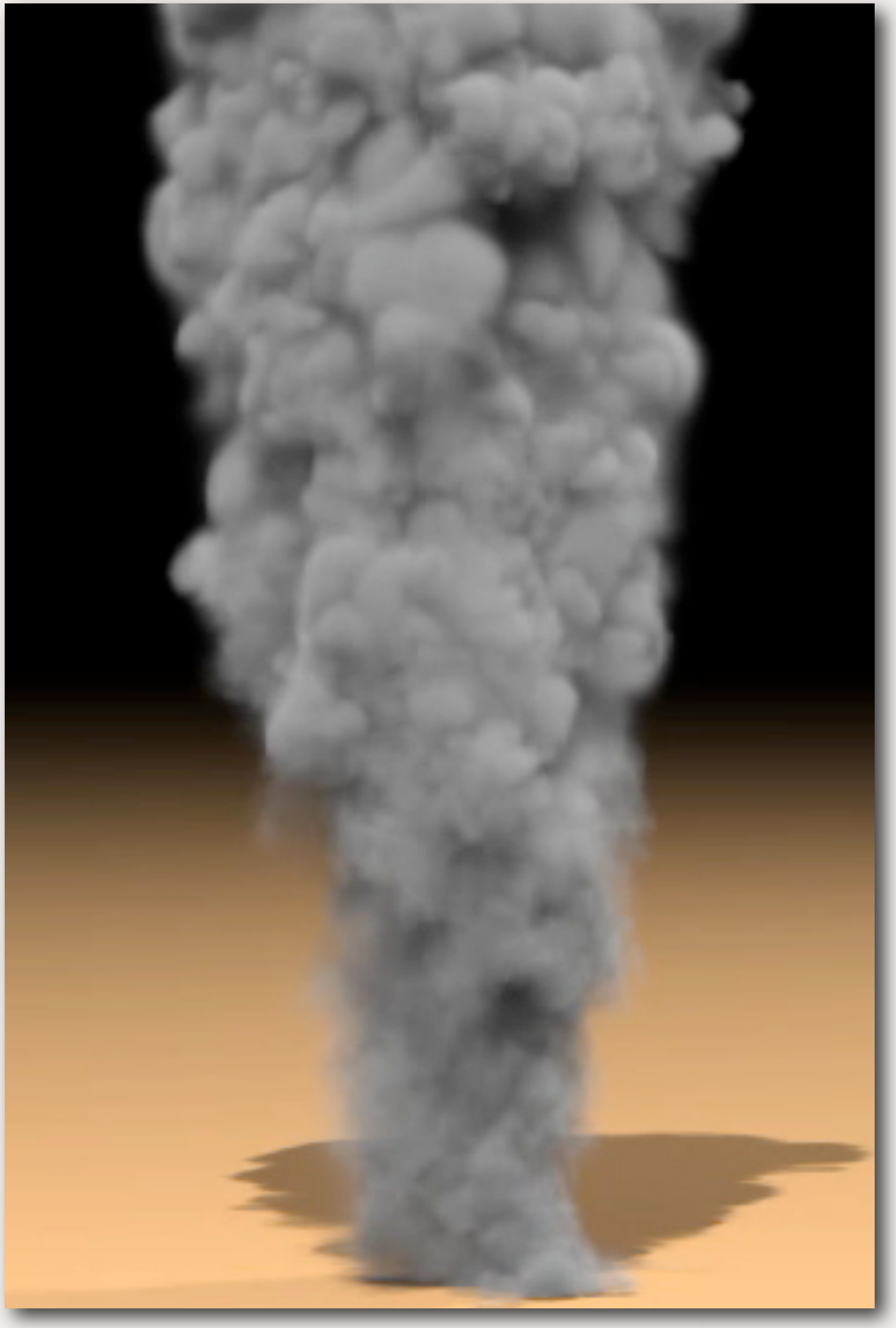
3D Fluid Simulation

Adrien Treuille



source: Selle *et al* 2005

Overview

- Last week's question.
- The physics of fluids.
- Simulating fluids.
- Heegun.
- This week's question.

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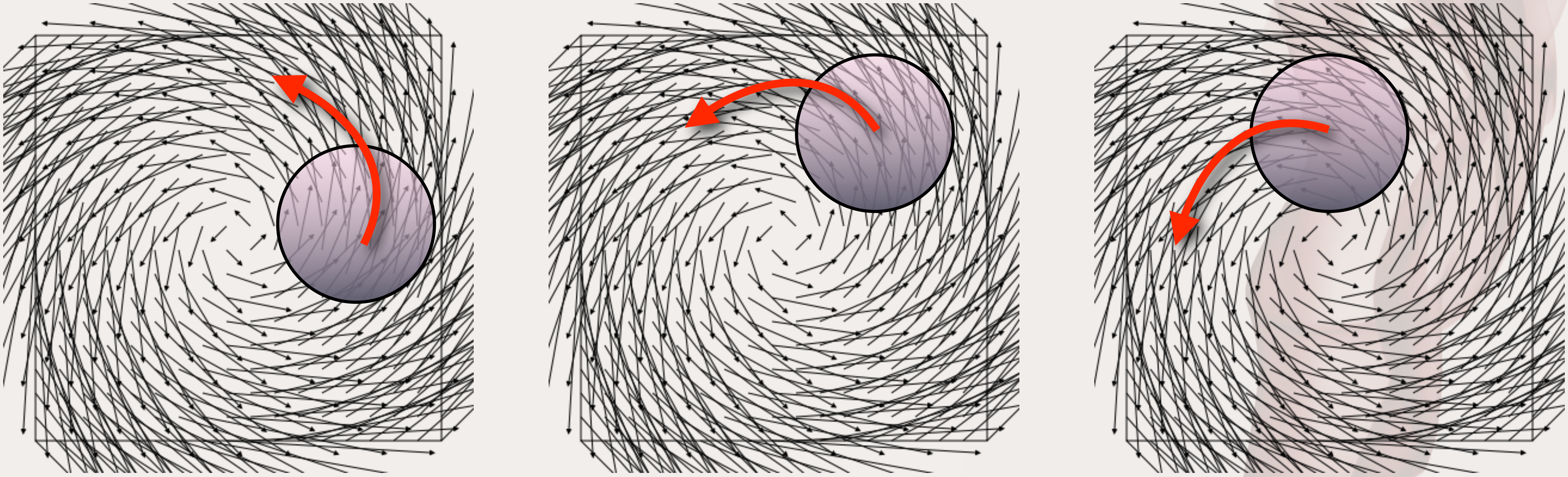
Question

- **How could you make a PDE that rotates...**

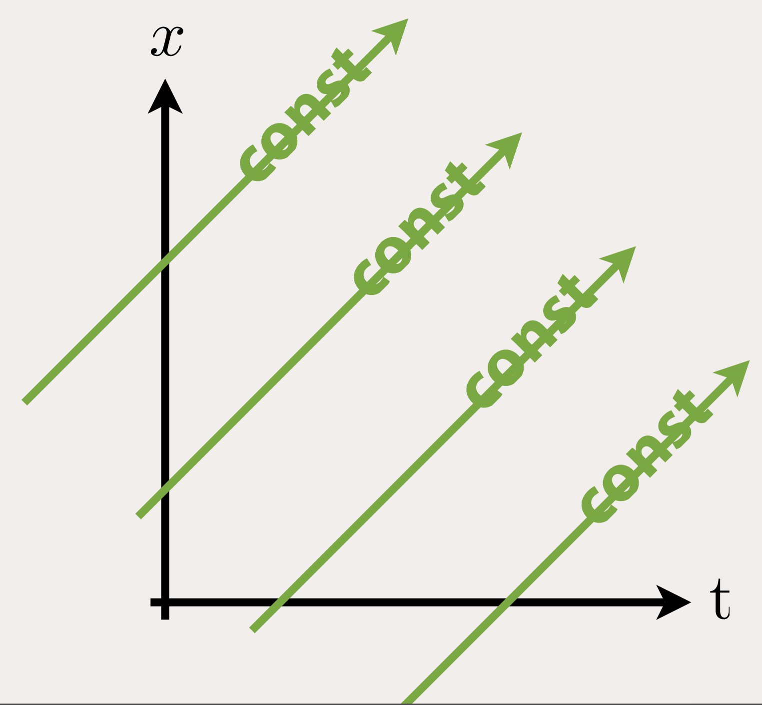


Recall...

$$f(x,t) = g(x-t)$$

- **Information propagates "to the right"**

$$f(x, t+\Delta t) = f(x-\Delta t, t)$$



Problem Solution

- Blackboard...

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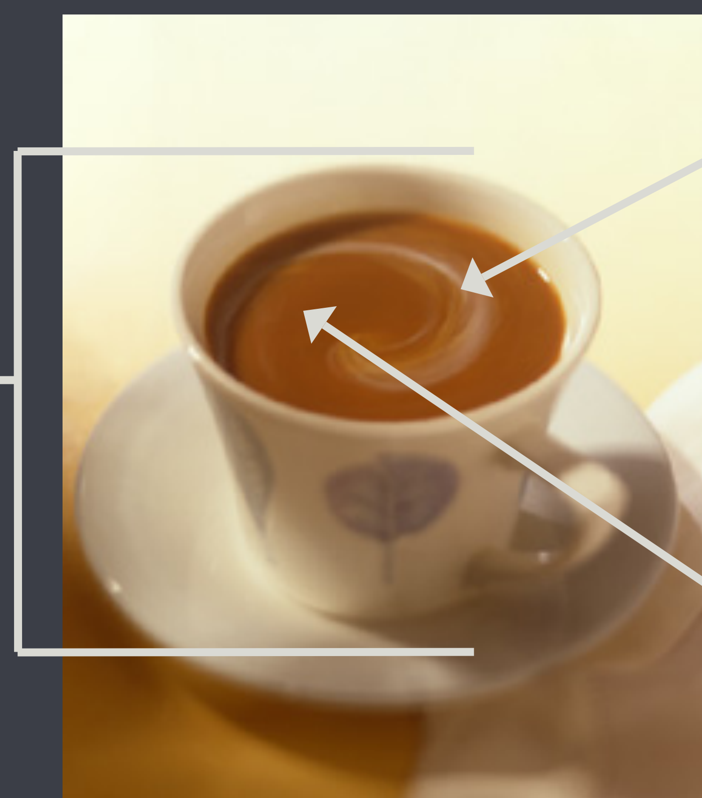
Incompressible Navier-Stokes Equations



Ω
(domain)

$\rho : \Omega \rightarrow [0, 1]$
(density)

$\mathbf{u} : \Omega \rightarrow \mathbb{R}^3$
(velocity)

~~"Coffee Cup"~~ Equations

Navier-Stokes

- Density

$$\frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho$$

- Velocity

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

Demo

Density Advection

$$\frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho$$

Video: Density Advection

Velocity Advection

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

Video: Velocity Advection

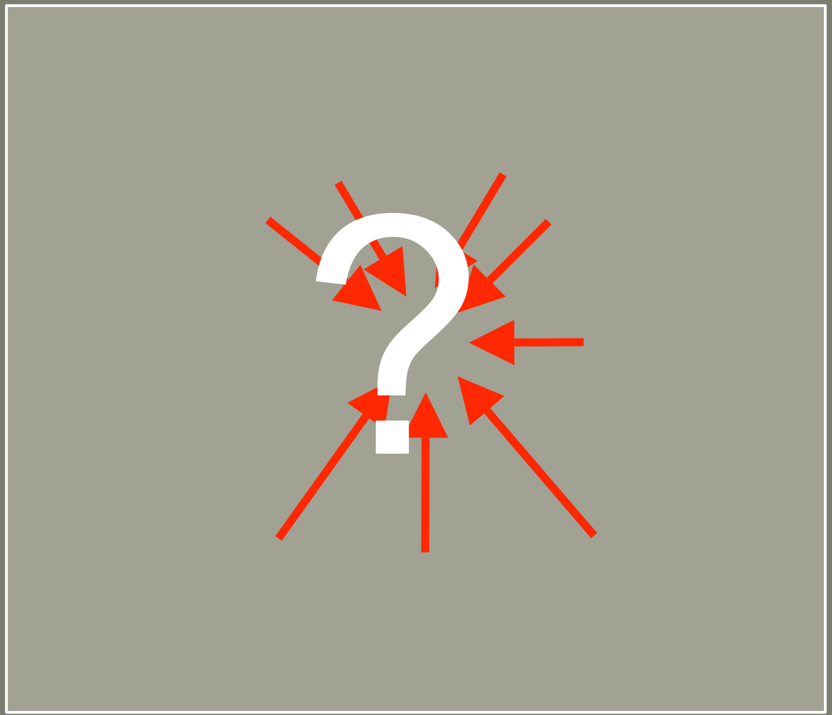
Projection

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

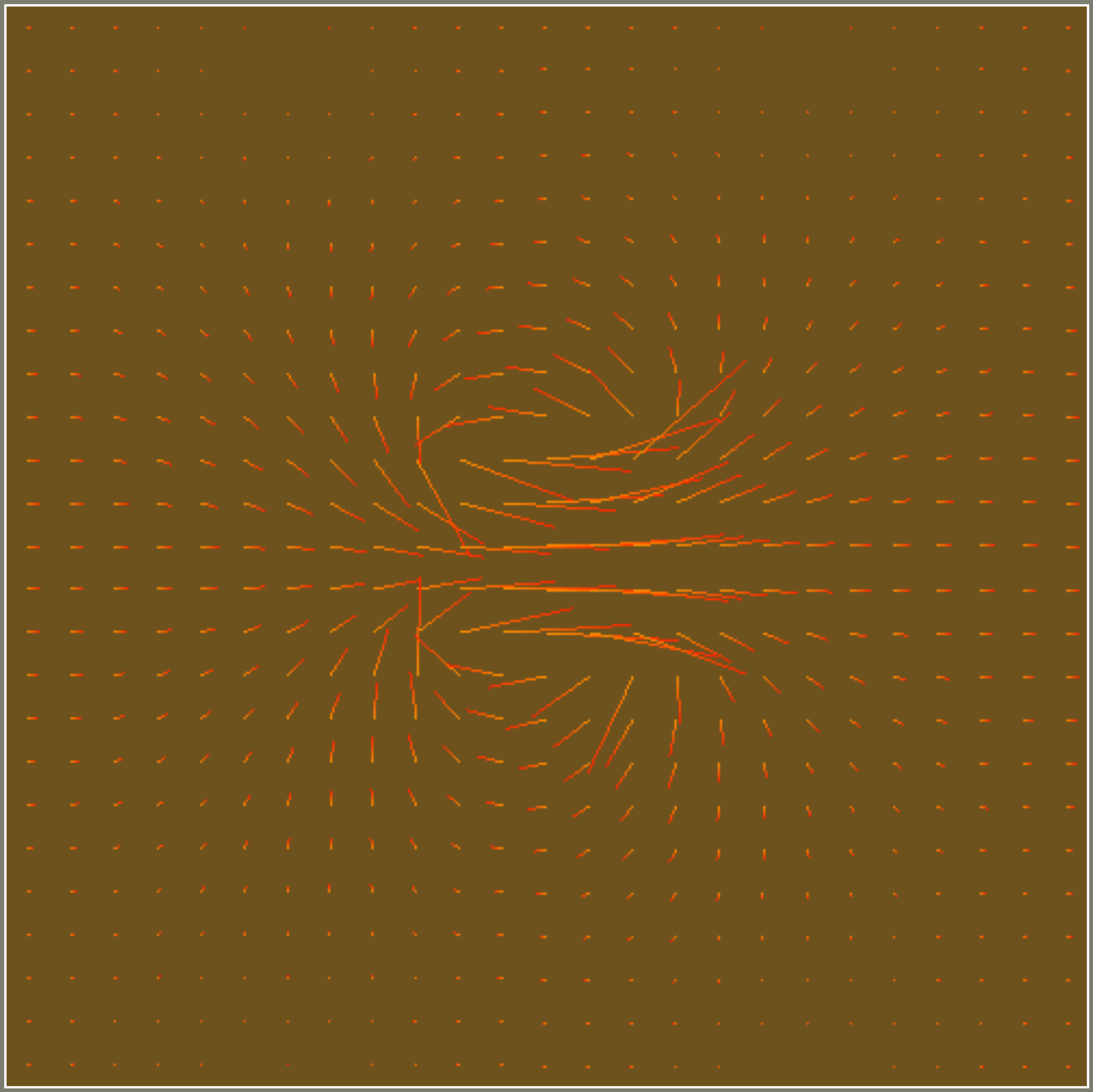
s.t. $\nabla \cdot \mathbf{u} = 0$

(divergence)

Div $\neq$ 0



Projection



Div $\neq 0$

Projection

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \boxed{\frac{1}{r}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

Video: Velocity Advection and Projection

Diffusion

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2 \mathbf{u} + \mathbf{f}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

External Forces

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

- Gravity
- Heat
- Surface Tension
- User-Created Forces (stirring coffee)

Physics Recap

- Physical quantities represented as fields.
- PDE describes the dynamics.
  - explains what we see in here…

- Much $$$ for analytic solution!

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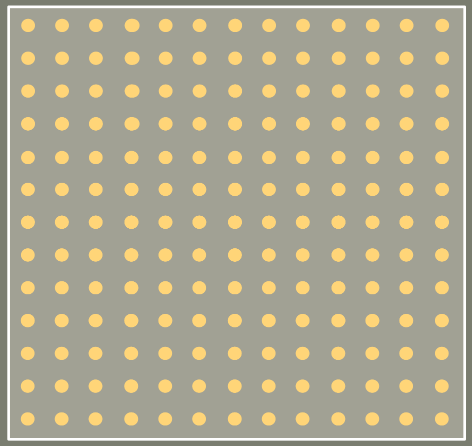
Simulation Representation

- Recall we're dealing with *fields*:

$$\rho : \Omega \rightarrow [0, 1]$$
(density)

$$\mathbf{u} : \Omega \rightarrow \mathbf{R}^3$$
(velocity)

- Grid Representation



  - Each grid cell represents integral over underlying quantities
  - Derivatives Easy to Implement

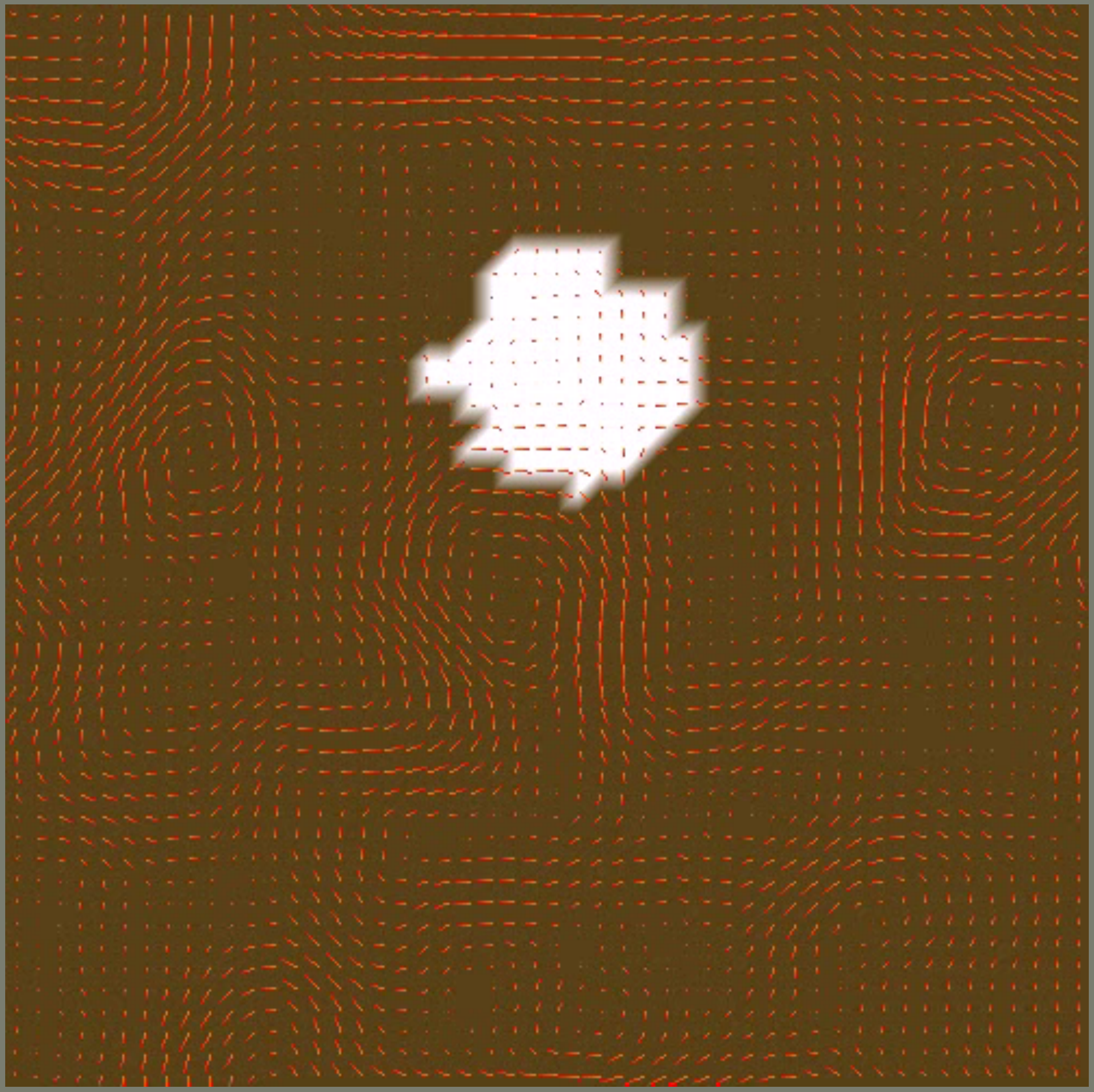
Explicit Integration

- Very simple method to “implement” physics

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2 \mathbf{u} + \mathbf{f}$$

$$x(t + \Delta t) \approx x(t) + (\Delta t) f(x(t))$$

Explicit Integration



Splitting Methods

- Suppose we had a system:

$$\frac{\partial x}{\partial t} = f(x) = g(t) + h(t)$$

- …and we define a *simulation* S_f.

$$S_f(x, \Delta t) \;:\; x(t) \mapsto x(t + \Delta t)$$

- Then we *could* define:

$$S_f(x, \Delta t) = S_g(x, \Delta t) \circ S_h(x, \Delta t)$$

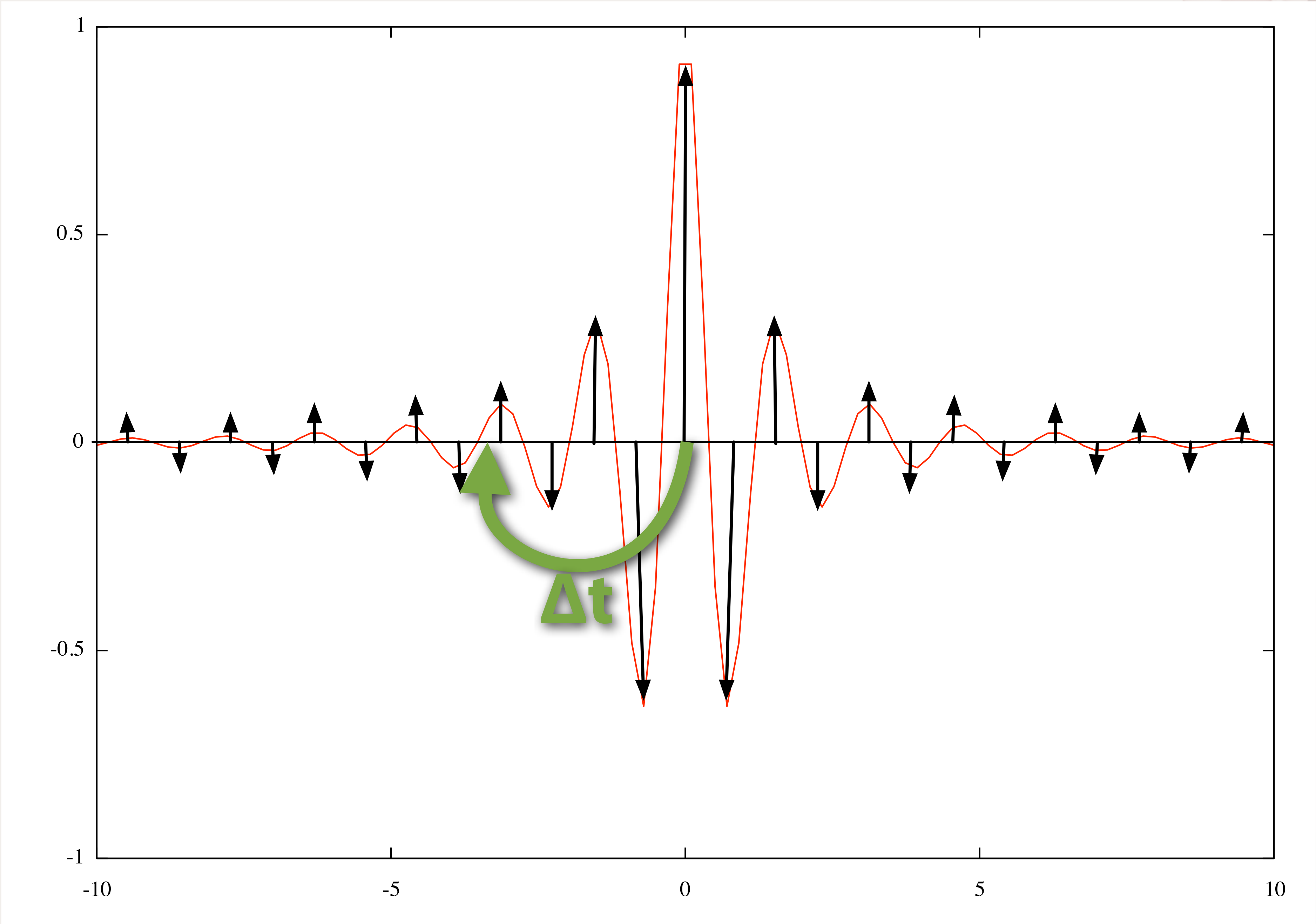
Splitting Methods

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

Advect → Project → Diffuse → Add Forces

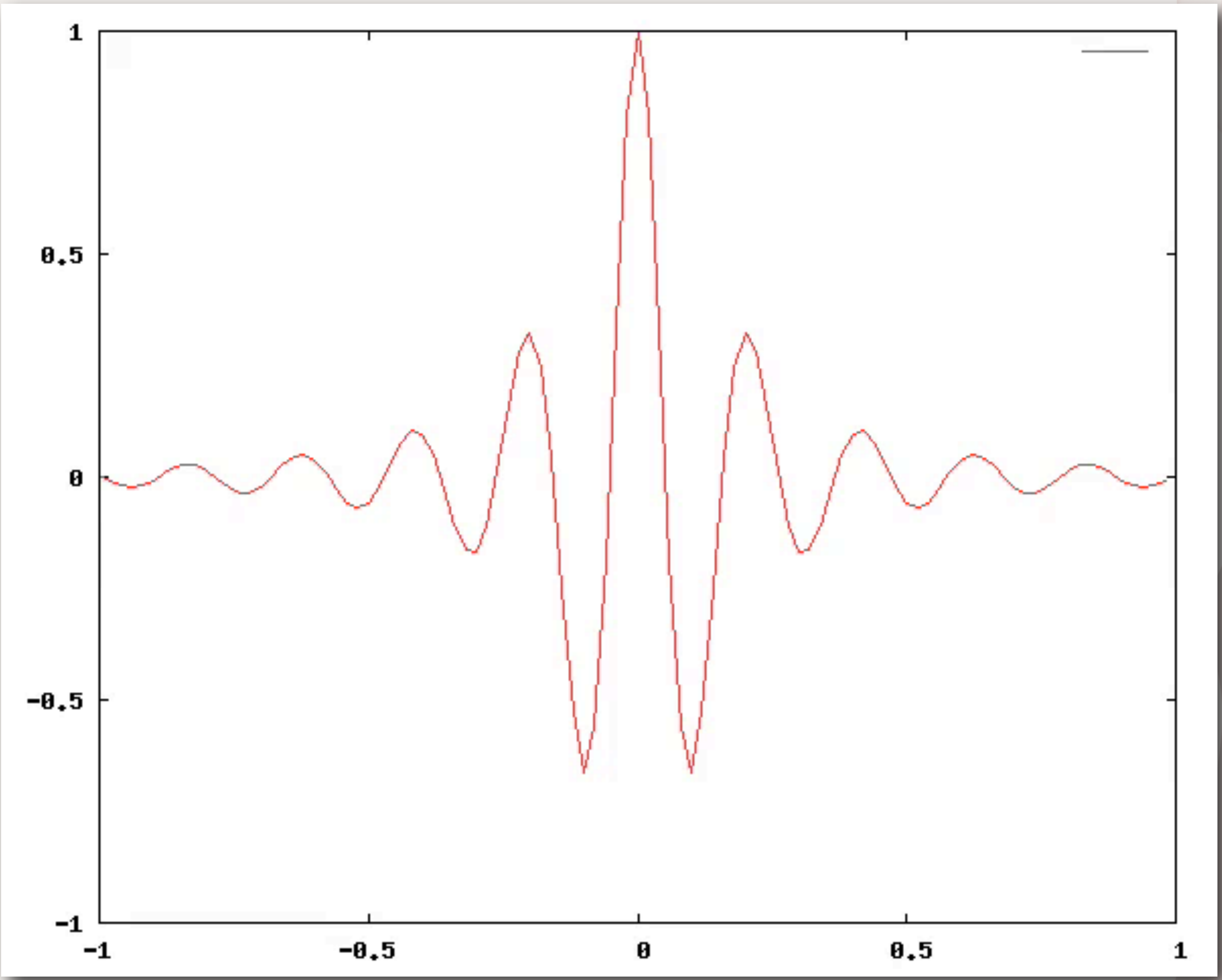
Semi-Lagrangian

$$f(x, t + \Delta t) = f(x - \Delta t, t)$$

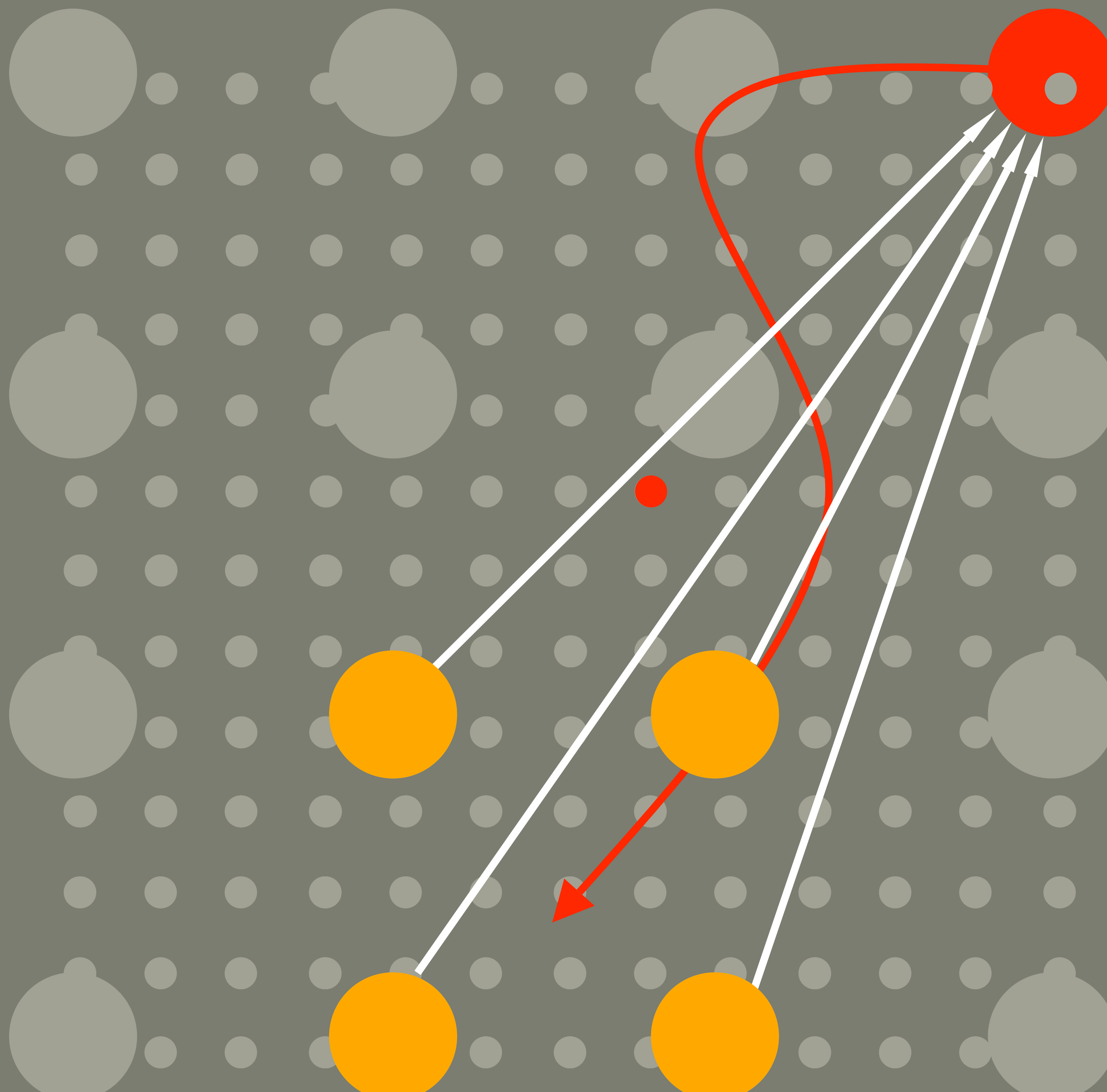


SL Advection

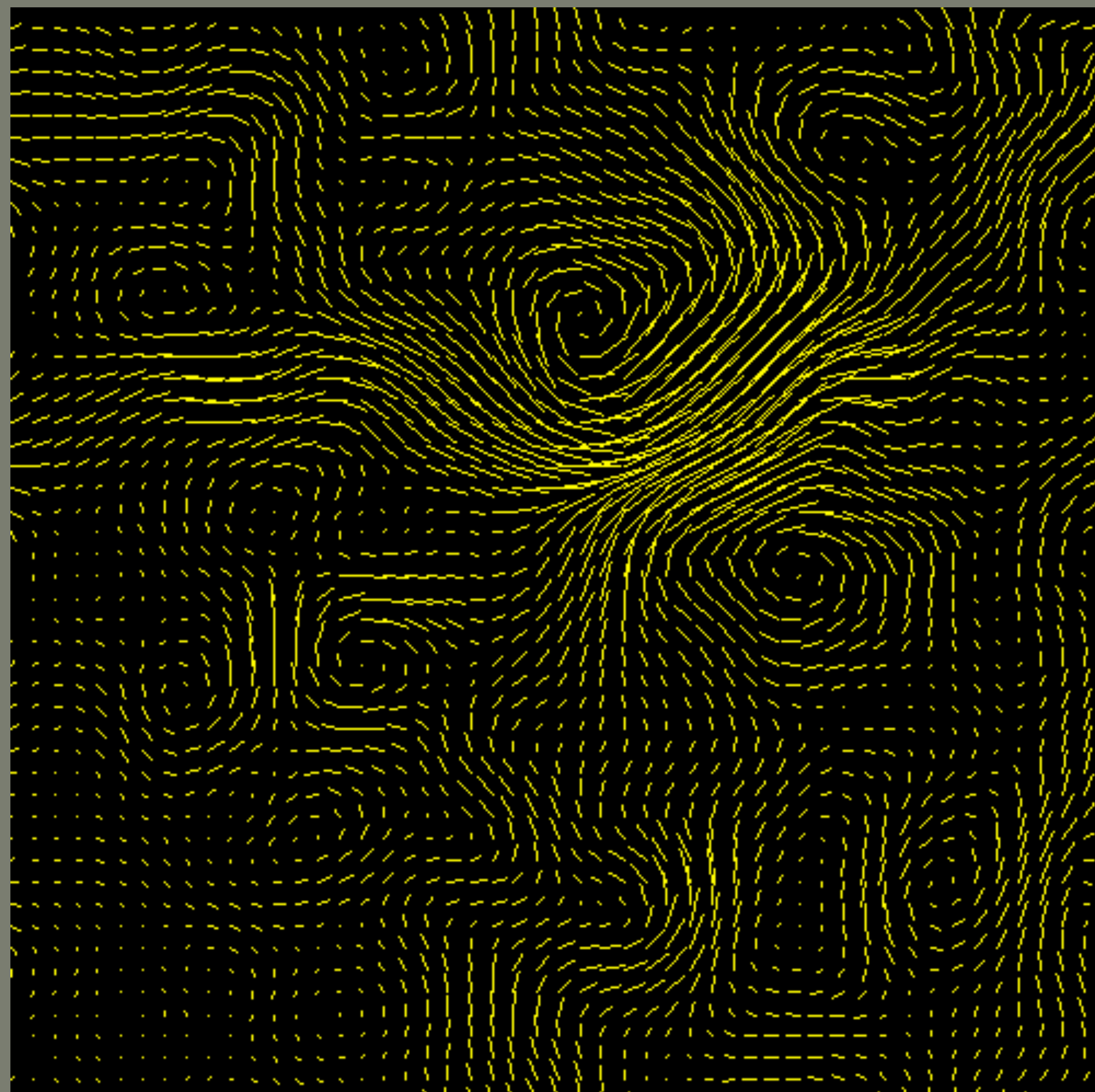
$$f(x, t + \Delta t) = f(x - \Delta t, t)$$



Advection



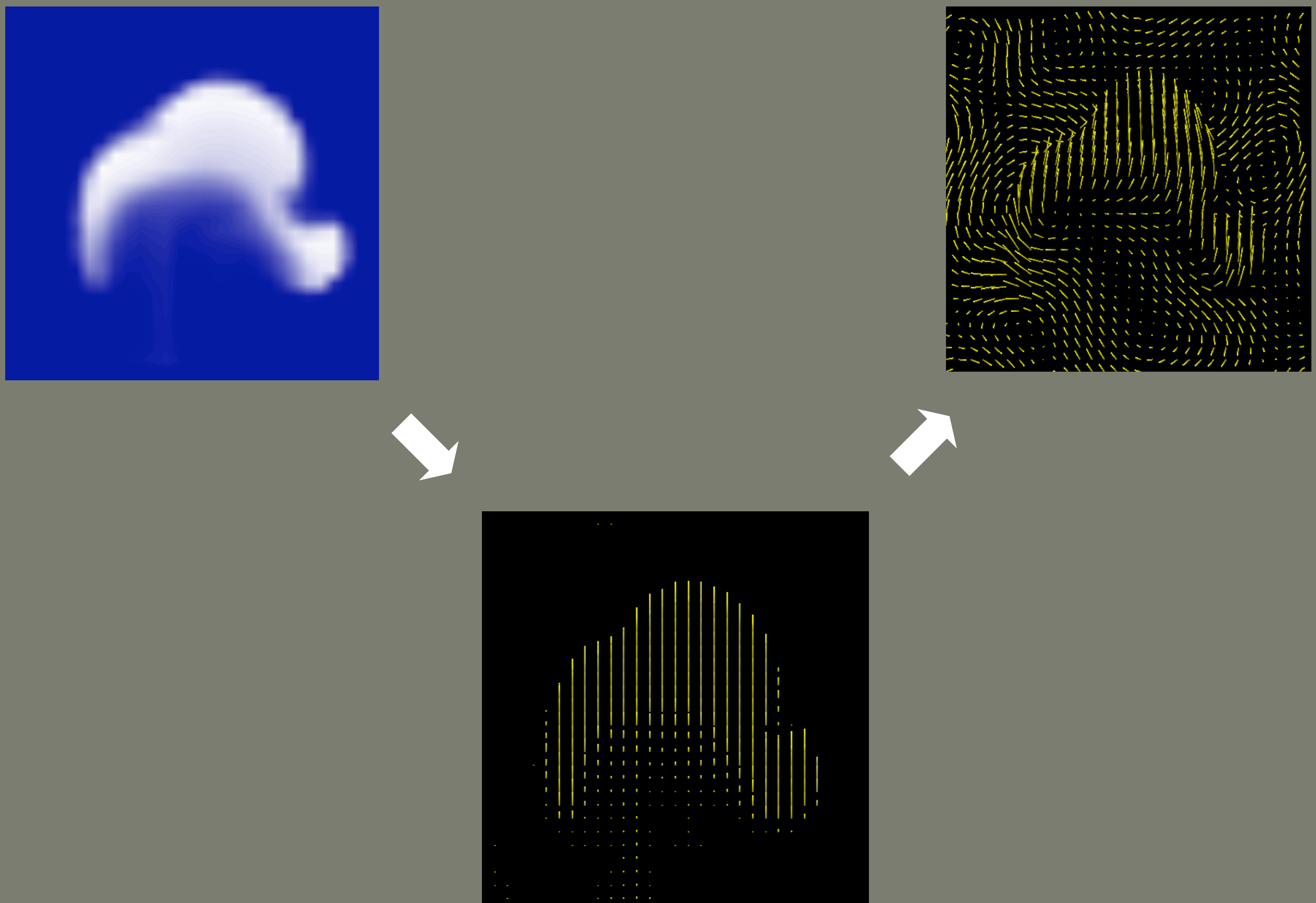
Projection

$P\Bigg($  $\Bigg)$

Diffusion

- Solved implicitly (like projection)
- I don’t have a picture of this.

Add Forces (e.g. heat)



Simulation Recap

- Decided Upon *grid-based* represenation.
- Explicit Methods will not work.
- Stable Fluids solves all our problems…
  - …maybe.

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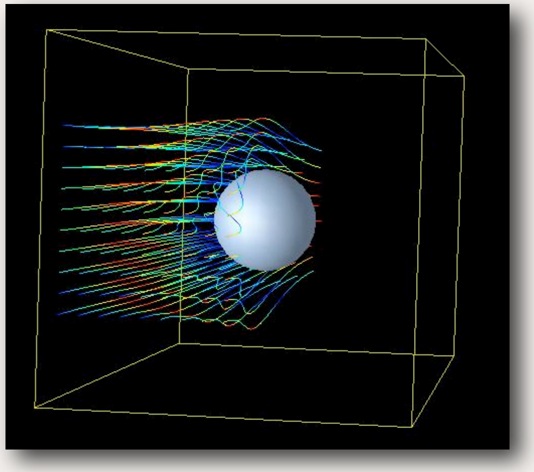
Questions

- How could we deal with fixed boundaries in the fluid?



- How can we deal with a free surface in the fluid?

- Hint: how can we modify the projection step?