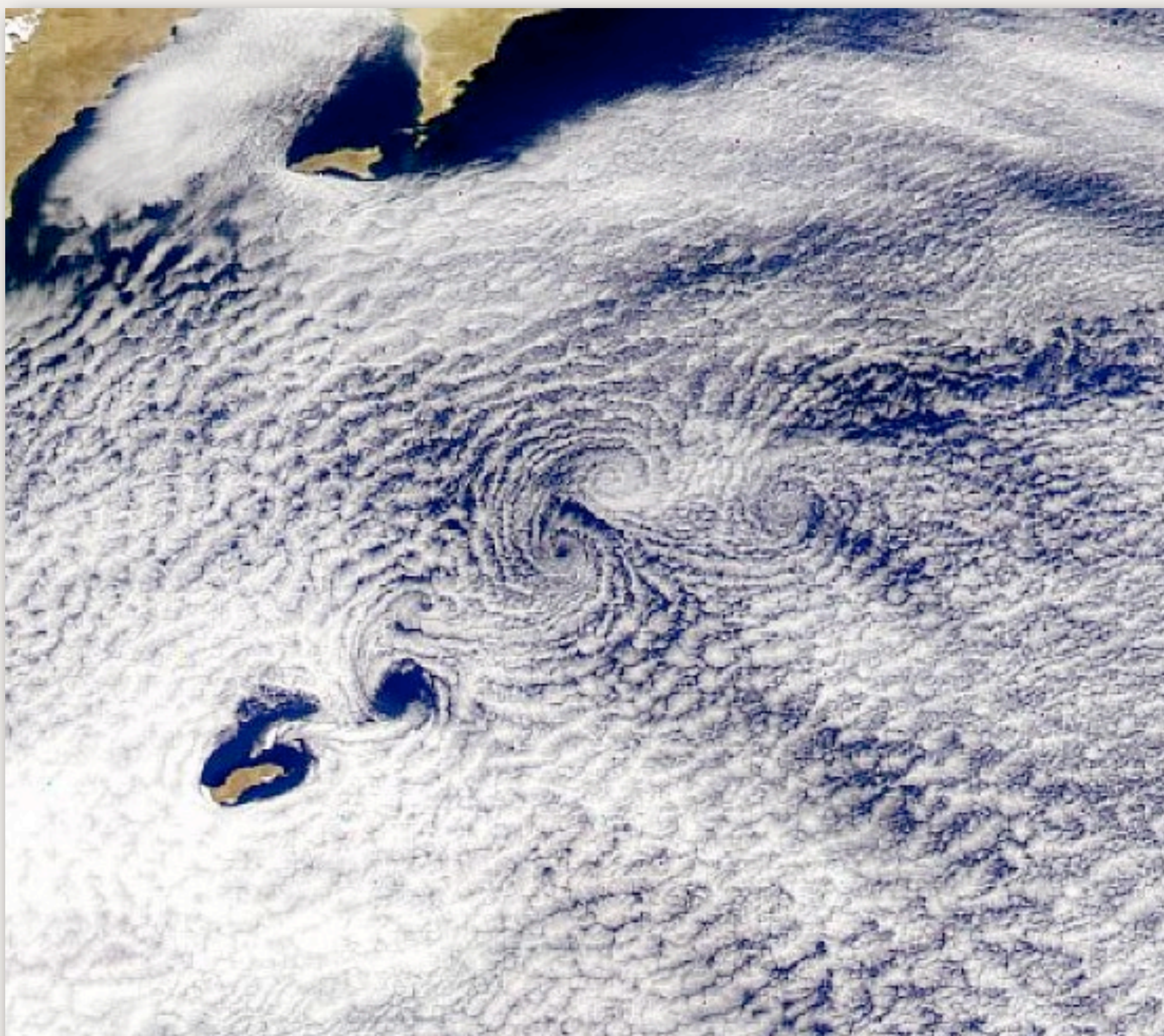
Partial Differential Equations

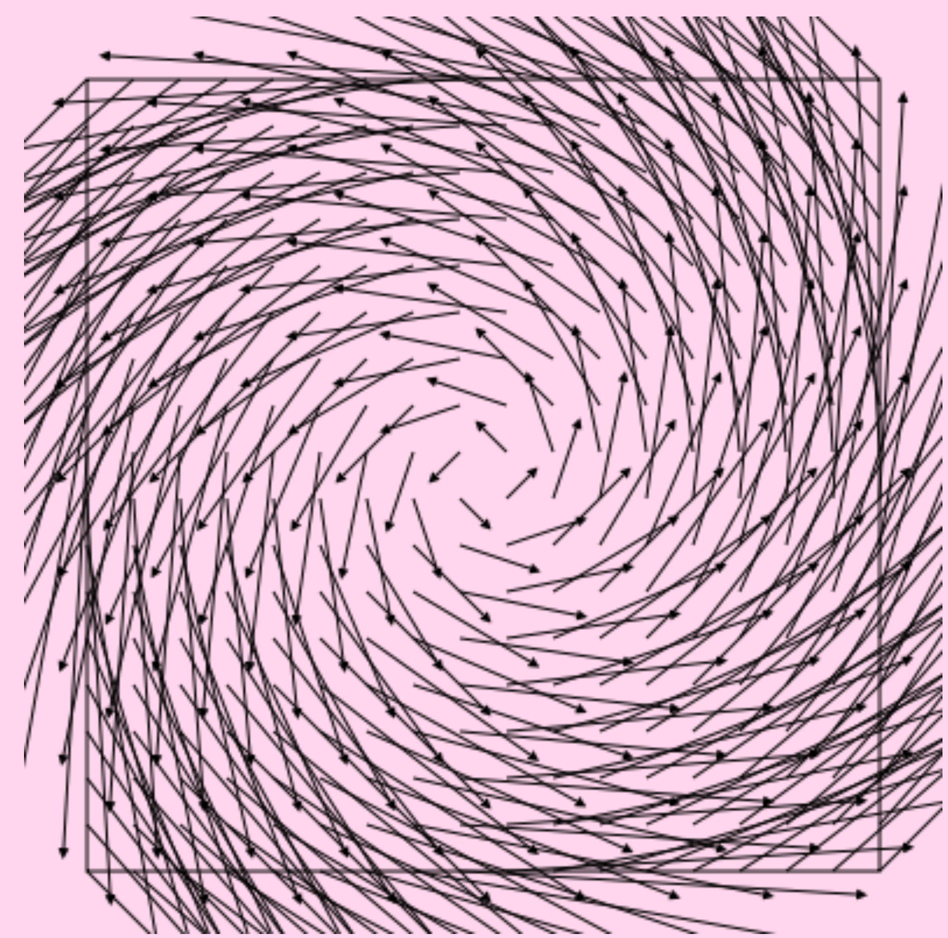
Adrien Treuille

source: http://talklikeaphysicist.com/

Velocity

- **How do we represent velocity...**
  - **As a *function* called a vector field.**





$\mathbf{u}(x, y)$

What is a PDE?

- **Ordinary Differential Equation:**

$$\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q})$$

- **Partial Differential Equation:**

$$\dot{q}(x, y) = f\left(q, \frac{\partial q}{\partial x}, \frac{\partial q}{\partial y}, \ldots\right)$$

What is a PDE?

- **Ordinary Differential Equation:**

$$\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q})$$

- **Partial Differential Equation:**

$$\dot{q}(x, y) = f\left(q, \frac{\partial q}{\partial x}, \frac{\partial q}{\partial y}, \ldots\right)$$

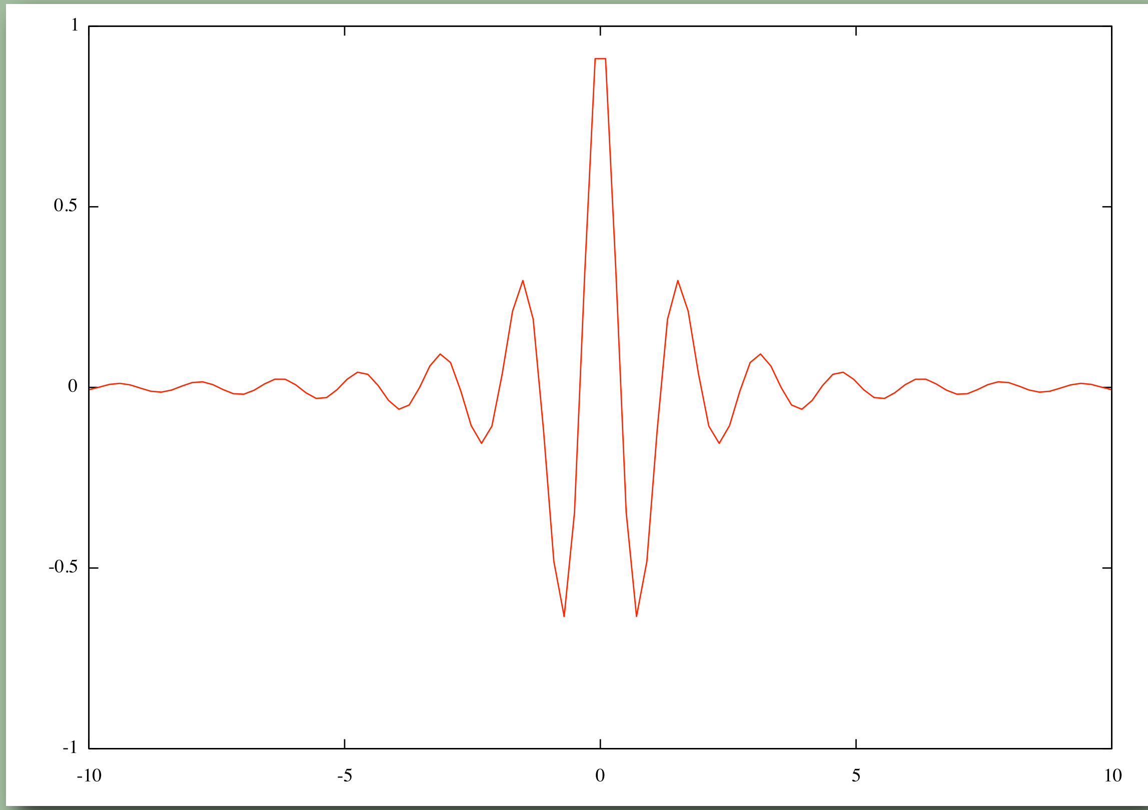
Example

$$\dot{f}(x,t) = -\frac{\partial f}{\partial x}$$

$$f(x,0) = g(x)$$

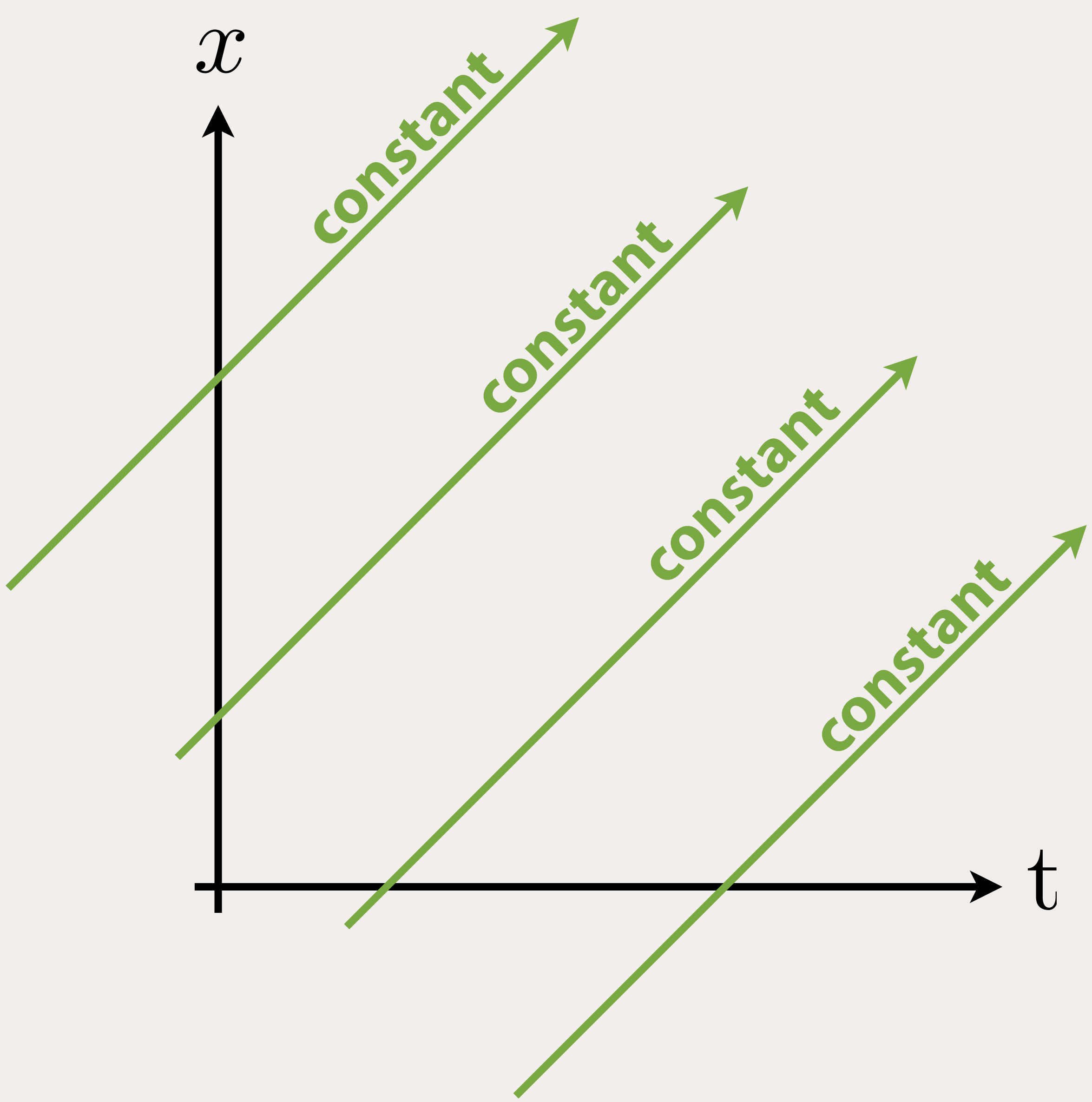
e.g...

g(x)=

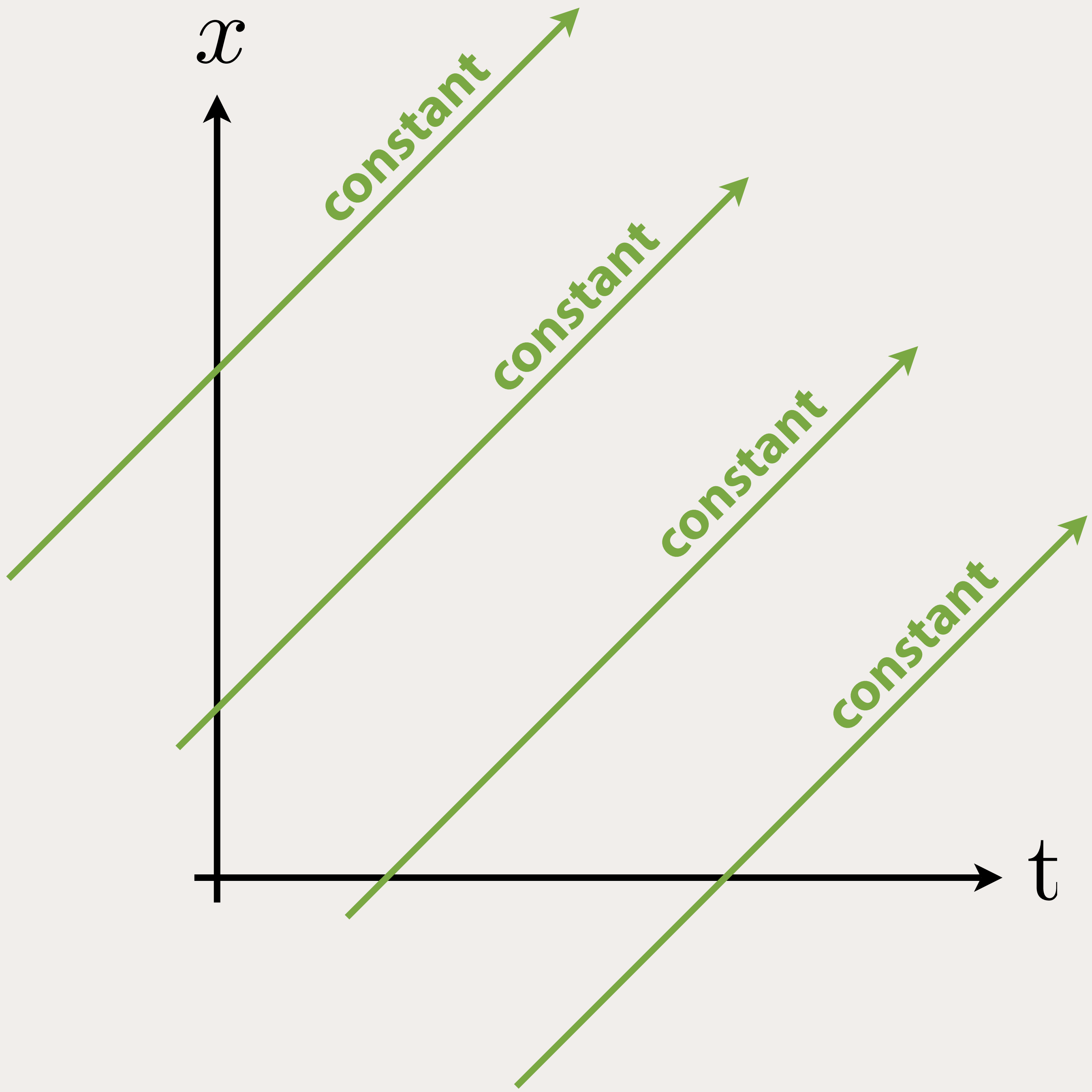


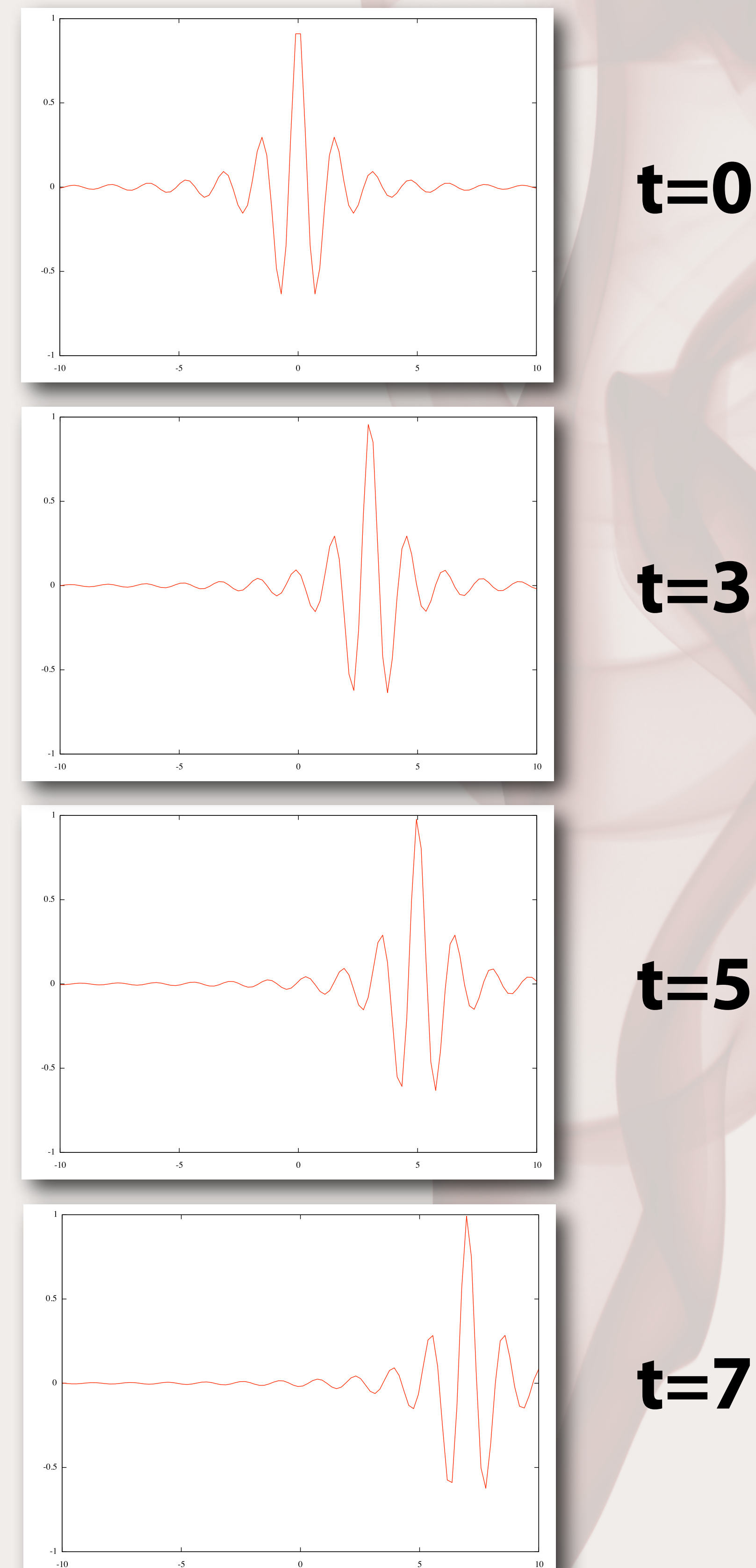
What is the solution?

$$\dot{f}(x,t) = -\frac{\partial f}{\partial x} \rightarrow [\partial_t f \;\; \partial_x f] \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$



What is the solution?



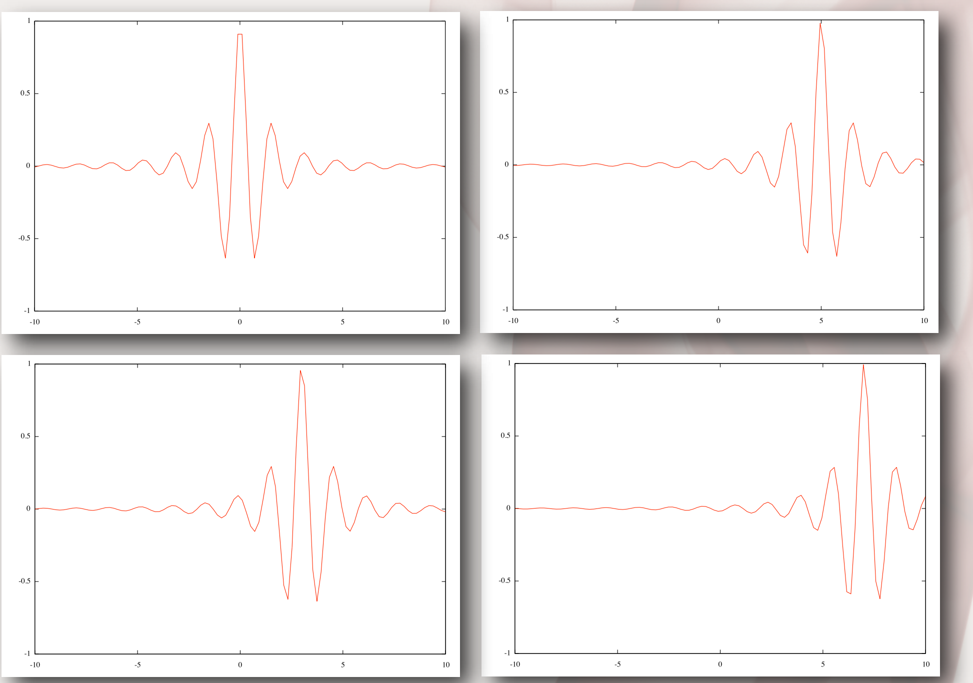


The Solution

If...

$$\dot{f}(x,t) = -\frac{\partial f}{\partial x}$$

$$f(x,0) = g(x)$$



then...

$$f(x,t) = g(x-t)$$

Simplified wave propogation.

Numerical Solutions

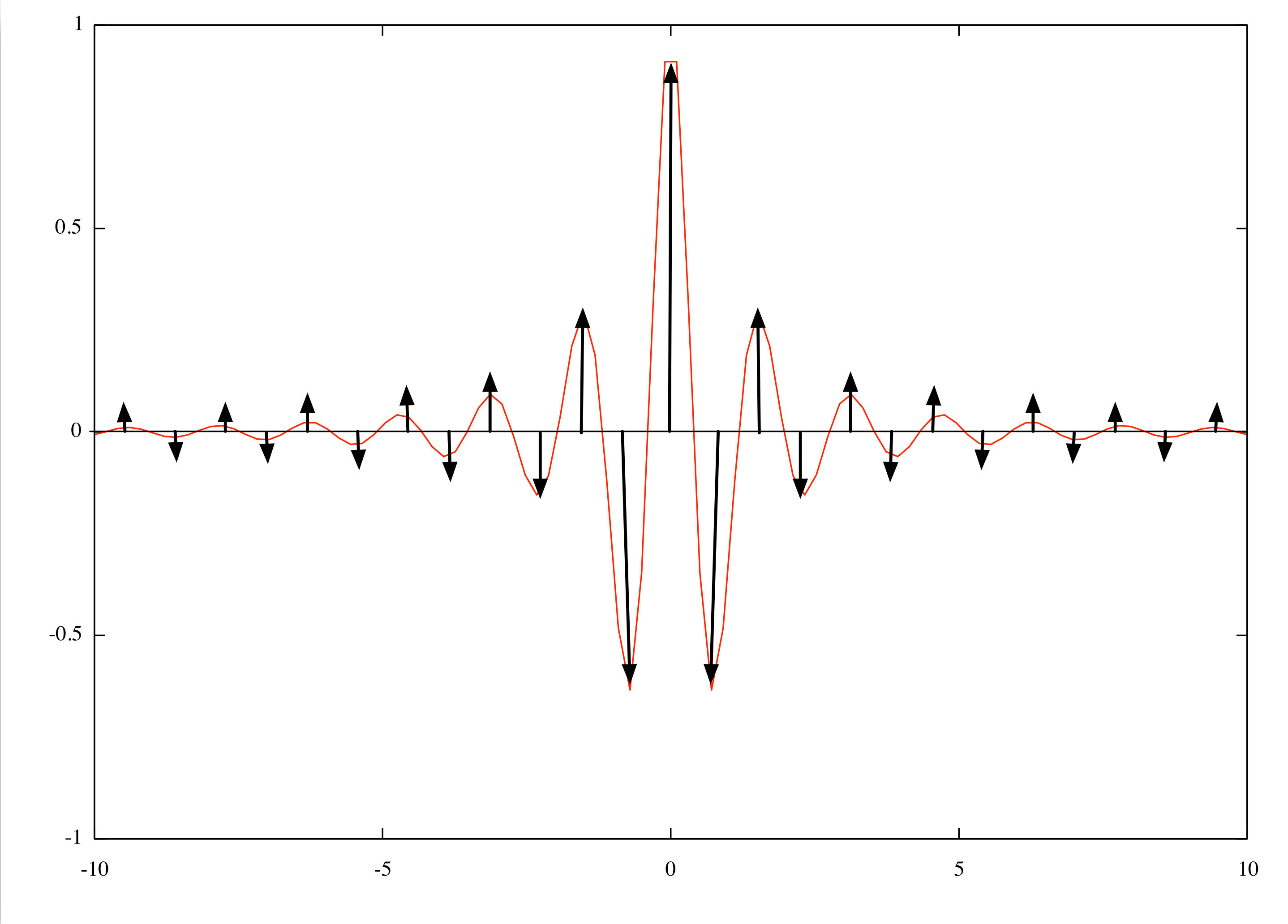
- **How can we solve this numerically?**

$$\dot{f}(x,t) = -\frac{\partial f}{\partial x}$$

- **Answer:**
  - ***First discretize in space.***
  - ***Then discretize in time.***

Discretize in Space

- **Turn our PDE into an ODE...**



f **becomes a "discrete function of space"...**

space between spikes = Δx

Discretize in Space

- **Now that we have a discrete function:**

$$\dot{f}_i = -\frac{\partial f}{\partial x}$$

- **What do we do about the derivative:**

$$\dot{f}_i = -\left(\frac{f_{i+1} - f_i}{\Delta x}\right)$$ **Forward Differencing**

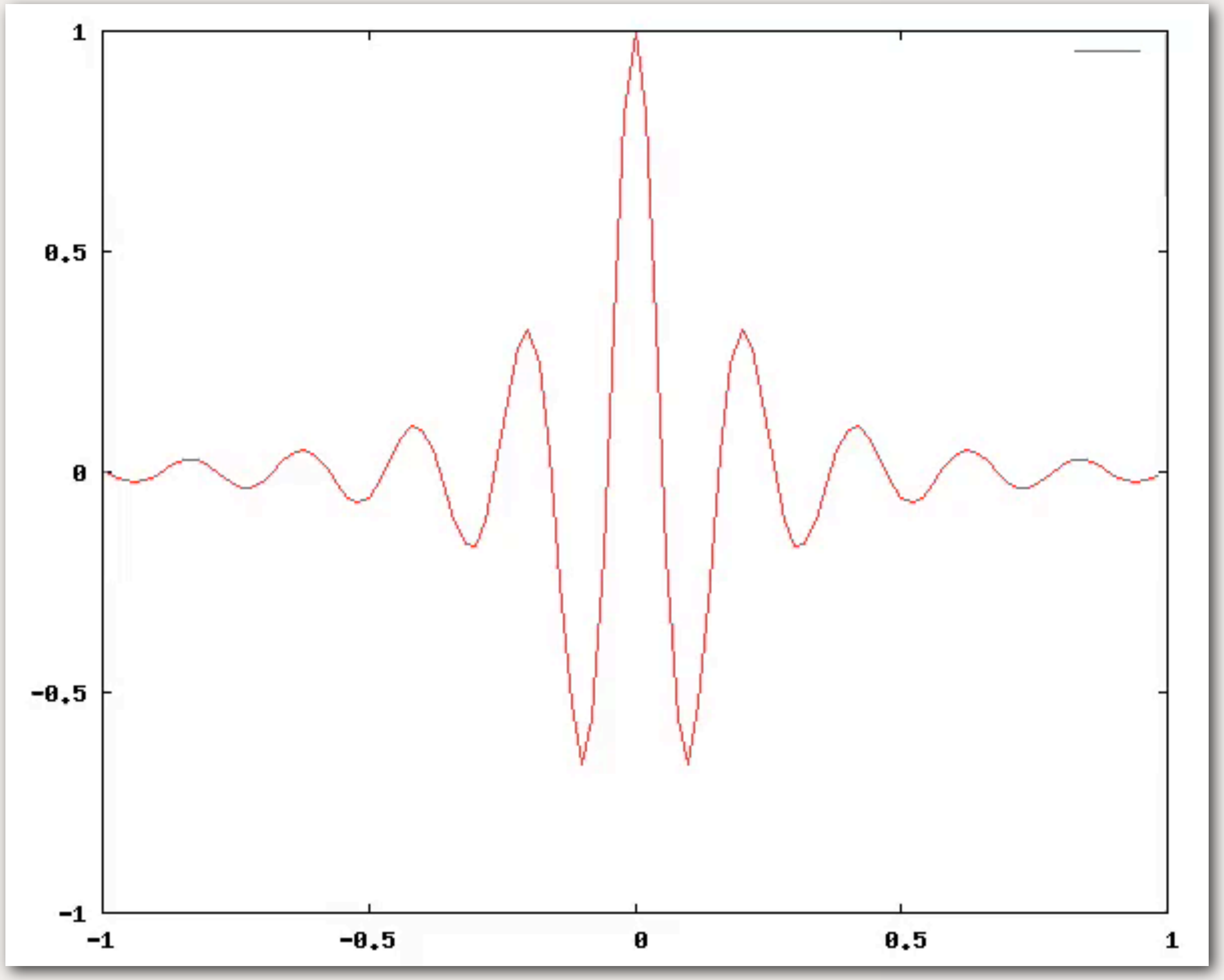
$$\dot{f}_i = -\left(\frac{f_i - f_{i-1}}{\Delta x}\right)$$ **Backward Differencing**

$$\dot{f}_i = -\left(\frac{f_{i+1} - f_{i-1}}{2\Delta x}\right)$$ **Central Differencing**

Discretize in Time

Euler's method backward differencing:

$$f_i^{t+1} = f_i^t - \Delta t \left(\frac{f_i - f_{i-1}}{\Delta x} \right)$$



Δt = 0.01

Discretize in Time

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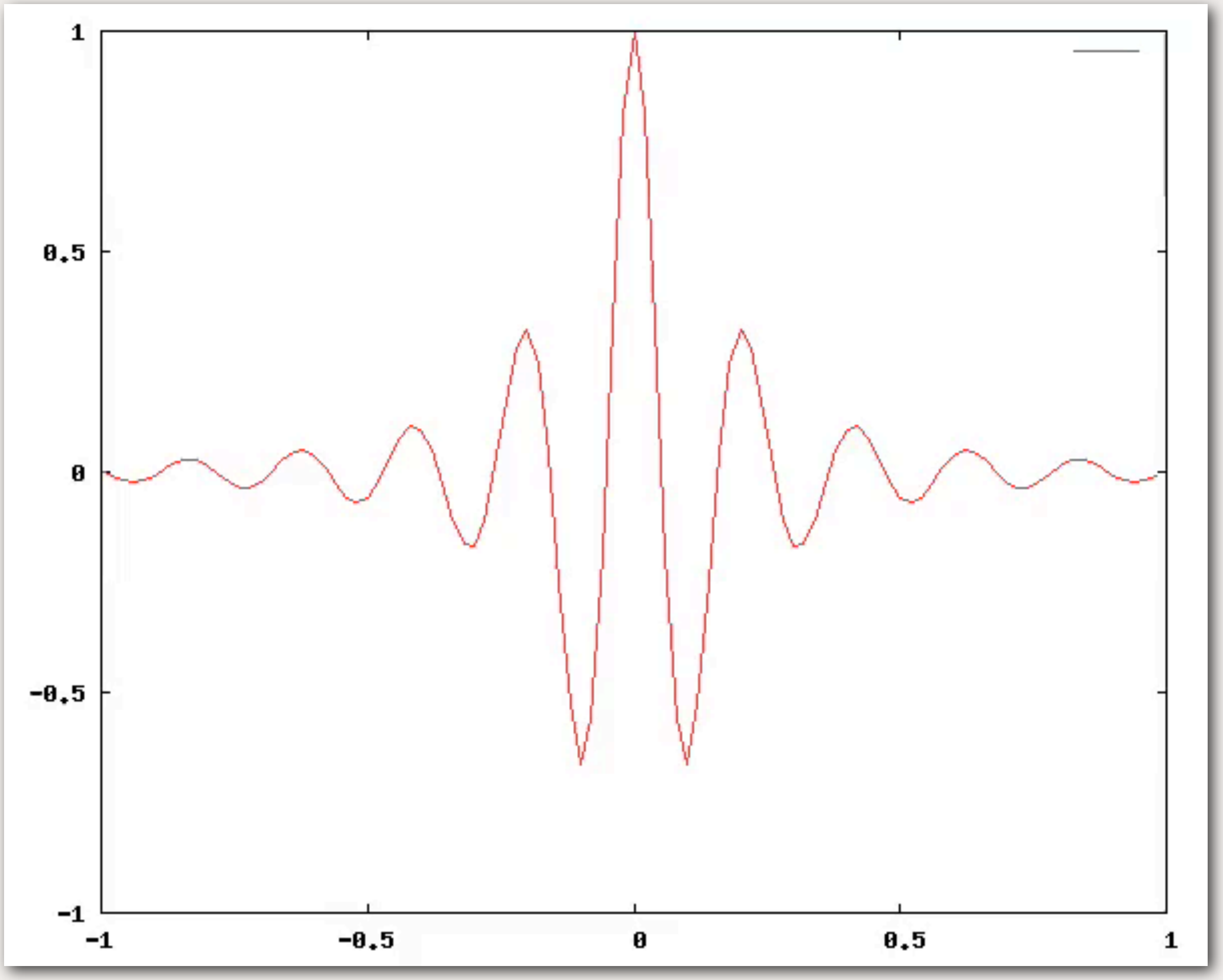

Δt = 0.1

Discretize in Time

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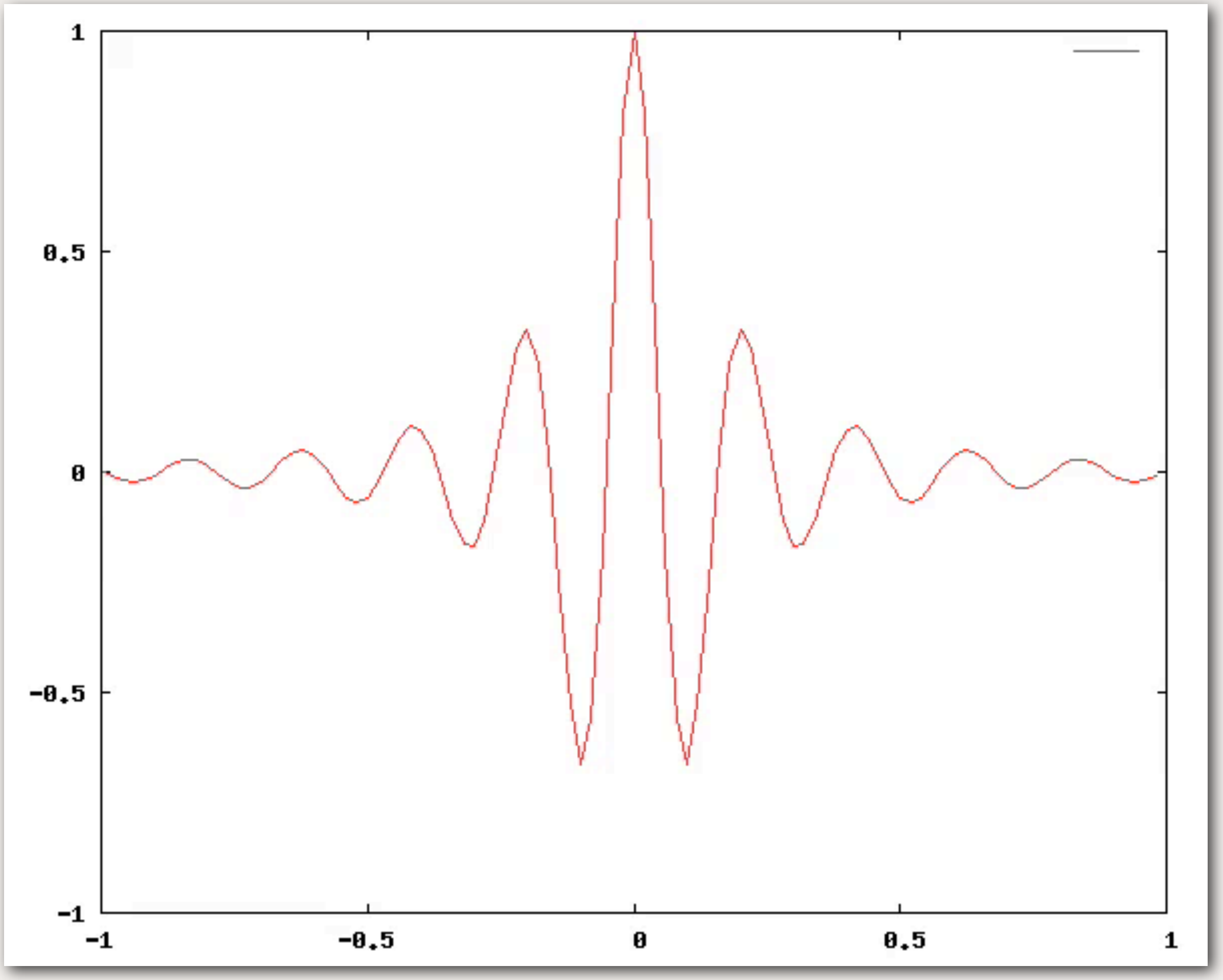

Δt = 1.0

Discretize in Time

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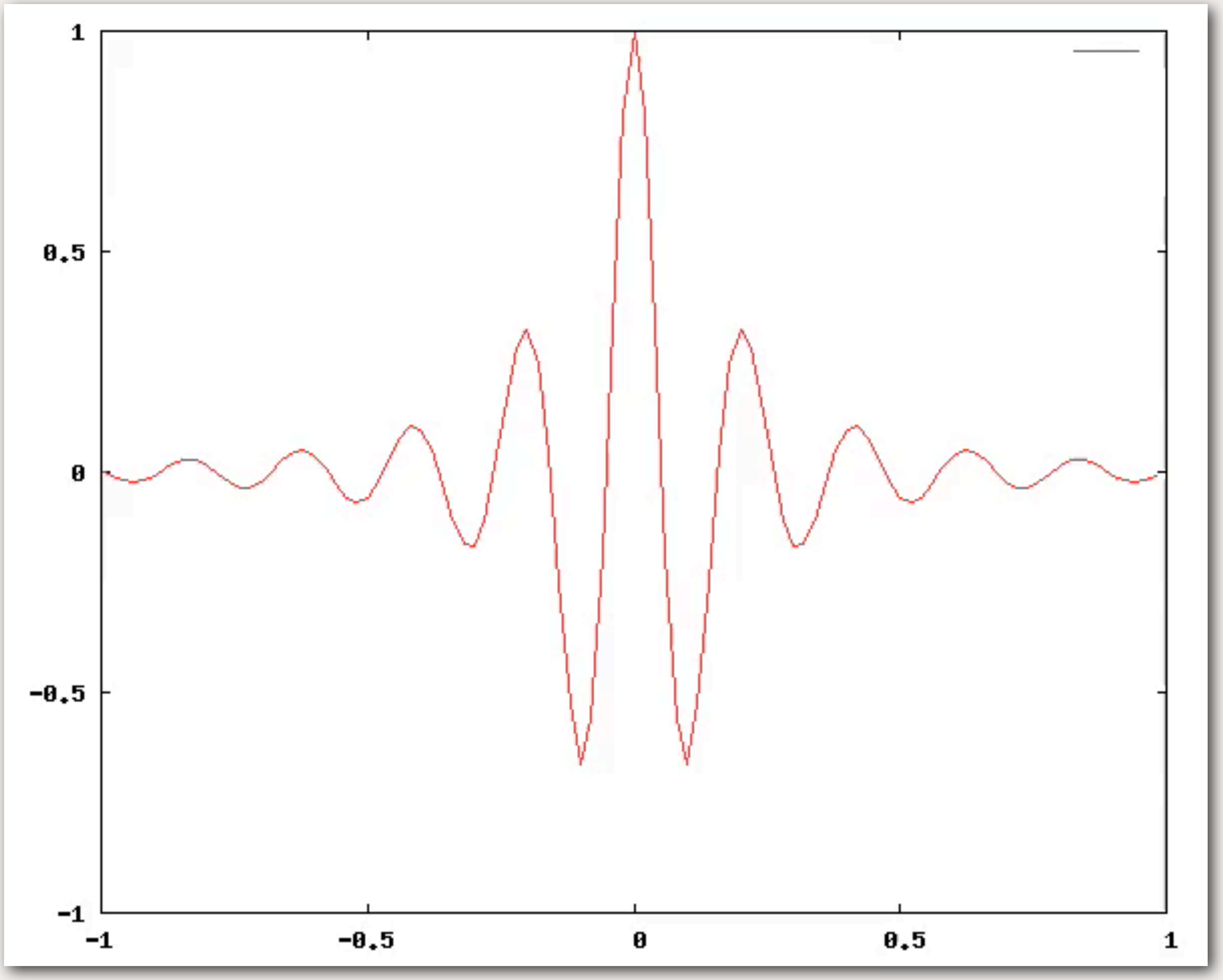

Δt = 2.0

Discretize in Time

Euler's method forward differencing:

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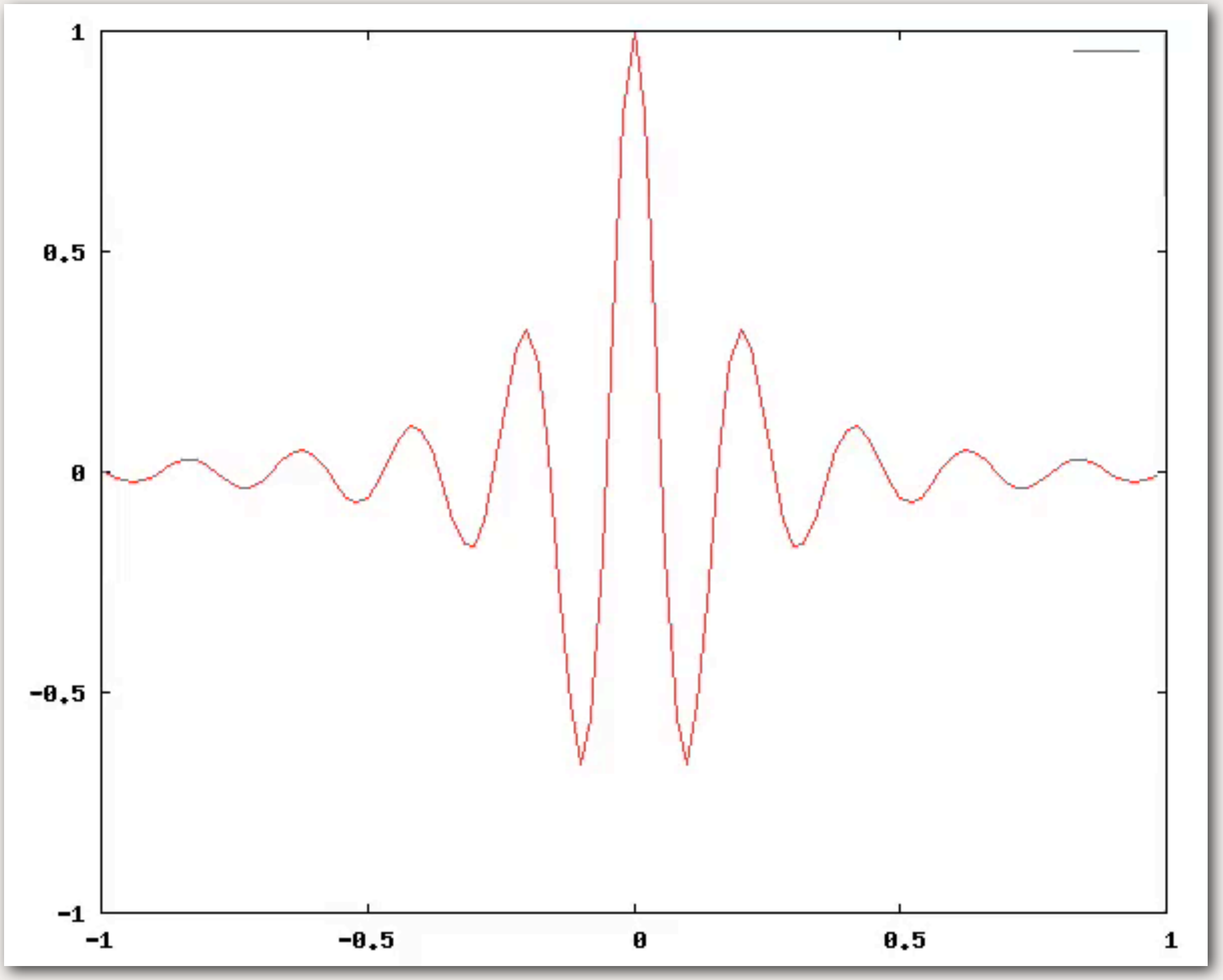

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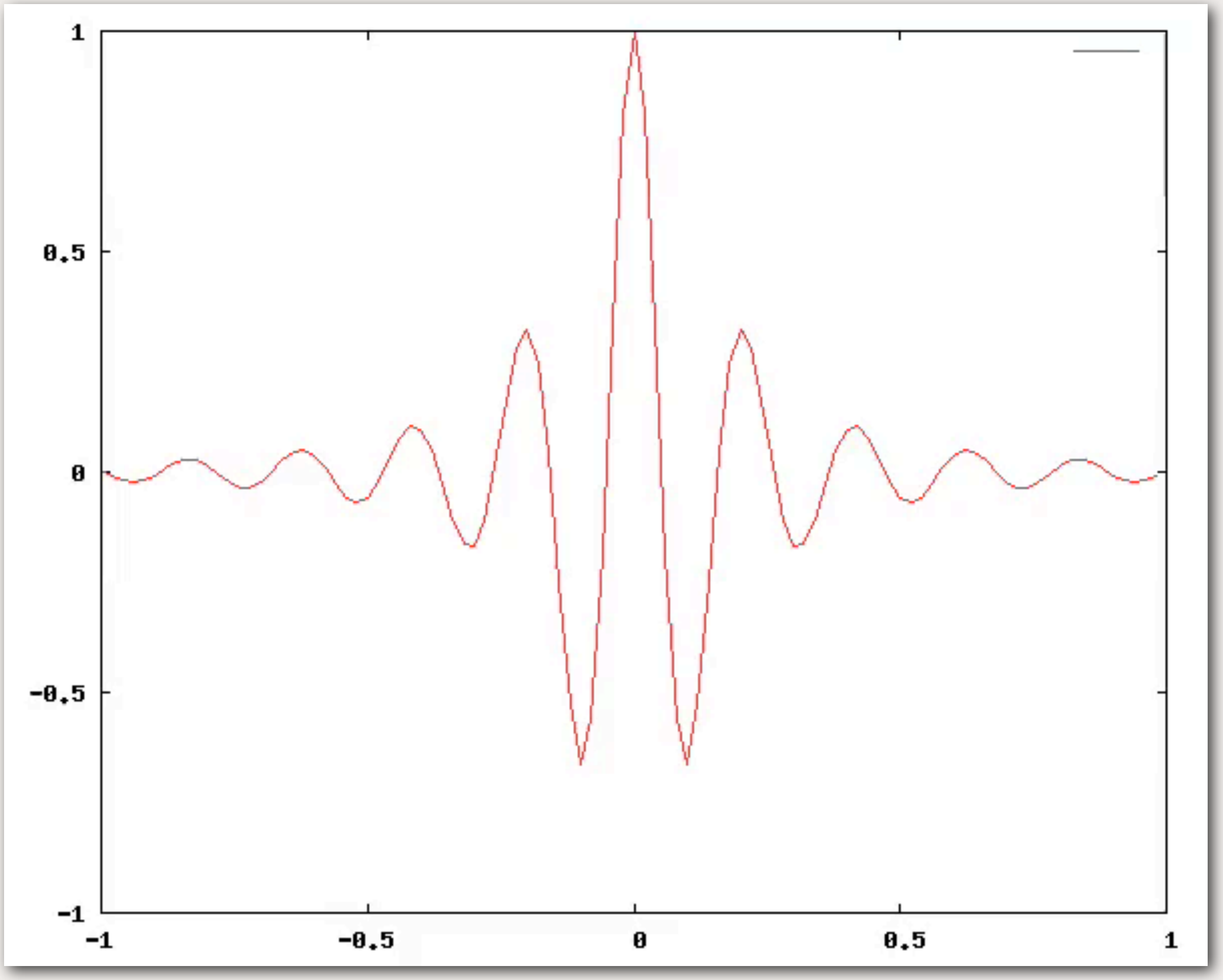

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Discretize in Time

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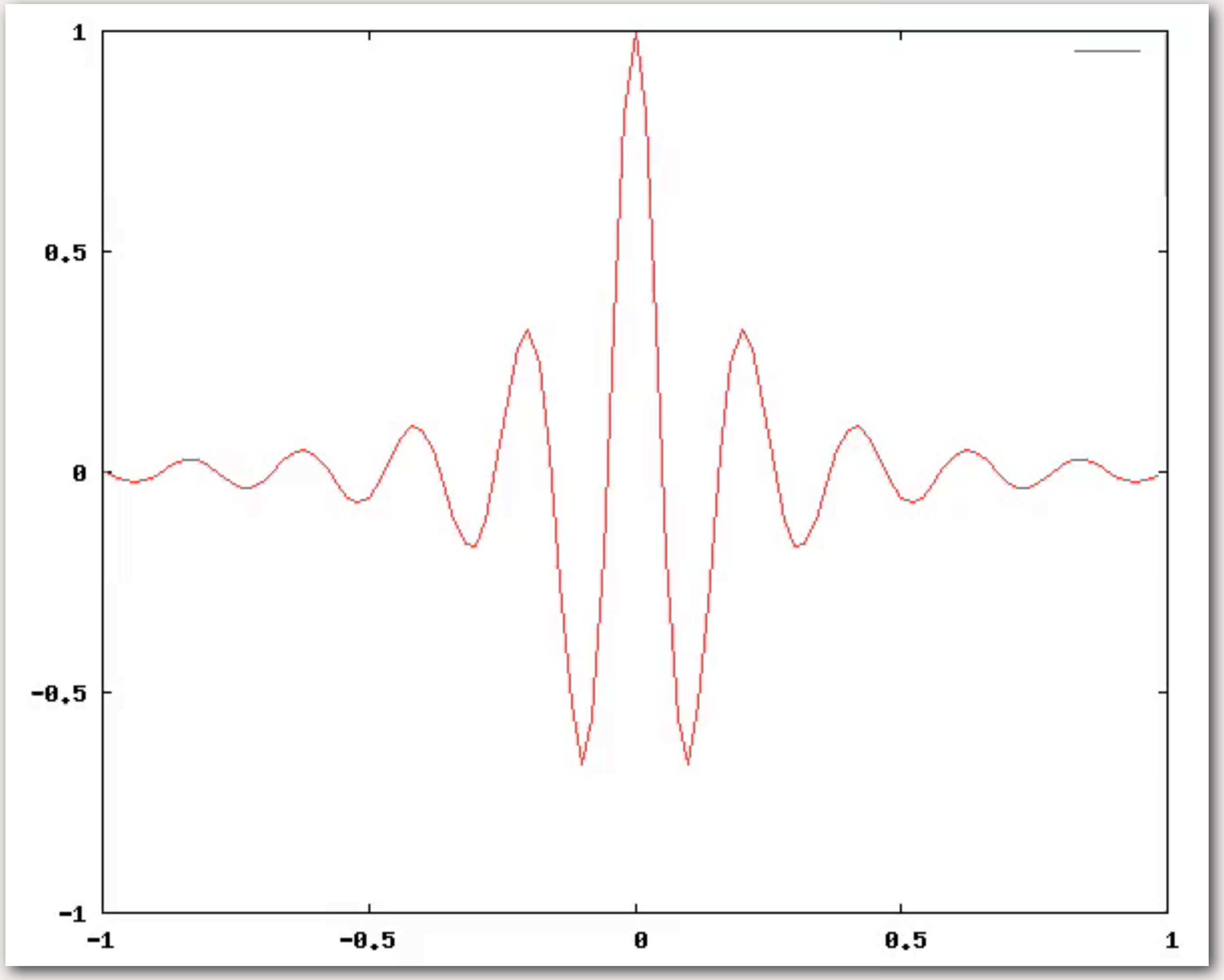

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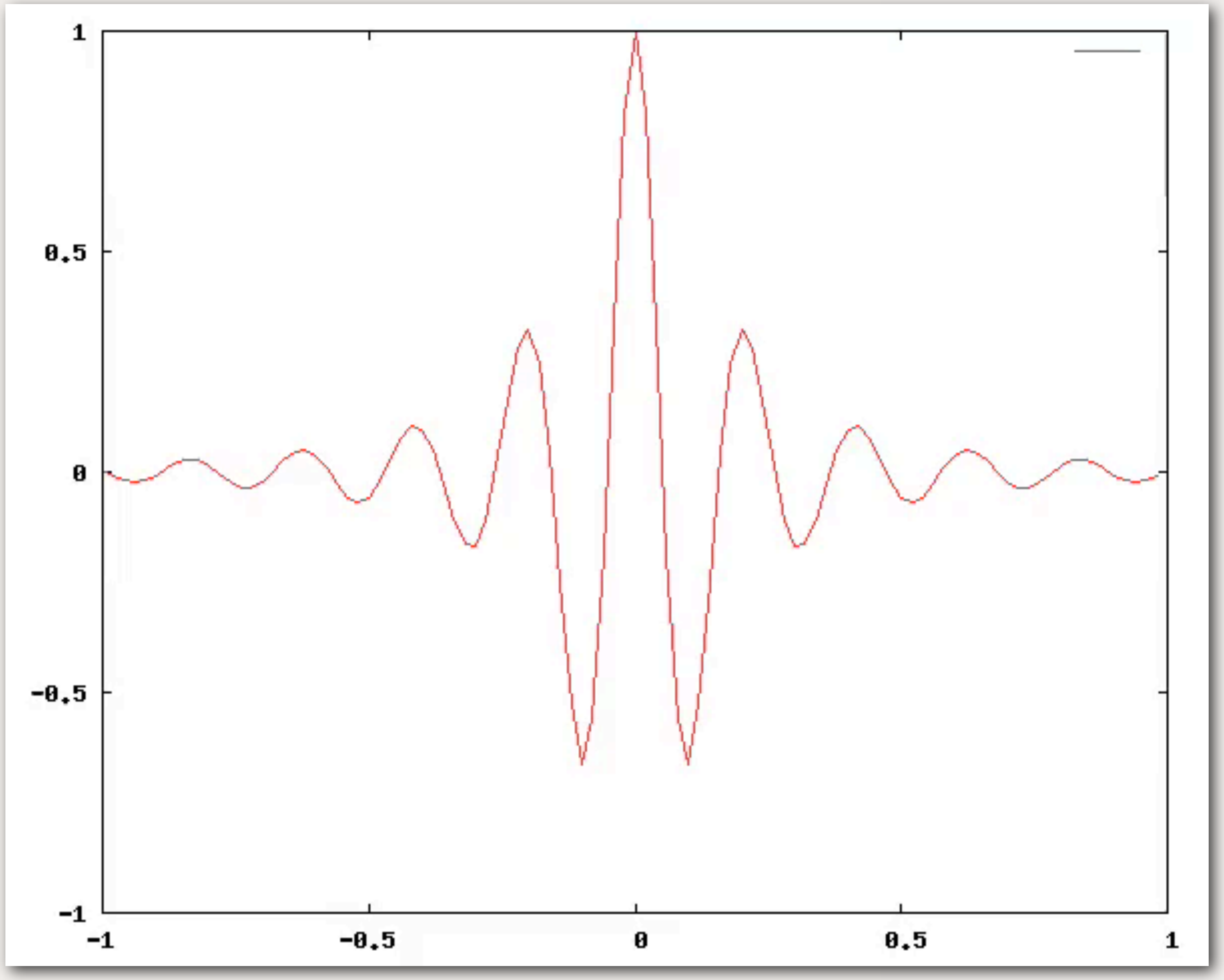


Δt = 0.01

In short.

$$\dot{f}(x,t) = -\frac{\partial f}{\partial x}$$

dt=	0.01	0.1	1.0	2.0
Backward	Diffusive	Diffusive	Perfect?	Crap.
Forward	Crap.	Crap.	Crap.	Crap.

Why?

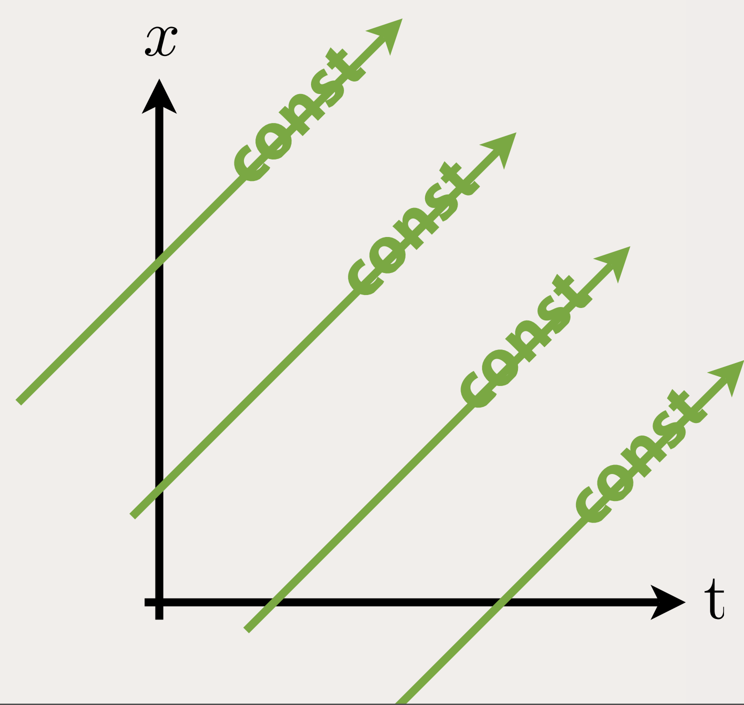
Can we do better?

- **Recall...**

$$f(x,t) = g(x-t)$$

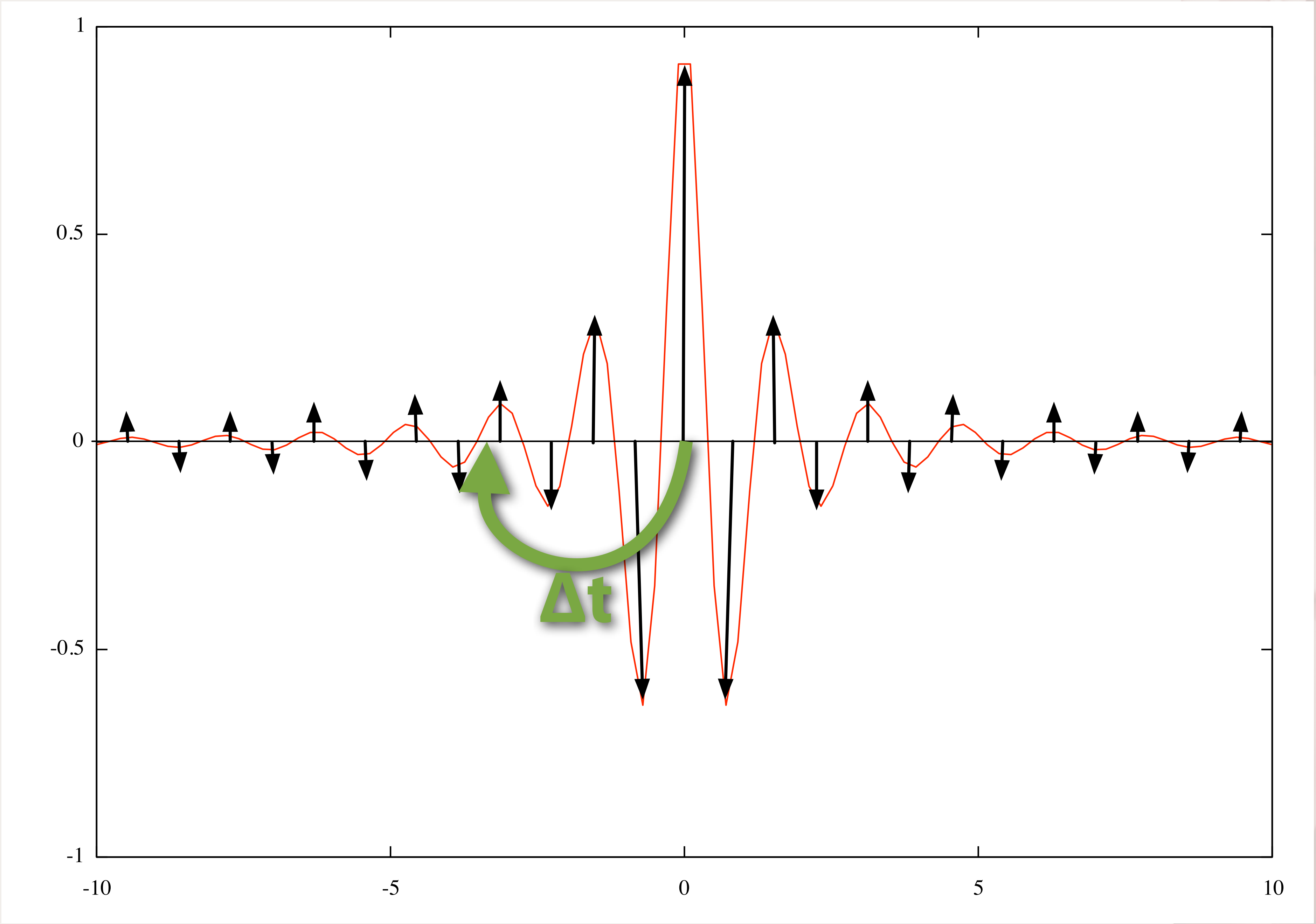
- **Information propagates "to the right"**

$$f(x, t+\Delta t) = f(x-\Delta t, t)$$



Semi-Lagrangian

$$f(x, t + \Delta t) = f(x - \Delta t, t)$$



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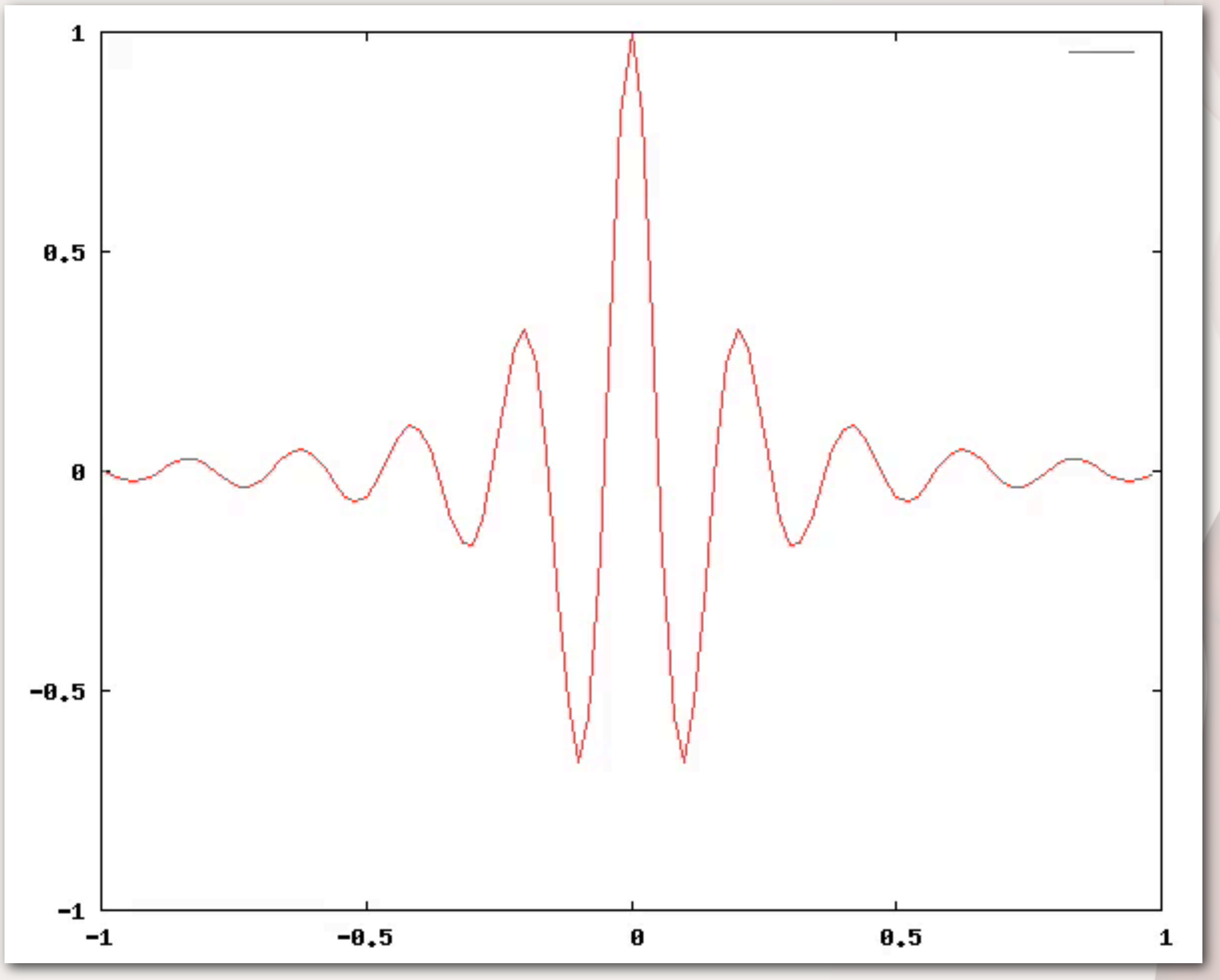
$$f(x, t + \Delta t) = f(x - \Delta t, t)$$



Question

- **How could you make a PDE that rotates...**