# Deformable Materials

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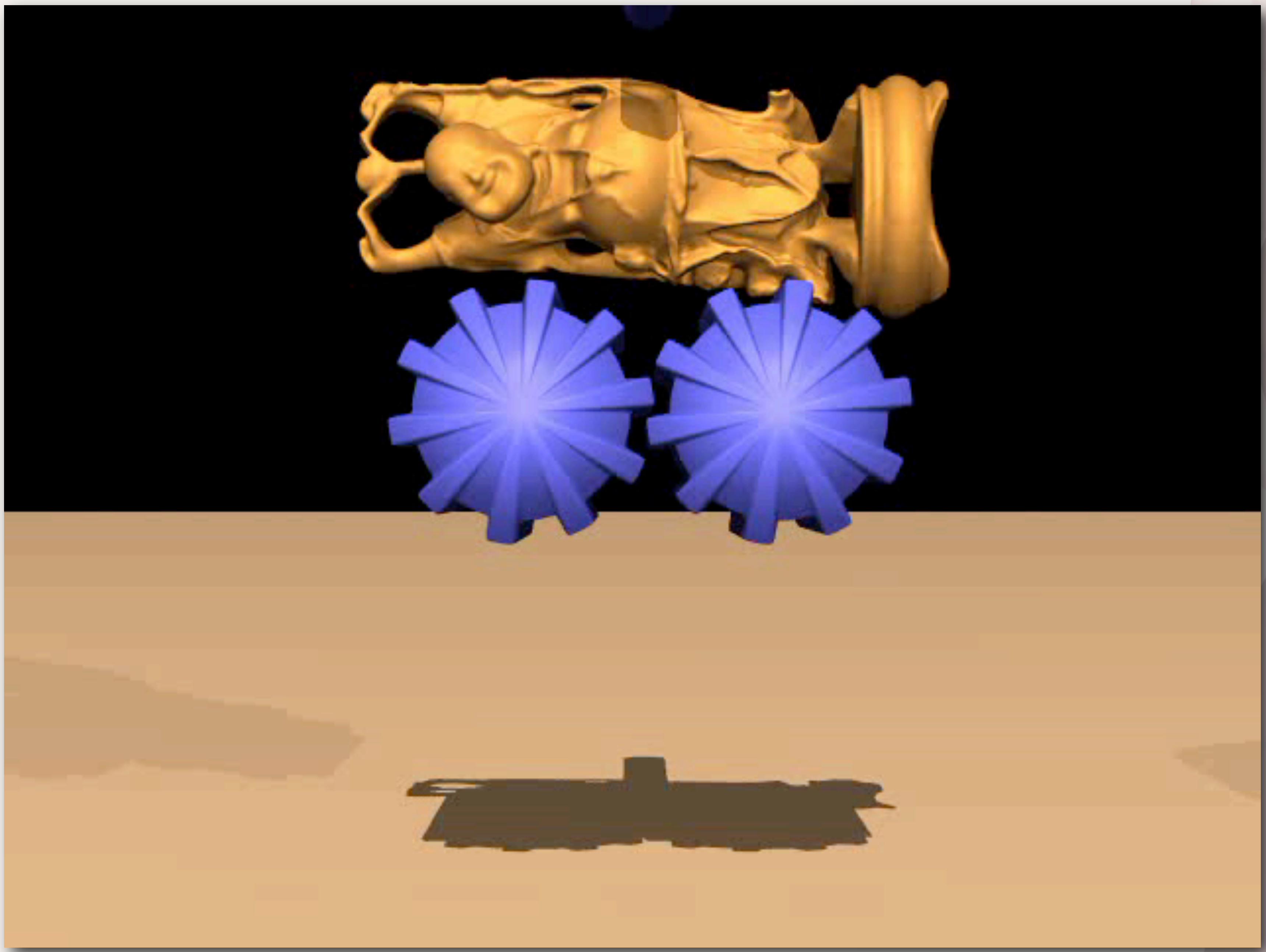
# TAing Undergrad Graphics?

# Deformable Materials



# Taking a Hard Look at Soft Things

## Spring with rest length 1:


## Deforms by Δx:


$$\text{energy} = E\frac{1}{2}(\Delta x)^2$$

$$\text{force} = -\frac{d\ \text{energy}}{dx}$$

$$\text{force} = -E\Delta x$$

# Deformations

## Spring deformed by Δx:


$$\text{force} = -E\Delta x$$

stress: σ

Young's modulus

strain: ε

## Hooke's Law:

$$\sigma = -E\epsilon$$

**Steel:** $E=10^{11}$ N/m$^2$

**Rubber:** $E=10^7 \sim 10^8$ N/m$^2$

# Hooke's Law

$$\sigma = -E\epsilon$$

- **Want to generalize in two ways:**
  - **Continuum Deformations**
  - **3D**

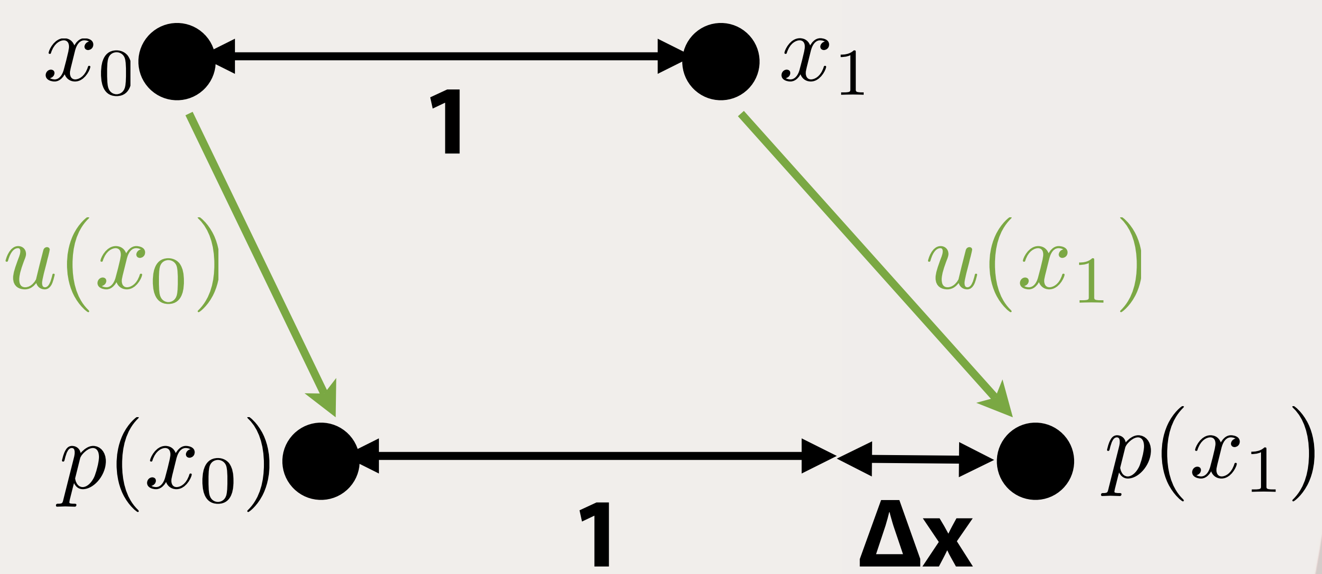
# Hooke's Law

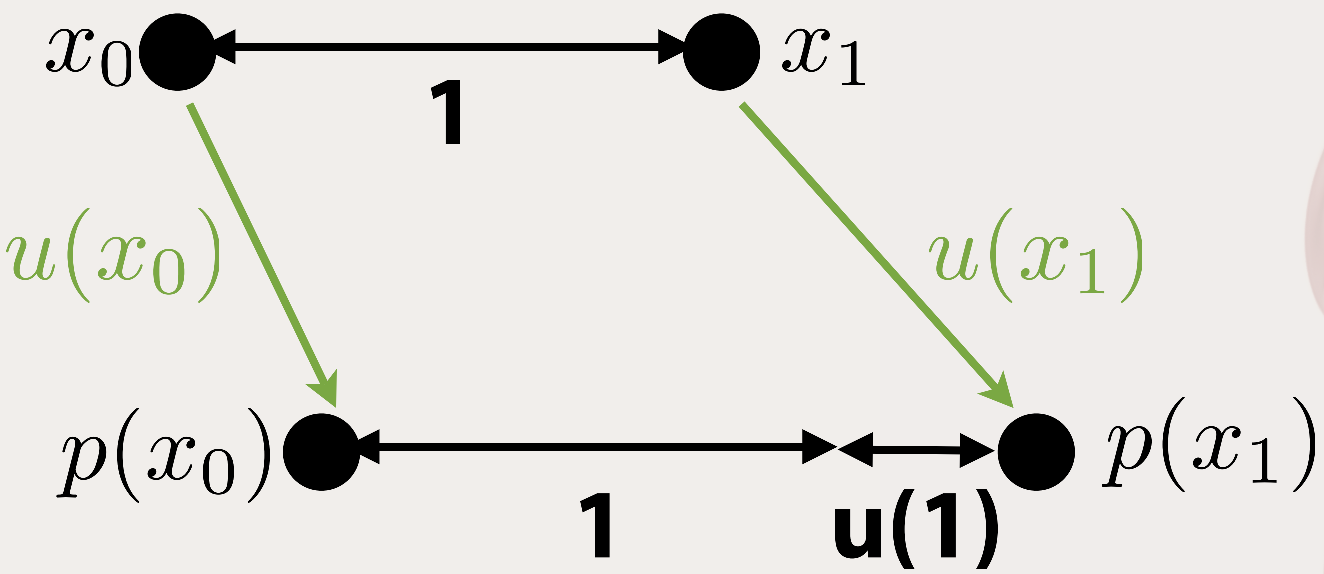
$$\sigma = -E\epsilon$$

- Want to generalize in two ways:
  - **Continuum Deformations**
  - 3D

# Continuum Deformations

- **Given a displacement field $p(x)$:**
- **Which defines a deformation field:**
  - $u(x) = p(x) - x$
  - **(Like velocities in a fluid.)**






- **Suppose: $x_0 = 0$ and $x_1 = 1$**
- **$p(0) = u(0)$**
- **$p(1) = 1 + u(1)$**
- **energy = ½ $E\,(p(1) - p(0) - 1)^2$**
- **$p(1) \approx 1 + u(0) + \nabla u(0)$**
- **energy ≈ ½ $E\,(1 + u(0) + \nabla u(0) - u(0) - 1)^2$**
- **energy ≈ ½ E $\nabla u^2$**
- **force = -E$\nabla u$**

# Therefore...

- **force = -*E* ∇*u***

$$\sigma = -E\epsilon$$

**In 1D, ε = ∇*u*.**

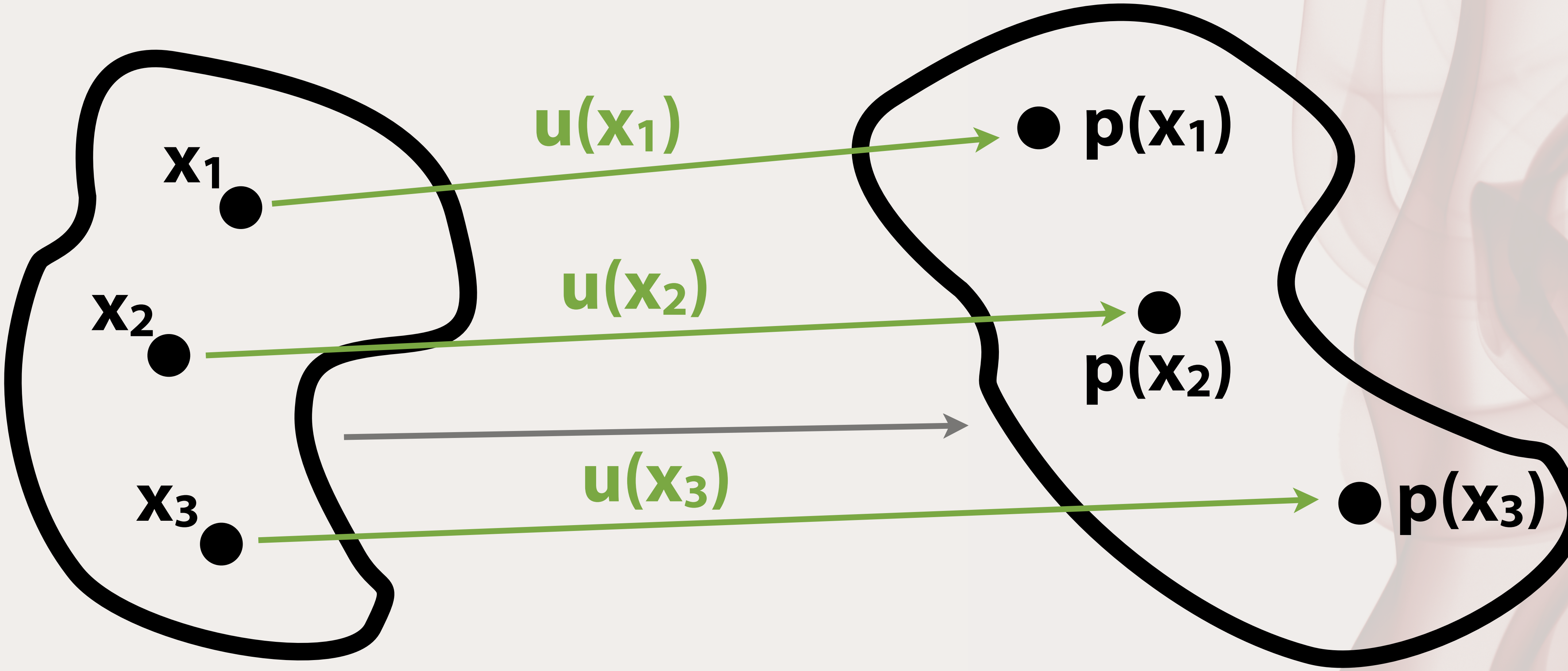
**(Would like to generalize to 3D.)**

# Hooke's Law

$$\sigma = -E\epsilon$$

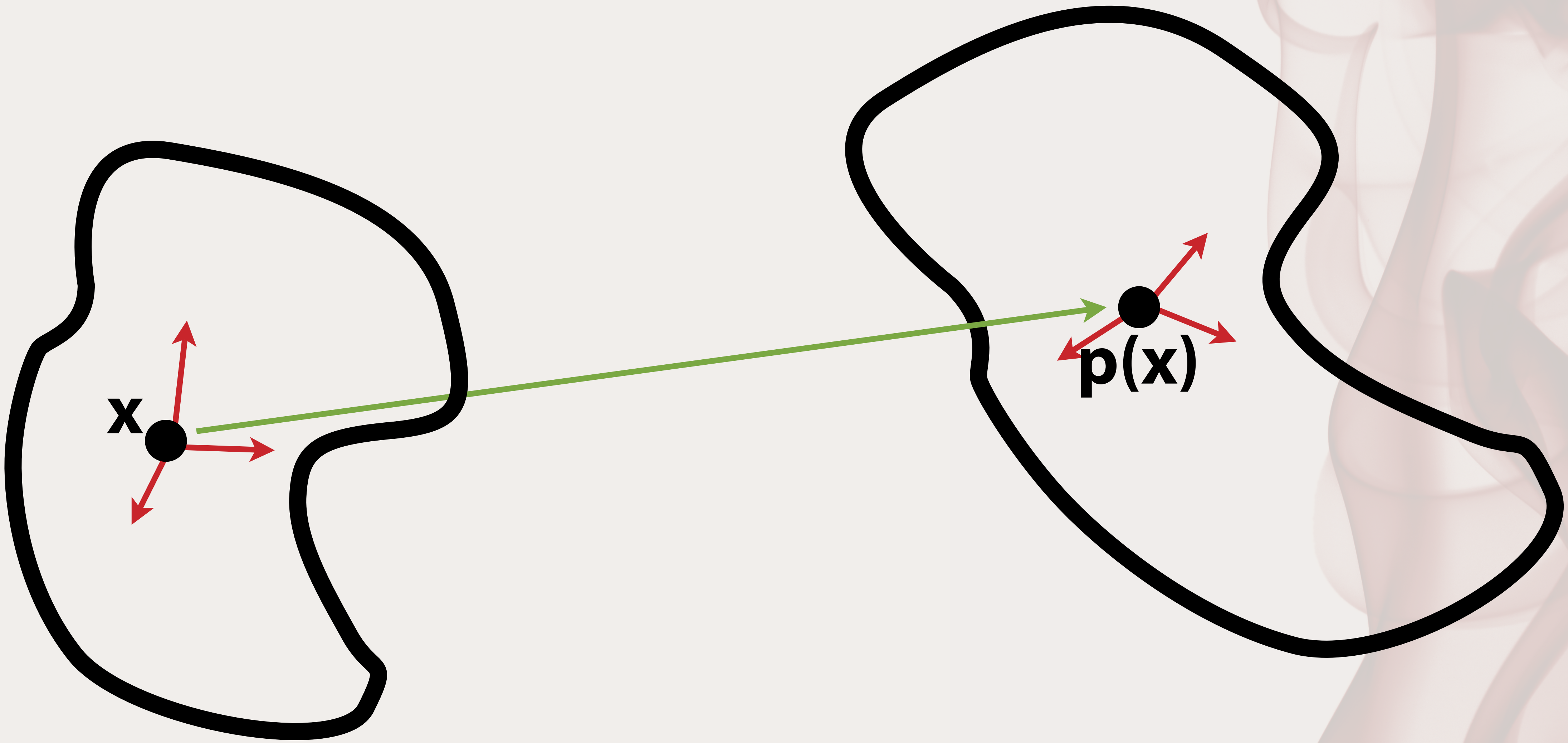
- Want to generalize in two ways:
  - Continuum Deformations
  - 3D (things will get a little silly)

# In 3D...




$$u(x) = p(x) - x$$

# Coord Sys Transform

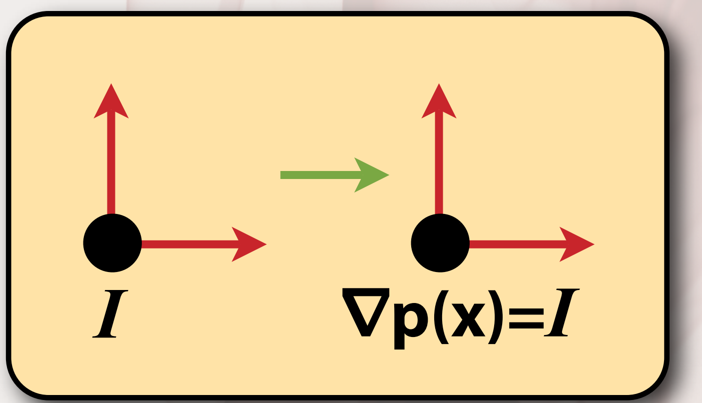



**point:** $x$
**local coordinates:** $I$

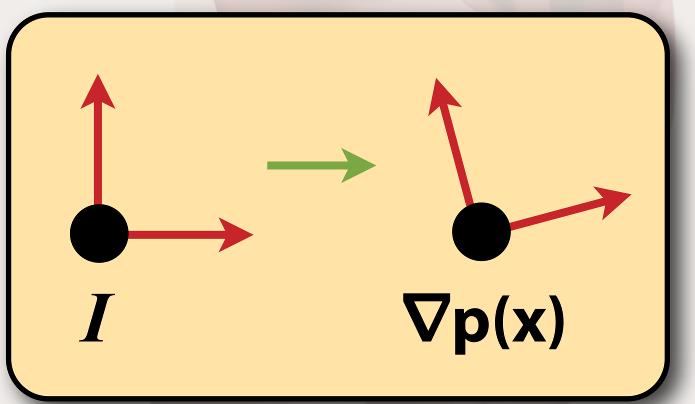
**point:** $p(x)$
**local coordinates:** $\nabla p$

# Defining Strain

- Strain is **invariant to translation**.
  - Ignore $p(x)$
  - Define in terms of local coordinate system transform: $\nabla p(x)$.




- Strain is **invariant to rotation**.
  - If $[\nabla p(x)]^T \nabla p(x) = I$,
  - Then $\varepsilon=0$




- Natural to define strain as:
  - $\varepsilon = ½([\nabla p(x)]^T \nabla p(x) - I)$
  - 6 DOFs

$$\epsilon = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$$

# Defining Strain

$$\epsilon = \frac{1}{2}\left[\nabla p(x)\right]^T \nabla p(x) - I$$

$$u(x) = p(x) - x$$

$$\nabla u(x) = \nabla p(x) - I$$

$$\epsilon = \frac{1}{2}\left[\nabla u(x) + I\right]^T \left[\nabla u(x) + I\right] - I$$

**Green's Strain:** $$\epsilon_G = \frac{1}{2}\left(\nabla u + [\nabla u]^T + [\nabla u]^T \nabla u\right)$$

**Cauchy's Strain:** $$\epsilon_C = \frac{1}{2}\left(\nabla u + [\nabla u]^T\right)$$ **(no rotation)**

**1D Strain:** $$\epsilon_{1D} = \nabla u$$

# Defining Strain

$$\epsilon = \frac{1}{2} \left[\nabla p(x)\right]^T \nabla p(x) - I$$

$$u(x) = p(x) - x$$

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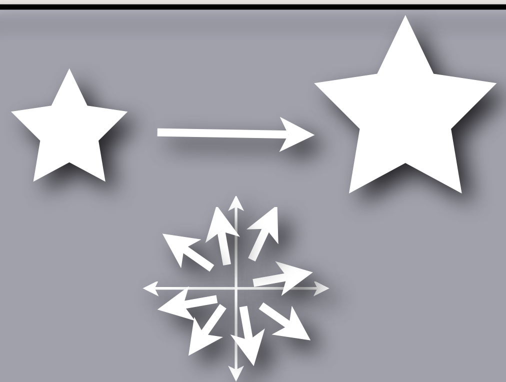
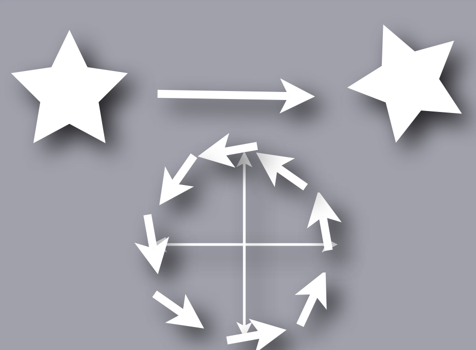
**Green's Strain:** $$\epsilon_G = \frac{1}{2} \left(\nabla u + [\nabla u]^T + [\nabla u]^T \nabla u\right)$$

**Cauchy's Strain:** $$\epsilon_C = \frac{1}{2} \left(\nabla u + [\nabla u]^T\right)$$ **(no rotation)**

**1D Strain:** $$\epsilon_{1D} = \nabla u$$

# Cauchy's vs. Green's Strain

| | Cauchy's Strain | Green's Strain |
|---|---|---|
| | $\epsilon_C = \frac{1}{2}\left(\nabla u + [\nabla u]^T\right)$ | $\epsilon_G = \frac{1}{2}\left(\nabla u + [\nabla u]^T + [\nabla u]^T \nabla u\right)$ |
|  $u(x) = x$ <br> $\nabla u(x) = I$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $\frac{3}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
|  $\nabla u = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ✖ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |

# Question