DiffEQ 1

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DiffEQ Integration

Differential Equation Basics

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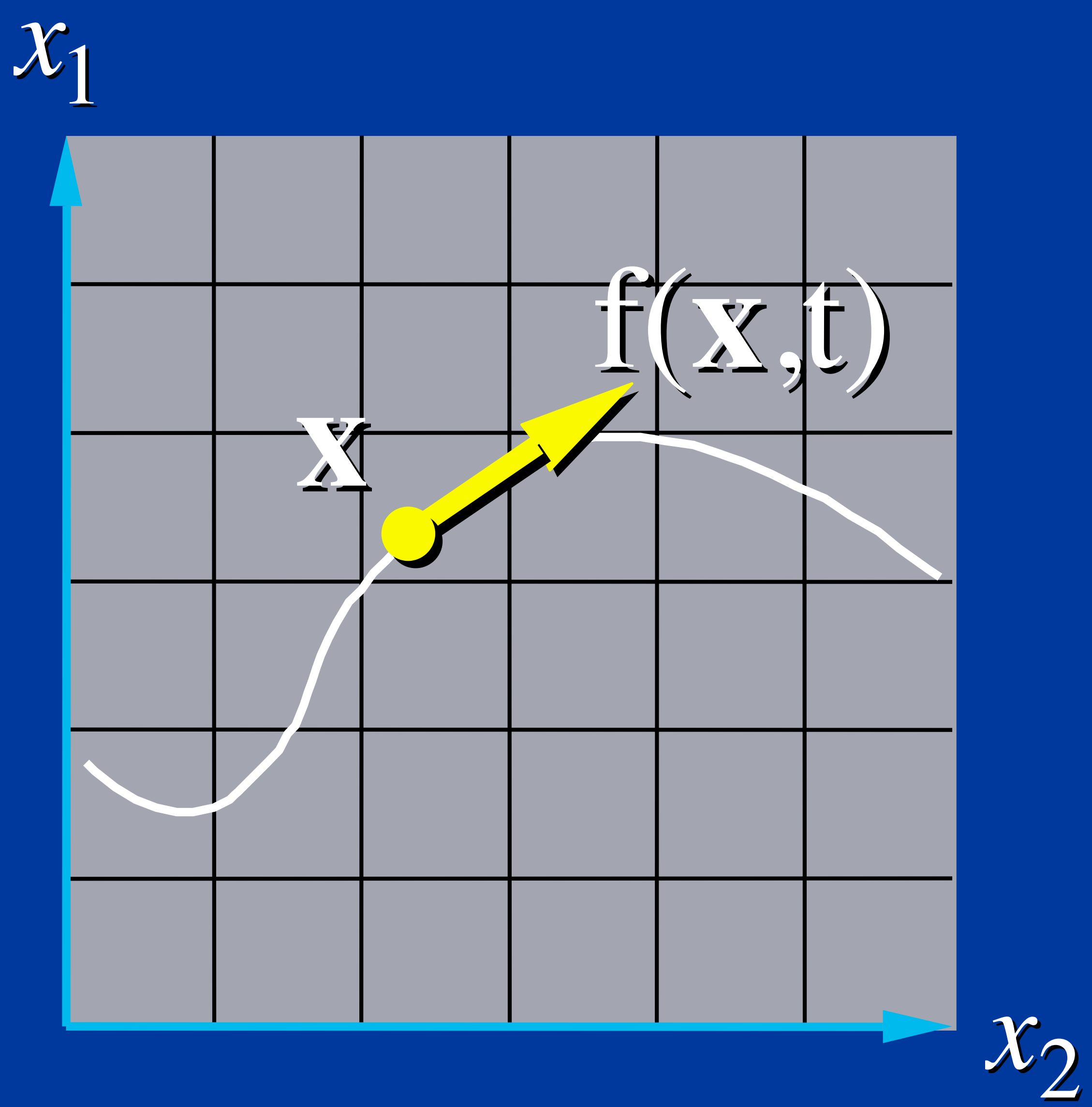
PIXAR
ANIMATION STUDIOS

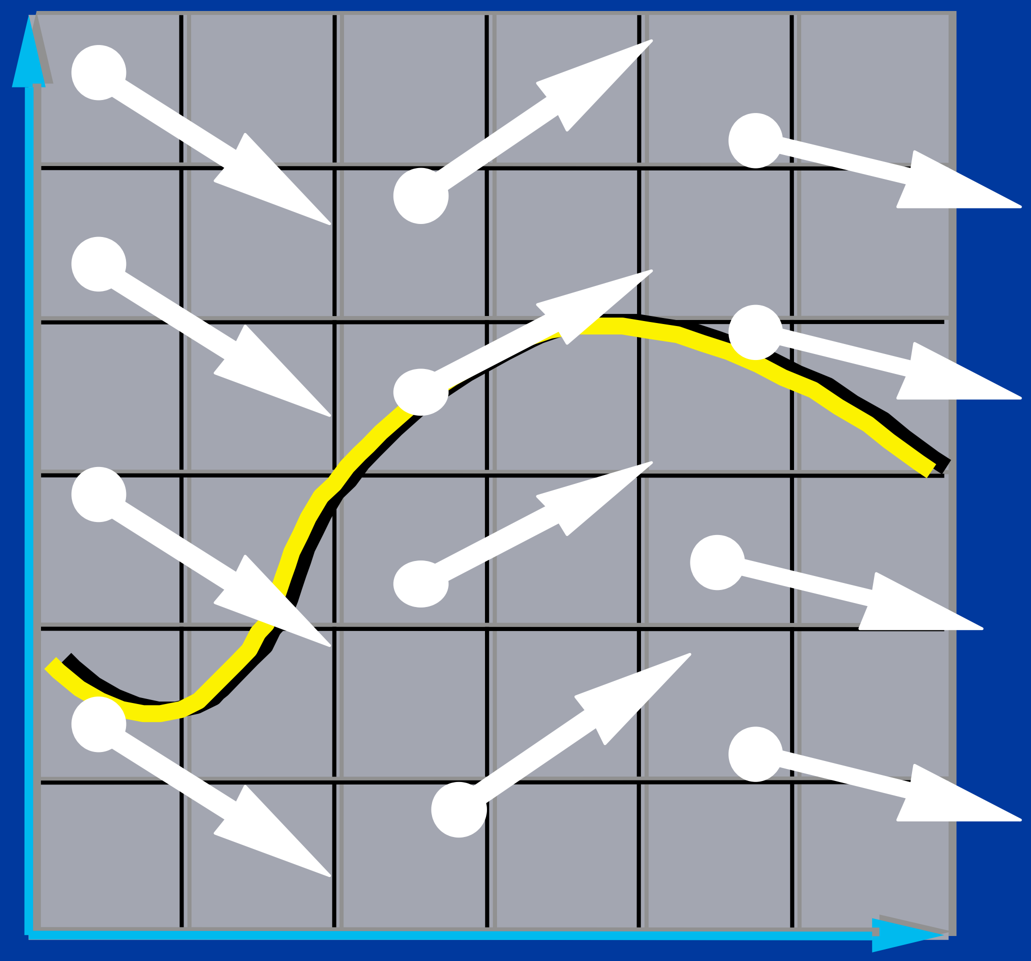
A Canonical Differential Equation



$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

- **x(*t*): a moving point.**
- **f(x,*t*): x's velocity.**

Vector Field



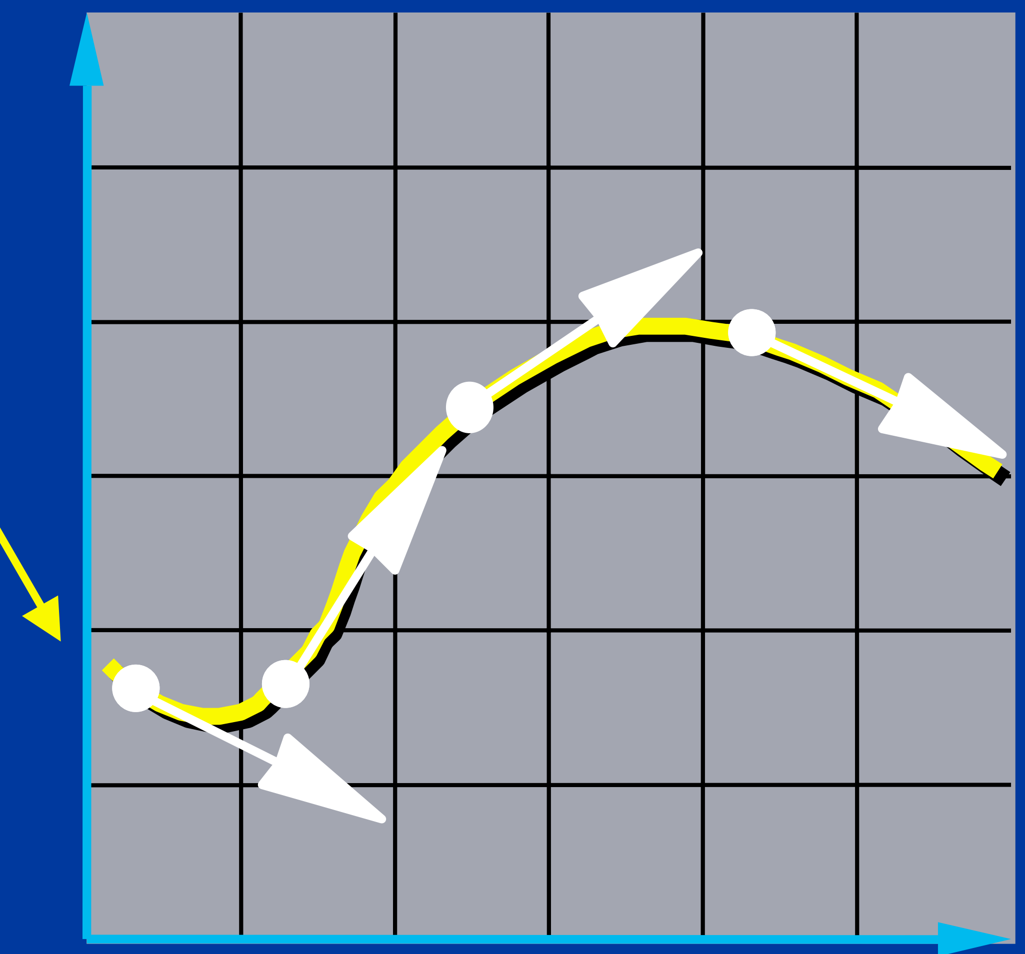
The differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

defines a vector field over x.

Integral Curves

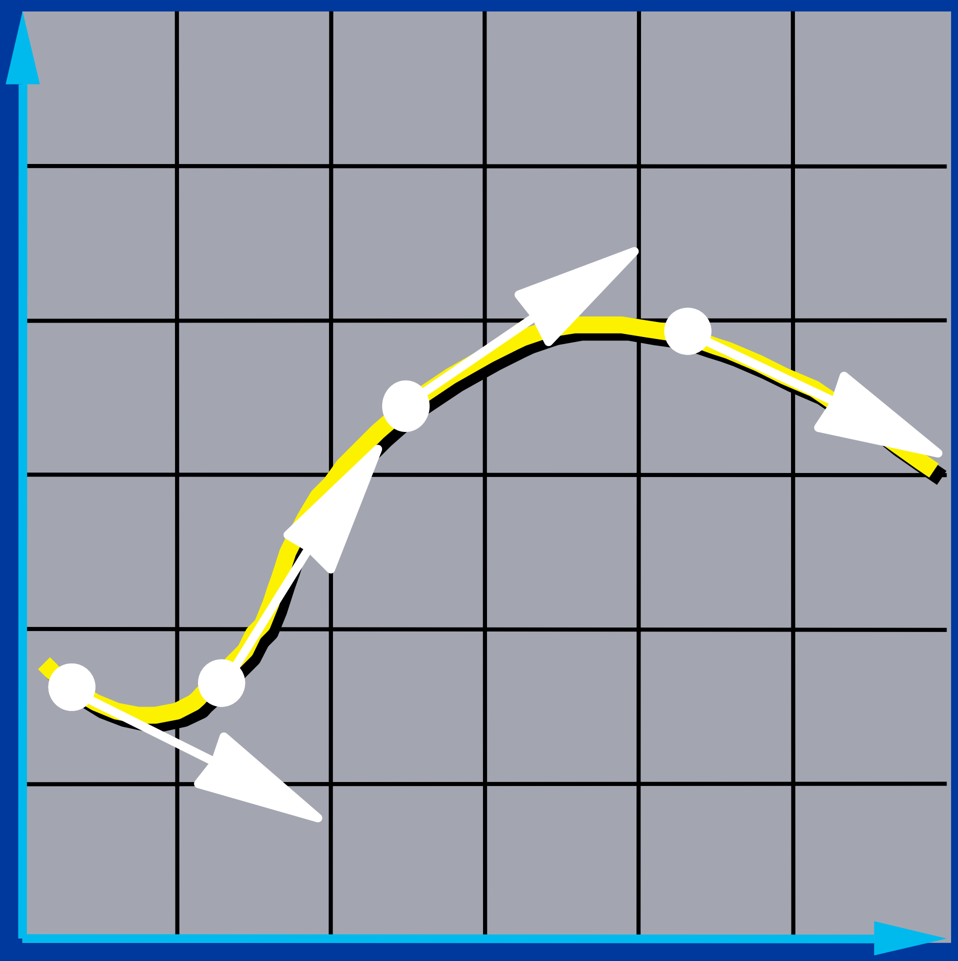
Start Here



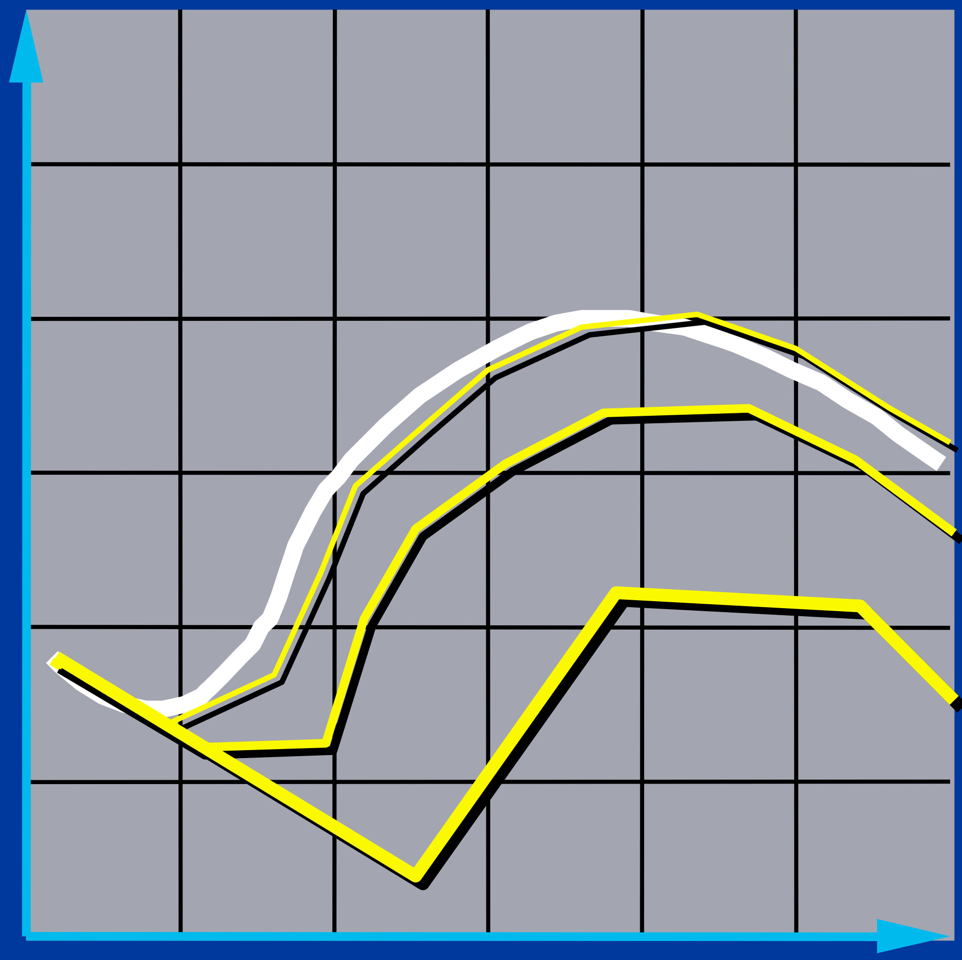
Pick any starting point, and follow the vectors.

Initial Value Problems

Given the starting point, follow the integral curve.



Euler's Method



- Simplest numerical solution method
- Discrete time steps
- Bigger steps, bigger errors.

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t\, \mathbf{f}(\mathbf{x}, t)$$

Two Problems

- Accuracy
- Instability

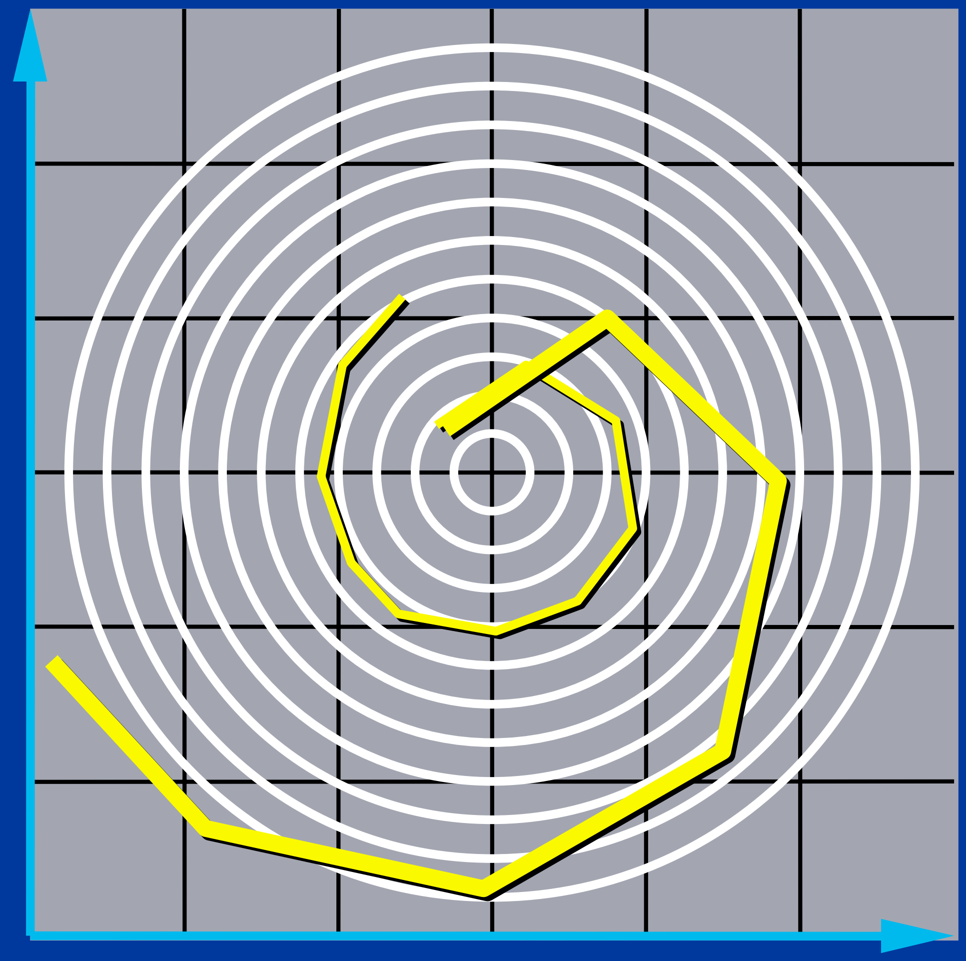
Accuracy

Consider the equation:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}$$

What do the integral curves look like?

Problem I: Inaccuracy



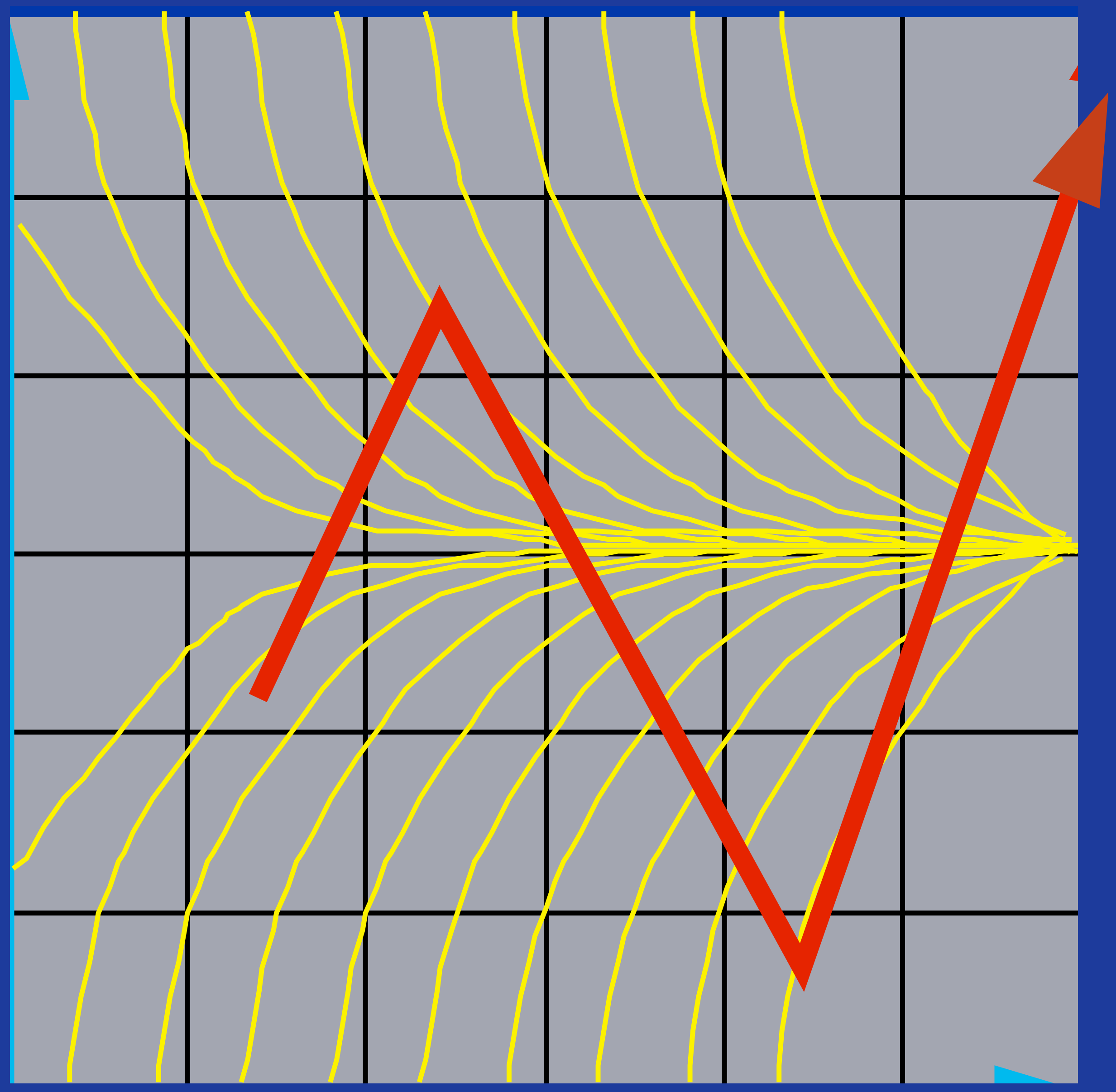
Error turns x(t) from a circle into the spiral of your choice.

Problem 2: Instability

- **Consider the following system:**

$$\dot{x} = -x$$
$$x(0) = 1$$

Problem 2: Instsability



To Neptune!

Accuracy of Euler Method

$$\dot{x} = f(x)$$

Consider Taylor Expansion about x(t)...

$$x(t+h) = \overbrace{\underbrace{x(t)}_{\text{constant}} + \underbrace{hf(x(t))}_{\text{linear}}}^{\text{Euler step}} + \overbrace{\underbrace{O(h^2)}_{\text{everything else}}}^{\text{error}}$$

Therefore, Euler's method has error $O(h^2)$... it is *first order.*

How can we get to $O(h^3)$ error?

The Midpoint Method

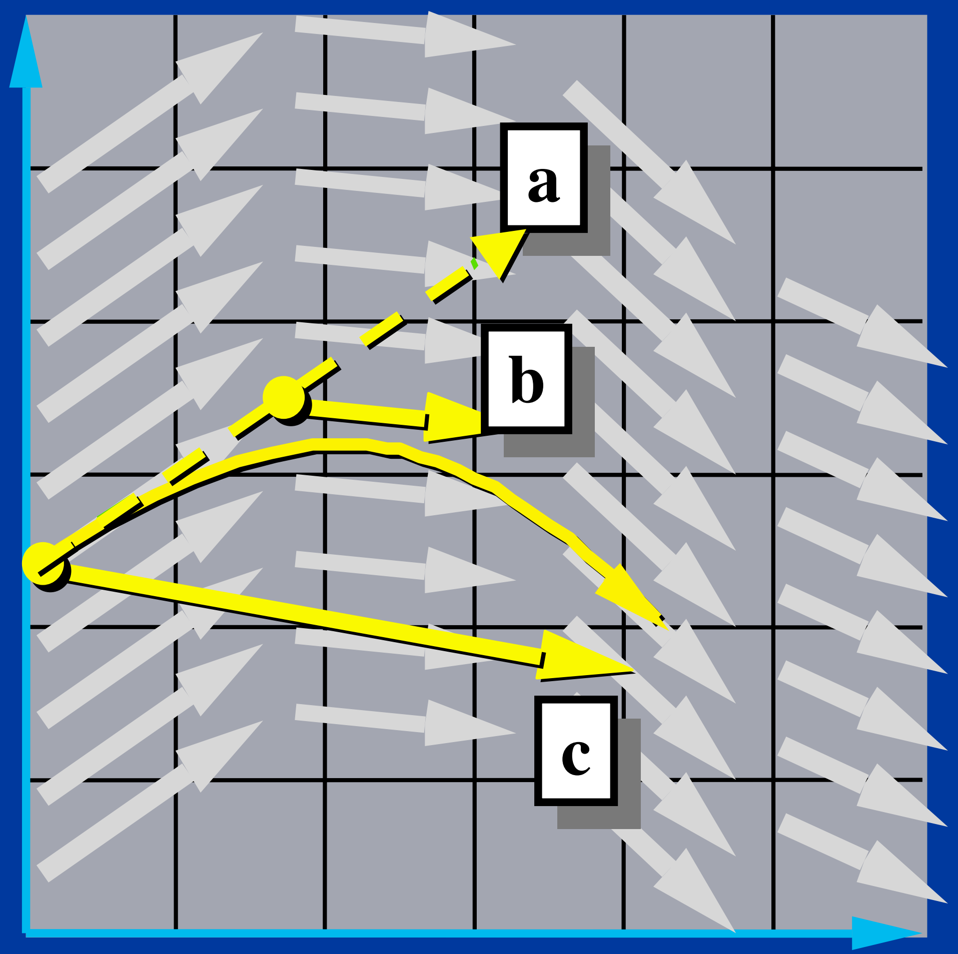
- **Also known as second order Runge-Kutte:**

$$k_1 = h(f(x_0, t_0)$$

$$k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2})$$

$$x(t_0 + h) = x_0 + k_2 + O(h^3)$$

The Midpoint Method



a. Compute an Euler step

$$\Delta \mathbf{x} = \Delta t\, \mathbf{f}(\mathbf{x}, t)$$

b. Evaluate f at the midpoint

$$\mathbf{f}_{\mathrm{mid}} = \mathbf{f}\left(\frac{\mathbf{x} + \Delta \mathbf{x}}{2}, \frac{t + \Delta t}{2}\right)$$

c. Take a step using the midpoint value

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t\, \mathbf{f}_{\mathrm{mid}}$$

q-Stage Runge-Kutta

General Form:

$$x(t_0 + h) = x_0 + h \sum_{i=1}^{q} w_i k_i$$

where:

$$k_i = f\left(x_0 + h \sum_{j=1}^{i-1} \beta_{ij} k_j \right)$$

Find the constant that ensure accuracty $O(h^n)$.

4th-Order Runge-Kutta

$$k_1 = hf(x_0, t_0)$$

$$k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2})$$

$$k_3 = hf(x_0 + \frac{k_2}{2}, t_0 + \frac{h}{2})$$

$$k_4 = hf(x_0 + k_3, t_0 + h)$$

$$x(t_0 + h) = x_0 + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + O(h^5)$$

Why so popular?

Order vs. Stages

Order	1	2	3	4	5	6	7	8
Stages	1	2	3	4	6	7	9	11

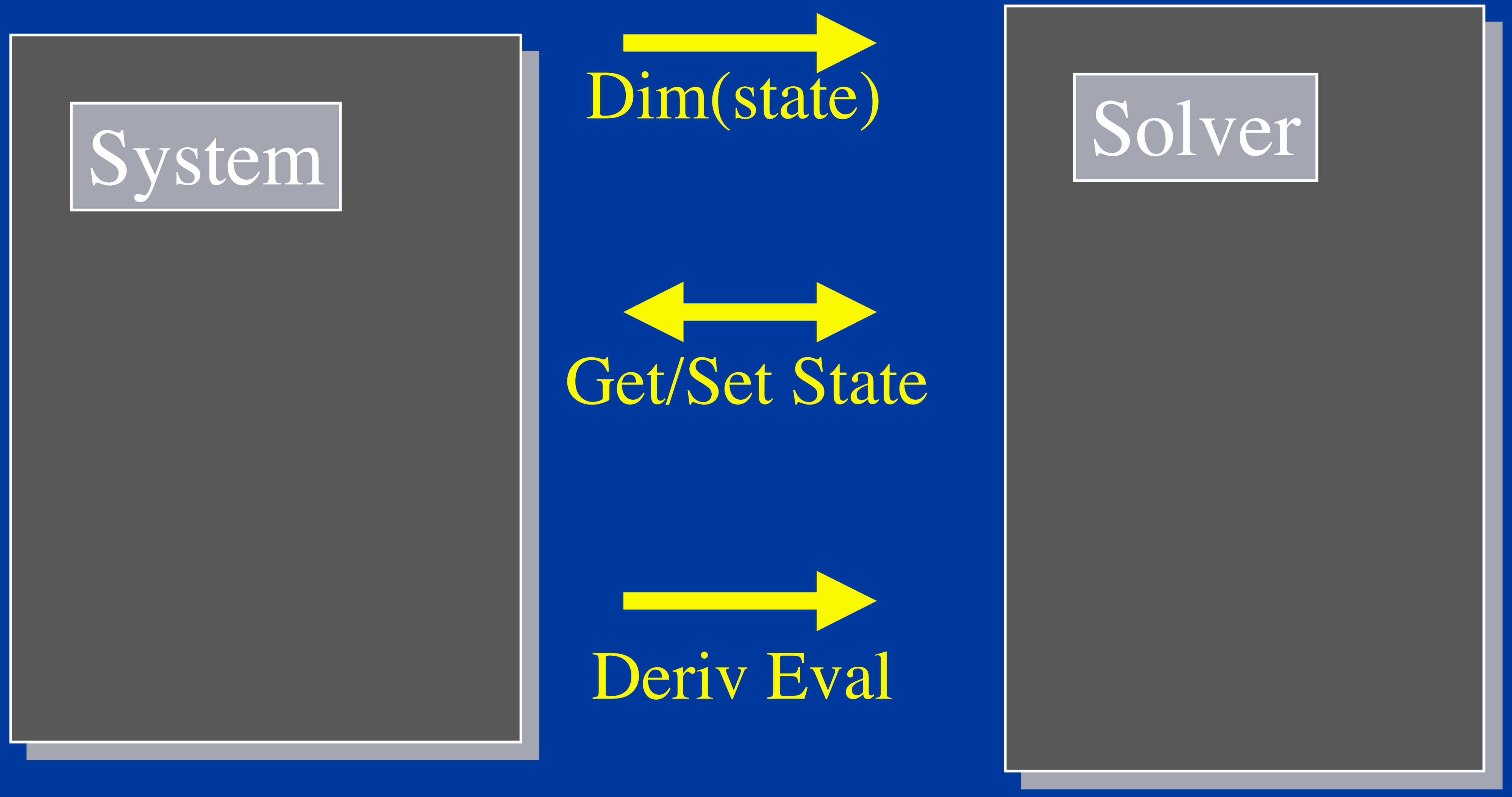
More methods…

- Euler's method is *1st Order.*
- The midpoint method is *2nd Order.*
- Just the tip of the iceberg. See *Numerical Recipes* for more.
- Helpful hints:
  - *Don't* use Euler's method (you will anyway.)
  - *Do* use adaptive step size.

Modular Implementation

- Generic operations:
  - Get dim(x)
  - Get/set x and t
  - Deriv Eval at current (x,t)
- Write solvers in terms of these.
  - Re-usable solver code.
  - Simplifies model implementation.

Solver Interface



A Code Fragment

```
void eulerStep(Sys sys, float h) {
   float t = getTime(sys);
   vector<float> x0, deltaX;

   t = getTime(sys);
   x0 = getState(sys);
   deltaX = derivEval(sys,x0, t);
   setState(sys, x0 + h*deltaX, t+h);
}
```