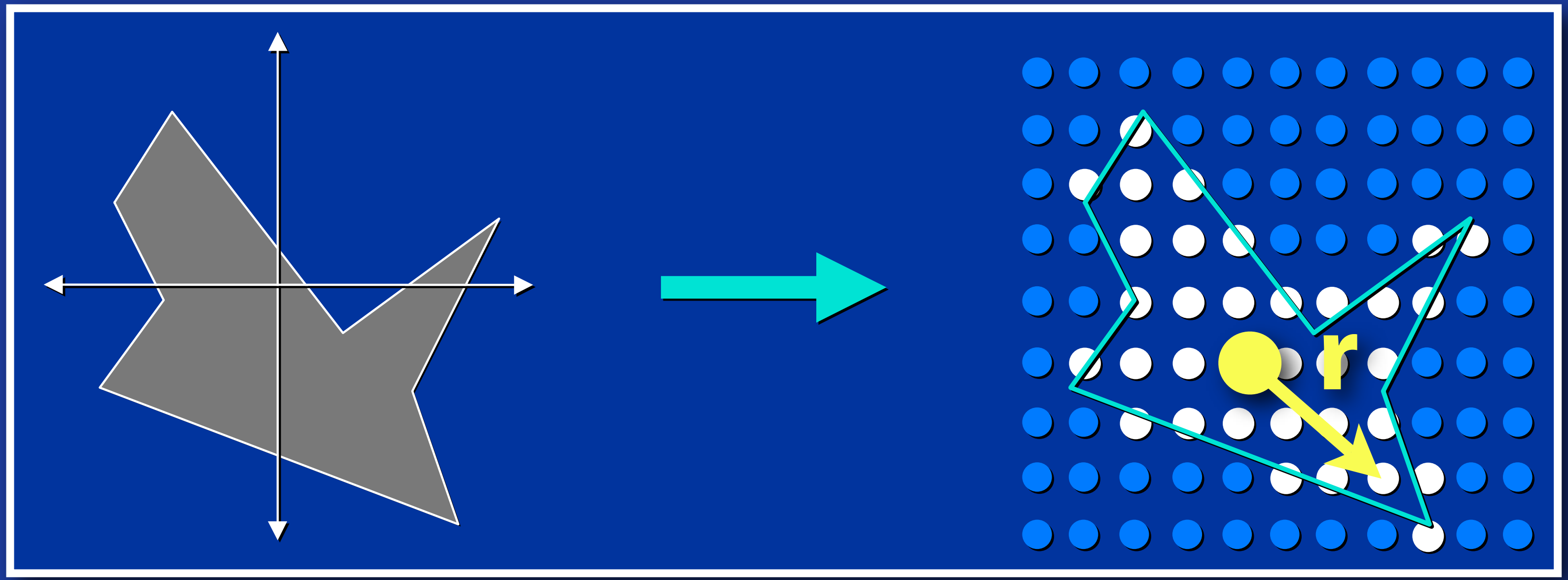
- **What is this?**

Discretized View



- **Total Mass:** $$M = \sum_i m_i$$
- **Center of Mass:** $$\bar{\mathbf{x}} = \frac{1}{M}\sum_i m_i \mathbf{x}_i$$
- **Relative Position:** $\mathbf{r}_i = \mathbf{x}_i - \bar{\mathbf{x}}$

Discretized View



- **Basic Principles:**
  - **Conservation of Linear Momentum**

$$\frac{d}{dt}\sum_i m_i \dot{\mathbf{x}}_i = 0$$

  - **Conservation of Angular Momentum**

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = 0$$

Conservation and Forces

Linear Momentum

$$\frac{d}{dt}\sum_i m_i \dot{\mathbf{x}}_i = \sum_i \mathbf{f}_i$$

||(def)

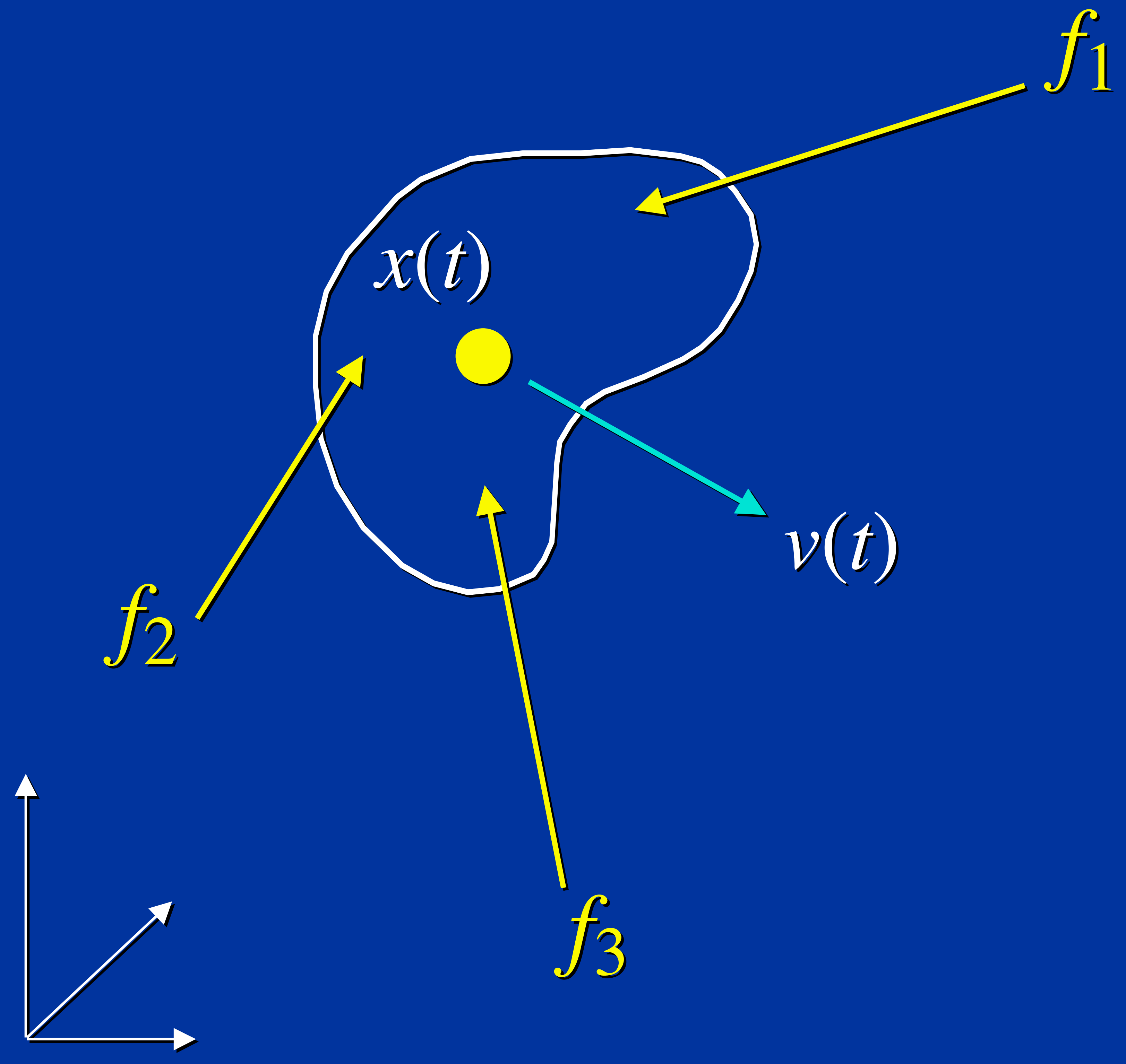
$$\sum_i m_i \ddot{\mathbf{x}}_i = \mathbf{F}$$

$$\left(\bar{\mathbf{x}} = \frac{1}{M}\sum_i m_i \mathbf{x}_i\right)$$

$$\left(M\ddot{\bar{\mathbf{x}}} = \sum_i m_i \ddot{\mathbf{x}}_i\right)$$

$$M\ddot{\bar{\mathbf{x}}} = \mathbf{F}$$

Net Force



$$F(t) = \sum f_i$$

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ R(t) \\ Mv(t) \\ \omega(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ ? \\ F(t) \\ ? \end{pmatrix}$$

- What are these?

Angular Velocity

We represent angular velocity as a vector $\omega(t)$, which encodes both the axis of the spin and the speed of the spin.

How are $\mathbf{R}(t)$ and $\omega(t)$ related?

Angular Velocity

$\dot{\mathbf{R}}(t)$ and $\omega(t)$ are related by:

$$\frac{d}{dt}\mathbf{R}(t) = \begin{pmatrix} 0 & -\omega_z(t) & \omega_y(t) \\ \omega_z(t) & 0 & -\omega_x(t) \\ -\omega_y(t) & \omega_x(t) & 0 \end{pmatrix} \mathbf{R}(t)$$

$$= \omega(t)^* \mathbf{R}(t)$$

w^* can be viewed as the matrix form of $-(w\times)$

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ \mathbf{R}(t) \\ Mv(t) \\ \langle \omega(t) \rangle \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^{*}\mathbf{R}(t) \\ F(t) \\ ? \end{pmatrix}$$

Need to relate $\dot{\omega}(t)$ and mass distribution to $F(t)$.

Conservation and Forces

Linear Momentum

$$\frac{d}{dt}\sum_i m_i \dot{\mathbf{x}}_i = \sum_i \mathbf{f}_i$$

|| (def)

$$\sum_i m_i \ddot{\mathbf{x}}_i = \mathbf{F}$$

$$\left(\bar{\mathbf{x}} = \frac{1}{M}\sum_i m_i \mathbf{x}_i\right)$$

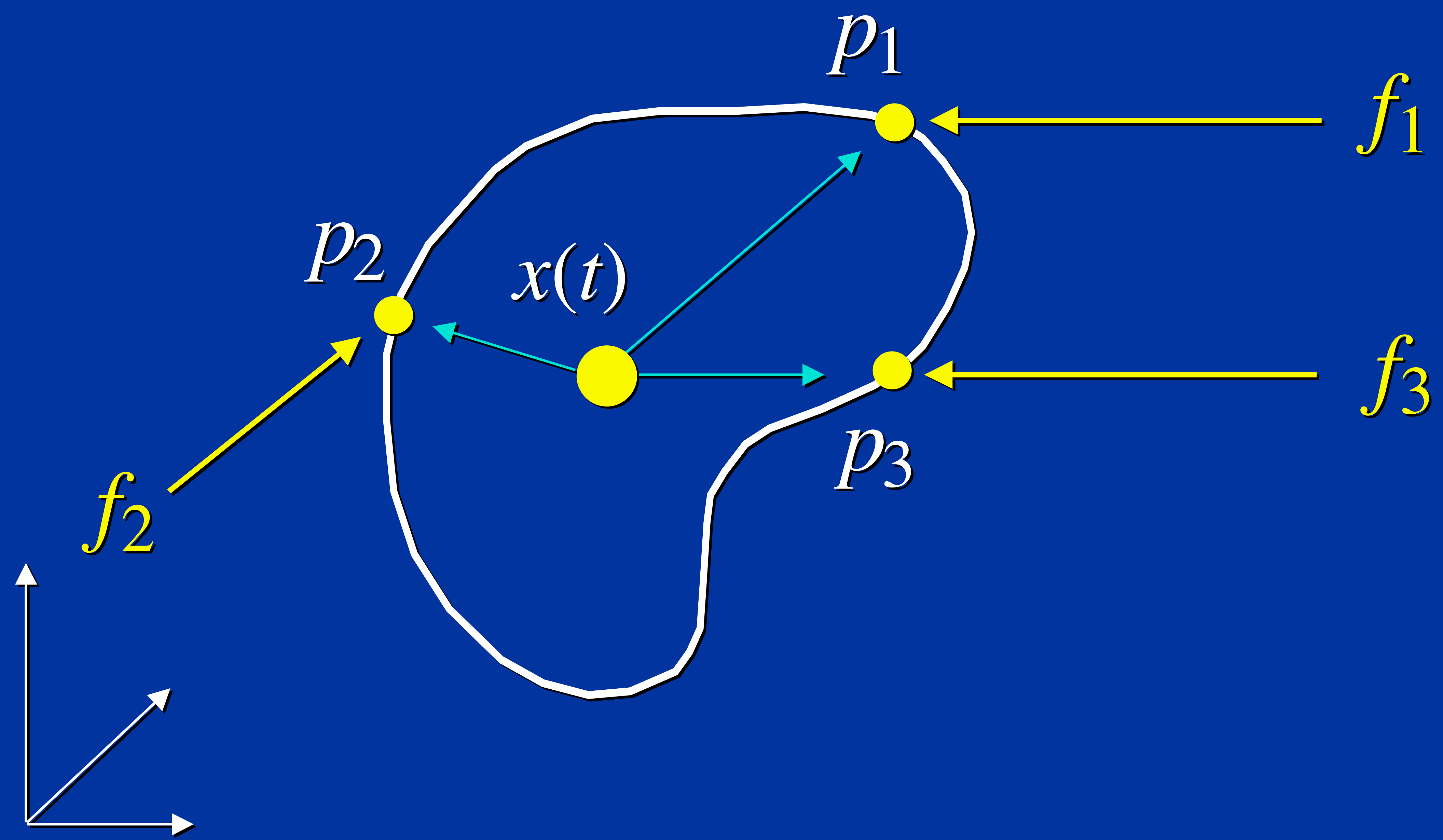
$$\left(M\ddot{\bar{\mathbf{x}}} = \sum_i m_i \ddot{\mathbf{x}}_i\right)$$

$$M\ddot{\bar{\mathbf{x}}} = \mathbf{F}$$

Angular Momentum

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \sum_i \mathbf{r}_i \times \mathbf{f}_i$$

Net Torque



$$\tau(t) = \sum \left(p_i - x(t) \right) \times f_i$$

Conservation and Forces

Linear Momentum

$$\frac{d}{dt}\sum_i m_i \dot{\mathbf{x}}_i = \sum_i \mathbf{f}_i$$

|| (def)

$$\sum_i m_i \ddot{\mathbf{x}}_i = \mathbf{F}$$

$$\left(\bar{\mathbf{x}} = \frac{1}{M}\sum_i m_i \mathbf{x}_i\right)$$

$$\left(M\ddot{\bar{\mathbf{x}}} = \sum_i m_i \ddot{\mathbf{x}}_i\right)$$

$$M\ddot{\bar{\mathbf{x}}} = \mathbf{F}$$

Angular Momentum

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \sum_i \mathbf{r}_i \times \mathbf{f}_i$$

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \tau$$

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \omega \times \mathbf{r}_i = \tau$$

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^* \omega = \tau$$

Discrete Inertia

$$I = \sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^*$$

$$I = \sum_i \left(m_i \begin{bmatrix} -y^2 - z^2 & xy & xz \\ xy & -x^2 - z^2 & yz \\ xz & yz & -x^2 - y^2 \end{bmatrix} \right)$$

Conservation and Forces

Linear Momentum

$$\frac{d}{dt}\sum_i m_i \dot{\mathbf{x}}_i = \sum_i \mathbf{f}_i$$

|| (def)

$$\sum_i m_i \ddot{\mathbf{x}}_i = \mathbf{F}$$

$$\left(\bar{\mathbf{x}} = \frac{1}{M}\sum_i m_i \mathbf{x}_i\right)$$

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$$M\ddot{\bar{\mathbf{x}}} = \mathbf{F}$$

Angular Momentum

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \sum_i \mathbf{r}_i \times \mathbf{f}_i$$

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \tau$$

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \omega \times \mathbf{r}_i = \tau$$

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^* \omega = \tau$$

$$\frac{d}{dt} I\omega = \tau$$

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ \mathbf{R}(t) \\ Mv(t) \\ \mathbf{I}(t)\omega(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^{*}\mathbf{R}(t) \\ F(t) \\ \tau(t) \end{pmatrix}$$

$P(t)$ – linear momentum

$L(t)$ – angular momentum

Discrete Inertia

$$I = \sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^*$$

$$I = \sum_i \left(m_i \begin{bmatrix} -y^2 - z^2 & xy & xz \\ xy & -x^2 - z^2 & yz \\ xz & yz & -x^2 - y^2 \end{bmatrix} \right)$$

Continuous Inertia

$$\mathbf{I}(t) = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

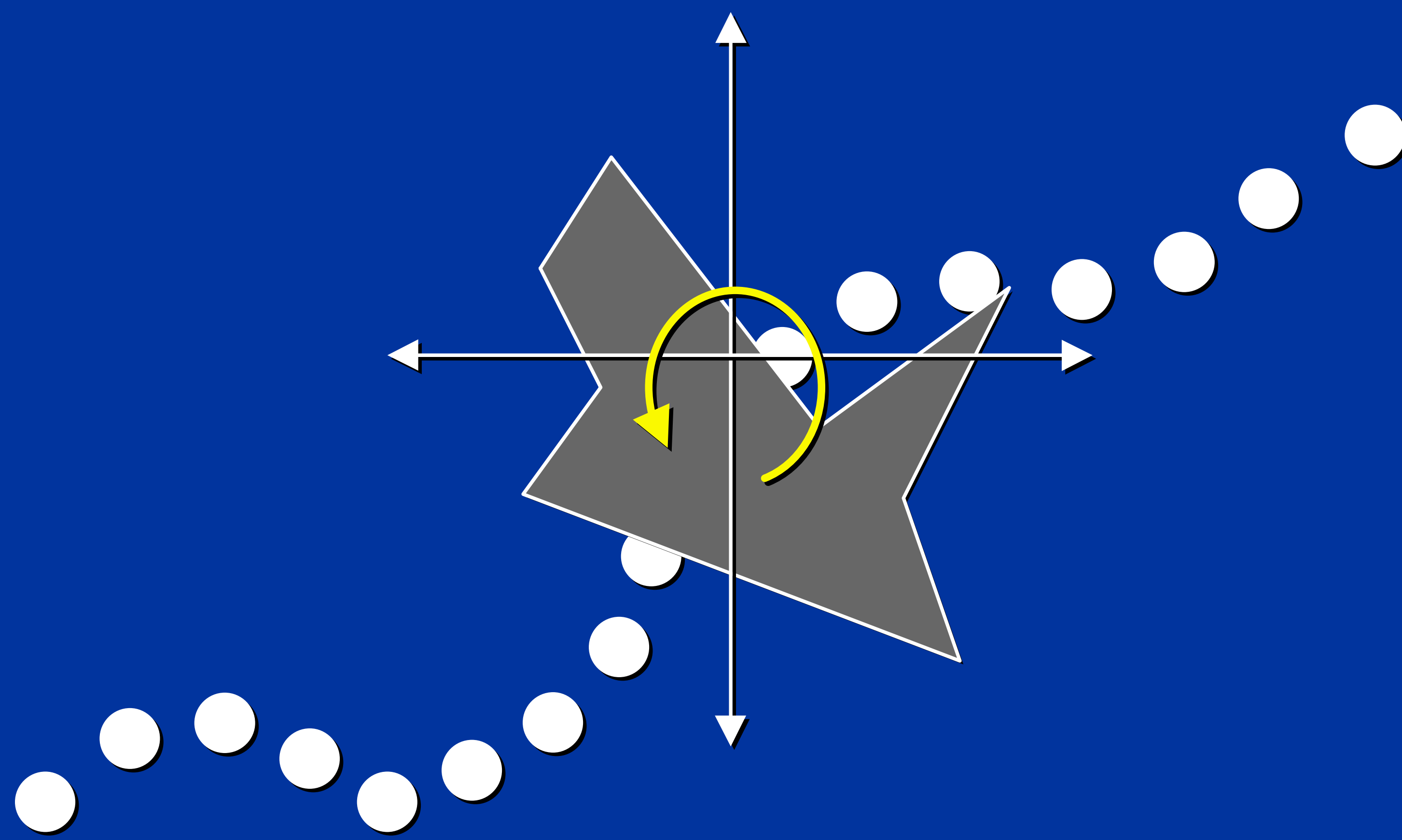
diagonal terms

$$I_{xx} = M \int_V (y^2 + z^2)\, dV$$

off-diagonal terms

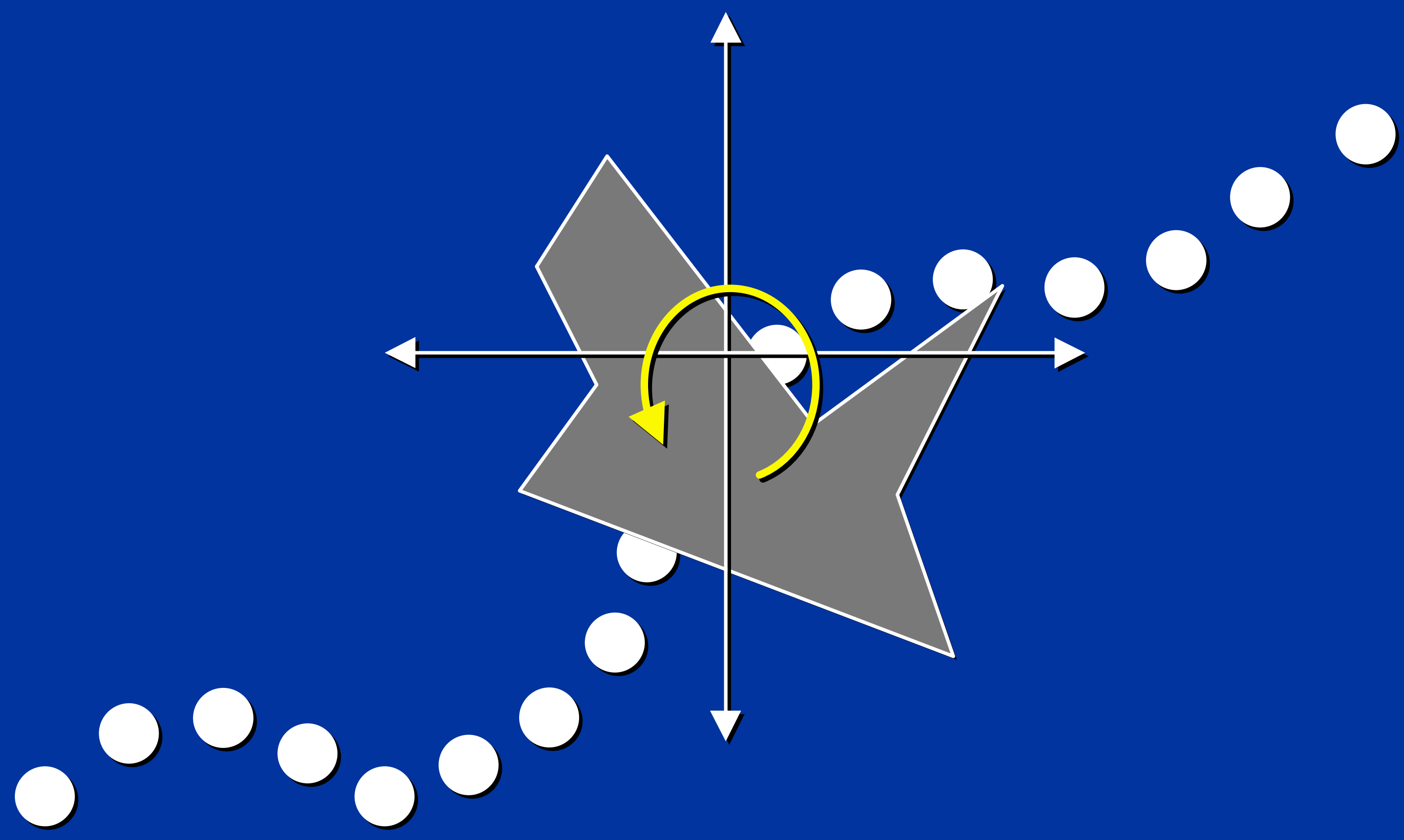
$$I_{xy} = -M \int_V xy\, dV$$

Inertia Tensors Vary in World Space...



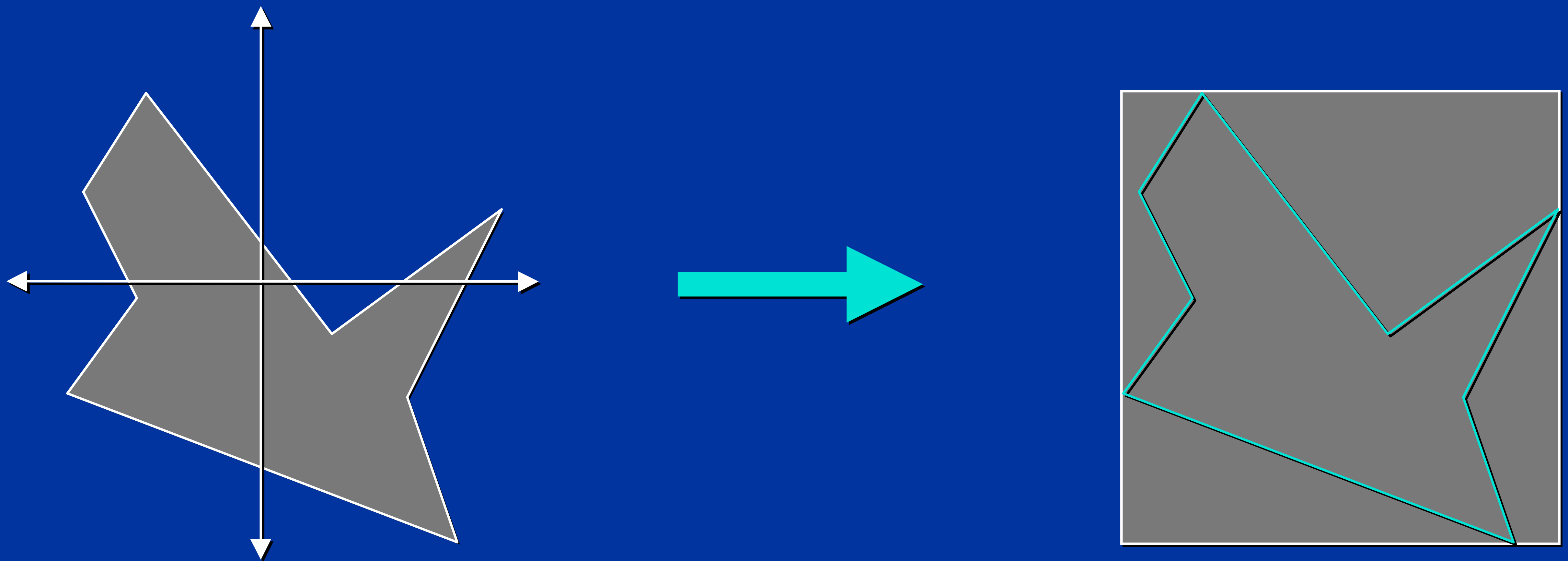
$$I_{xx} = M\int_V (y^2 + z^2)\, dV \qquad I_{xy} = -M\int_V xy\, dV$$

... but are Constant in Body Space



$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{\text{body}}\mathbf{R}(t)^T$$

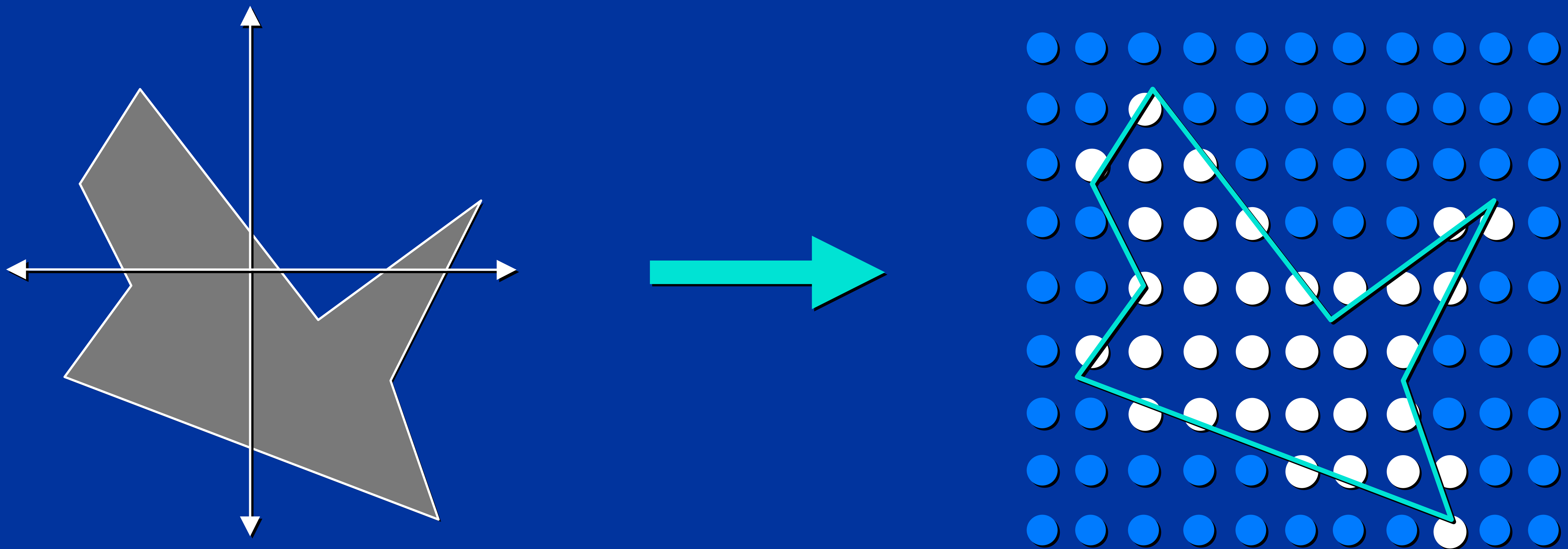
Approximating I_{body}: Bounding Boxes



Pros: Simple.

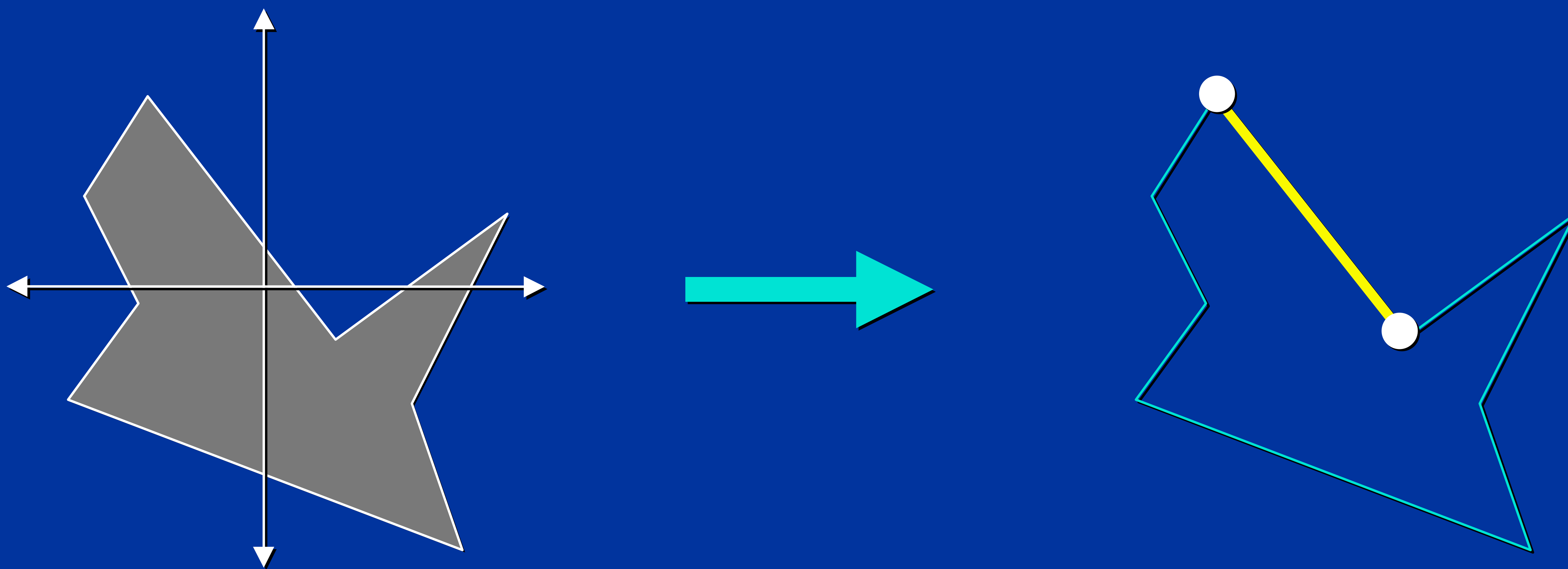
Cons: Bounding box may not be a good fit.
Inaccurate.

Approximating I_{body}: Point Samping



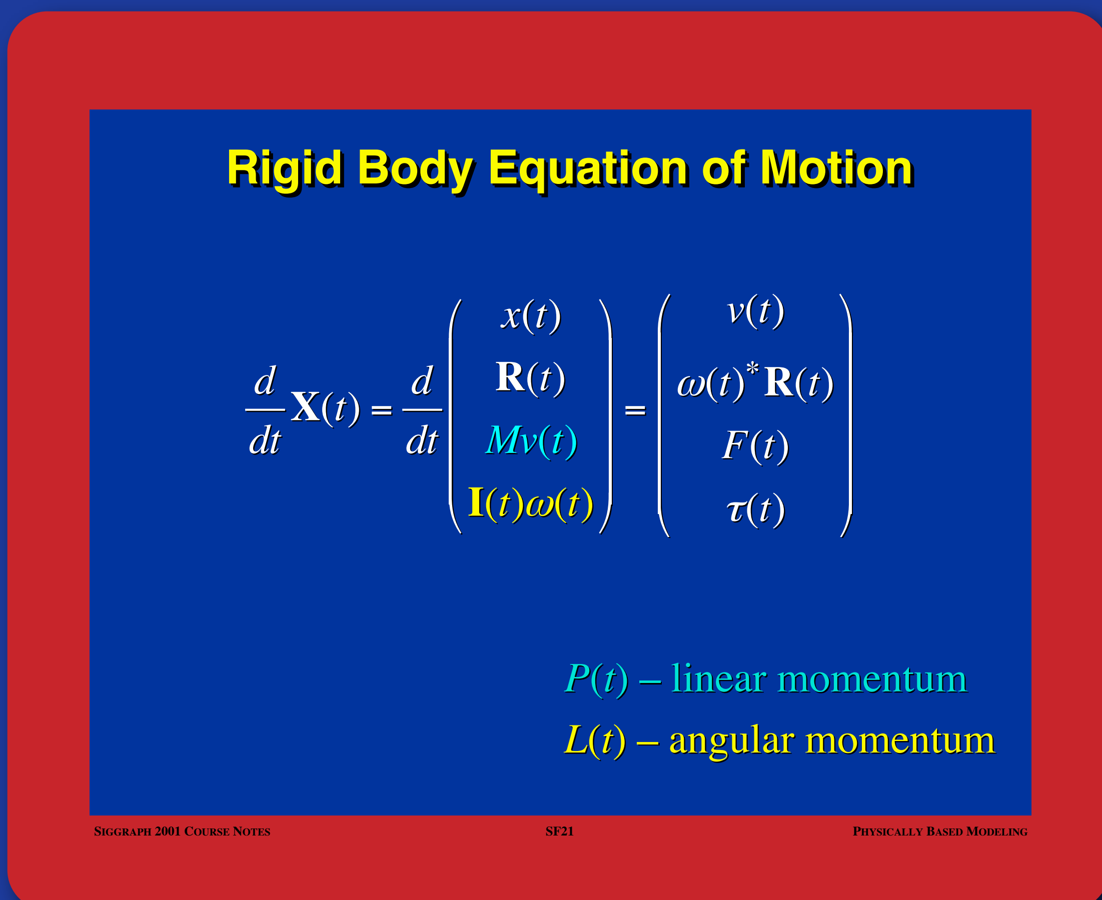
Pros: Simple, fairly accurate, no B-rep needed.
Cons: Expensive, requires volume test.

Computing I_{body}: Green's Theorem (2x!)



Pros: Simple, exact, no volumes needed.
Cons: Requires boundary representation.
Code: http://www.acm.org/jgt/papers/Mirtich96

Summary


Rigid Body Equation of Motion
$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ \mathbf{R}(t) \\ Mv(t) \\ \mathbf{I}(t)\omega(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^{*}\mathbf{R}(t) \\ F(t) \\ \tau(t) \end{pmatrix}$
$P(t)$ – linear momentum
$L(t)$ – angular momentum
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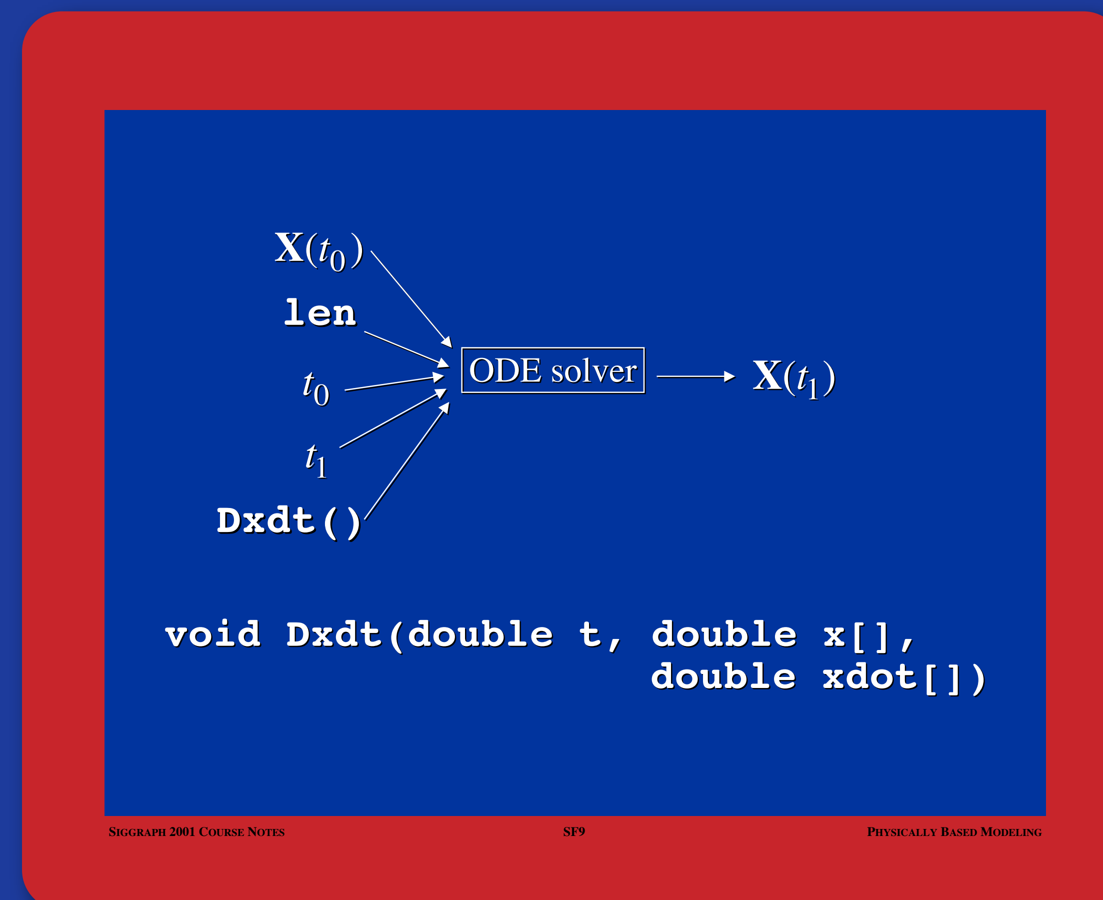
$\mathbf{X}(t_0)$
len
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PHYSICALLY BASED MODELING

What's in the Course Notes

1. Implementation of `Dxdt()` for rigid bodies (bookkeeping, data structures, computations)
2. Quaternions—derivations and code
3. Miscellaneous formulas and examples
4. Derivations for force and torque equations, center of mass, inertia tensor, rotation equations, velocity/acceleration of points

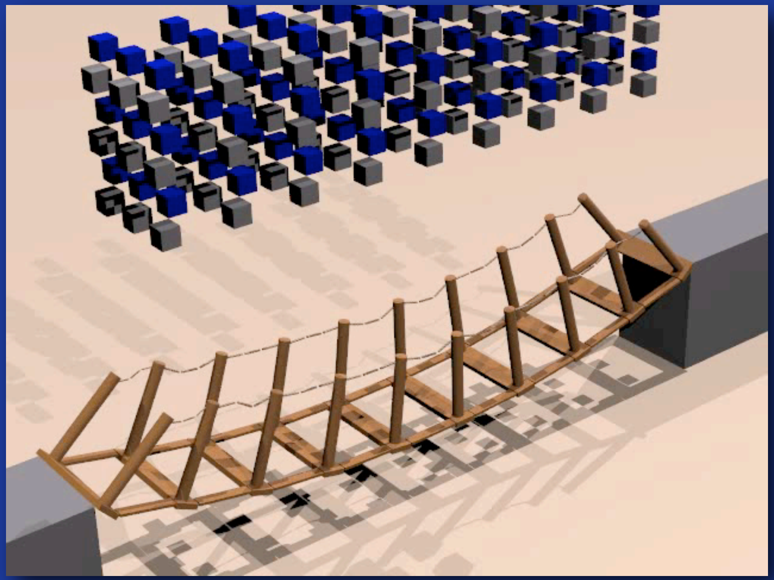
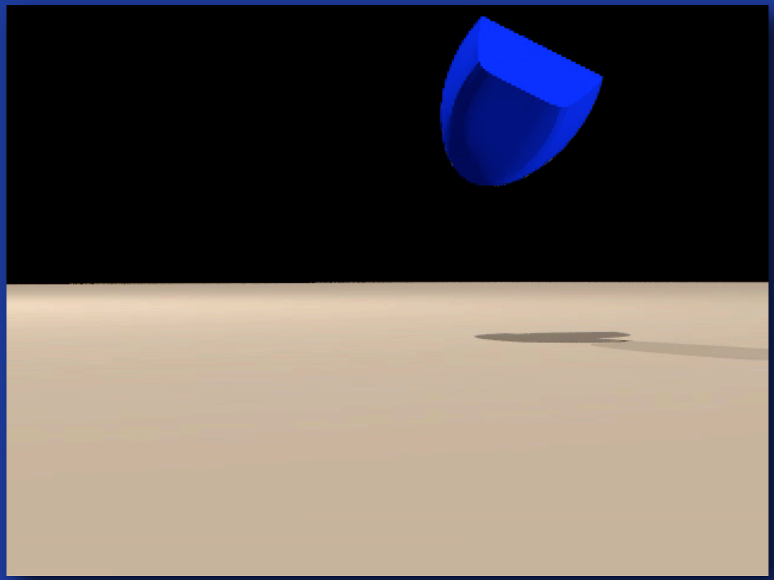
Example

Example

These simulations could never have been created by hand.

Question

- What Kind of Collisions Are Possible?
  - Geometrically?
  - Physically?

- How can these be detected?
- What algorithm can handle them?