Bidirectional path tracing
Course announcements

• Take-home quiz 8 posted, due Wednesday 3/29 at 3:00.

• Programming assignment 4 posted, due Friday 3/31 at 23:59.
  - How many of you have looked at/started/finished it?
  - Any questions?
Overview of today’s lecture

• Types of light paths.

• Light tracing.

• Bidirectional path tracing.
Slide credits

Most of these slides were directly adapted from:

• Wojciech Jarosz (Dartmouth).
Light Paths
Light Paths

Express light paths in terms of the surface interactions that have occurred

A light path is a chain of linear segments joined at event “vertices”
Heckbert’s Classification

Classification of “vertices”:

- $L$: a light source
- $E$: the eye
- $S$: a specular reflection
- $D$: a diffuse reflection
Heckbert’s Classification
Heckbert’s Classification

- Diffuse
- Specular

Light source

Eye

Image plane

$LE$

Specular

Diffuse
Heckbert’s Classification

- **Diffuse**
- **Specular**

Diagram showing the light source, eye, image plane, and the paths LE and LDE.
Heckbert’s Classification

Diffuse

Specular

Light source

Eye

Image plane

$LDE$

$LE$

$LSE$
 Heckbert’s Classification

Light source

Image plane

Diffuse

Specular

Eye

LDDE

LDE

LE

LSE
Heckbert’s Classification

Light source

Specular

Diffuse

Eye

Image plane

LDE

LE

LSE

LDDE

LSSDE
Heckbert’s Classification

Can express arbitrary classes of paths using a regular expression type syntax:

- $k^+$: one or more of event $k$
- $k^*$: zero or more of event $k$
- $k?$: zero or one $k$ events
- $(k|h)$: a $k$ or $h$ event
Heckbert’s Classification

Direct illumination: \( L(D \| S)E \)

Indirect illumination: \( L(D \| S)(D \| S)^*E \)
Heckbert’s Classification

Direct illumination: $L(D | S)E$

Indirect illumination: $L(D | S)(D | S)^*E$

Full global illumination: $L(D | S)^*E$
Diffuse inter-reflections: $LDD^+E$
Caustics: $LS^+DE$

source: Flickr
Subsurface Scattering
A Simple Scene

10 paths/pixel

Henrik Wann Jensen
+ Glass/Mirror Material

10 paths/pixel
Path Tracing Caustics

Glass sphere
Path Tracing Caustics
Path Tracing Caustics

Random sampling of hemisphere will rarely hit the light source
Random sampling of hemisphere will **never** hit the light source.
Let’s just give it more time...

Nature $\sim 2 \times 10^{33}$ / second

Fastest GPU ray tracer $\sim 2 \times 10^8$ / second

".. if we'd rendered [Gravity] on a single processor instead of having a room full of computers, we would have had to start rendering in 5000 BC to finish in time to deliver the film. At the dawn of Egyptian civilisation."

Tim Webber, Gravity VFX supervisor
Let’s just give it more time...

1 image ~ 8 core years (parallelized on a cluster)
Path Tracing - Summary

✓ Full solution to the rendering equation
✓ Simple to implement
✗ Slow convergence
   – requires 4x more samples to half the error
✗ Robustness issues
   – does not handle some light paths well (or not at all), e.g. caustics ($LS^+DE$)
✗ No reuse or caching of computation
✗ General sampling issue
   – makes only locally good decisions
Today’s agenda

Measurement Equation

Path Integral Framework

Solving the Rendering Equation

- Light tracing
- Bidirectional path tracing
Can we simulate this better?
Light transport is symmetric

Dual Photography [Sen et al. 2005]
Dual Photography

Pradeep Sen*  Billy Chen*  Gaurav Garg*  Stephen R. Marschner†
Mark Horowitz*  Marc Levoy*  Hendrik P.A. Lensch*

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SIGGRAPH 2005
Duality of Radiance and Importance
Measurement Equation

Rendering equation describes radiative equilibrium at point $x$:

$$L_o(x, \bar{\omega}_o) = L_e(x, \bar{\omega}_o) + \int_{H^2} f_r(x, \bar{\omega}_i, \bar{\omega}_o) L_i(x, \bar{\omega}_i) \cos \theta_i \, d\bar{\omega}_i$$

We are interested in the total radiance contributing to pixel $j$:

$$I_j = \int_{A_{film}} \int_{H^2} W_e(x, \bar{\omega}) L_i(x, \bar{\omega}) \cos \theta \, d\bar{\omega} \, dx$$

response of the sensor at film location $x$ to radiance arriving from direction $\bar{\omega}$ (often referred to as emitted importance)
Radiometry as Measurements

Weighted integral of 5D radiance function

\[ \int_V \int_{H^2} W_e(x, \omega) L(x, \omega) \, d\omega \, dx \]

Other radiometric quantities are measurements

- expressing *irradiance* in terms of radiance:
  \[ \int_{H^2} L(x, \omega) \cos \theta \, d\omega = E(x) \]
  Integrate radiance over hemisphere

- expressing *flux/power* in terms of radiance:
  \[ \int_A \int_{H^2} L(x, \omega) \cos \theta \, d\omega \, dA(x) = \Phi(A) \]
  Integrate radiance over hemisphere and area
Radiance vs. Importance

Radiance
- emitted from light sources
- describes *amount of light* traveling within a differential beam

Importance
- “emitted” from sensors
- describes the *response of the sensor* to radiance traveling within a differential beam
Duality of Radiance & Importance

\[ I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(x, \omega)L_i(x, \omega) \cos \theta \, d\omega \, dx \]
Duality of Radiance & Importance

\[ I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(x, \omega) L_i(x, \omega) \cos \theta \, d\omega \, dx \]

\[ = \int_{A_{\text{film}}} \int_{A} W_e(x, y) G(x, y) L_o(y, x) \, dy \, dx \]

outgoing quantities

Let’s expand \( L_o \) and consider direct illumination only
Duality of Radiance & Importance

\[ I_j = \int_{A_{\text{film}}} \int_{\mathcal{H}^2} W_e(x, \omega)L_i(x, \omega) \cos \theta \, d\omega \, dx \]

\[ = \int_{A_{\text{film}}} \int_{A} W_e(x, y)G(x, y)L_o(y, x) \, dy \, dx \]

\[ = \int_{A_{\text{film}}} \int_{A} \int_{A_{\text{light}}} W_e(x, y)G(x, y)f(y, z, x)G(y, z)L_e(z, y) \, dz \, dy \, dx \]

emitted quantities with identical measure

Let’s swap the inner and outer integral
Duality of Radiance & Importance

\[ I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(x, \omega)L_i(x, \omega) \cos \theta \, d\omega \, dx \]

\[ = \int_{A_{\text{film}}} \int_{A} W_e(x, y)G(x, y)L_o(y, x) \, dy \, dx \]

\[ = \int_{A_{\text{film}}} \int_{A} \int_{A_{\text{light}}} W_e(x, y)G(x, y)f(y, z, x)G(y, z)L_e(z, y) \, dz \, dy \, dx \]

\[ = \int_{A_{\text{light}}} \int_{A} \int_{A_{\text{film}}} W_e(x, y)G(x, y)f(y, z, x)G(y, z)L_e(z, y) \, dx \, dy \, dz \]

symmetric functions
Duality of Radiance & Importance

\[
I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(x, \omega)L_i(x, \omega) \cos \theta \, d\omega dx
\]

\[
= \int_{A_{\text{film}}} \int_{A} W_e(x, y) G(x, y)L_o(y, x) \, dy dx
\]

\[
= \int_{A_{\text{film}}} \int_{A} \int_{A_{\text{light}}} W_e(x, y) G(x, y)f(y, z, x)G(y, z)L_e(z, y) \, dz dy dx
\]

\[
= \int_{A_{\text{light}}} \int_{A} \int_{A_{\text{film}}} W_e(x, y) G(y, x)f(y, x, z)G(z, y)L_e(z, y) \, dx dy dz
\]

symmetric functions
Duality of Radiance & Importance

\[ I_j = \int_{A_{\text{film}}} \int_H W_e(x, \bar{\omega}) L_i(x, \bar{\omega}) \cos \theta \, d\bar{\omega} \, dx \]

\[ = \int_{A_{\text{film}}} \int_A W_e(x, y) G(x, y) L_o(y, x) \, dy \, dx \]

\[ = \int_{A_{\text{film}}} \int_A \int_{A_{\text{light}}} W_e(x, y) G(x, y) f(y, z, x) G(y, z) L_e(z, y) \, dz \, dy \, dx \]

\[ = \int_{A_{\text{light}}} \int_A \int_{A_{\text{film}}} W_e(x, y) G(y, x) f(y, x, z) G(z, y) L_e(z, y) \, dx \, dy \, dz \]

\[ = \int_{A_{\text{light}}} \int_A W_o(y, z) G(z, y) L_e(z, y) \, dy \, dz \]
Duality of Radiance & Importance

\[ I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(x, \bar{\omega}) L_i(x, \bar{\omega}) \cos \theta \, d\bar{\omega} \, dx \]

\[ = \int_{A_{\text{film}}} \int_{A} W_e(x, y) G(x, y) L_o(y, x) \, dy \, dx \]

\[ = \int_{A_{\text{film}}} \int_{A} \int_{A_{\text{light}}} W_e(x, y) G(x, y) f(y, z, x) G(y, z) L_e(z, y) \, dz \, dy \, dx \]

\[ = \int_{A_{\text{light}}} \int_{A} \int_{A_{\text{film}}} W_e(x, y) G(y, x) f(y, x, z) G(z, y) L_e(z, y) \, dx \, dy \, dz \]

\[ = \int_{A_{\text{light}}} \int_{A} W_o(y, z) G(z, y) L_e(z, y) \, dy \, dz \]

\[ = \int_{A_{\text{light}}} \int_{H^2} W_i(z, \bar{\omega}) L_e(z, \bar{\omega}) \cos \theta \, d\bar{\omega} \, dz \]
Duality of Radiance & Importance

\[ I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(x, \bar{\omega}) L_i(x, \bar{\omega}) \cos \theta \, d\bar{\omega} \, dx \]

\[ = \int_{A_{\text{light}}} \int_{H^2} W_i(z, \bar{\omega}) L_e(z, \bar{\omega}) \cos \theta \, d\bar{\omega} \, dz \]
Duality of Radiance & Importance

Path tracing
start from *film*, search for *radiance* at light

\[
I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(x, \omega) L_i(x, \omega) \cos \theta \, d\omega \, dx
\]

\[
= \int_{A_{\text{light}}} \int_{H^2} W_i(z, \omega) L_e(z, \omega) \cos \theta \, d\omega \, dz
\]

Light tracing
start from *light*, search for *importance* at sensor
Light Tracing
Light Tracing

Shoot multiple paths from light sources hoping to randomly hit the sensor

- Optionally: at each path vertex, connect to the image using next-event estimation (a.k.a. shadow rays in PT)
Light Tracing with NEE

Splat to the image at each vertex
Path Tracing Caustics
Light Tracing Caustics
Path vs. Light Tracing

Path tracing

Light tracing

Why is this glass sphere black?

Images courtesy of F. Suykens
Can we combine them?
Measurement Equation

\[ I_j = \int_A \int_A W_e(x_0, x_1) G(x_0, x_1) L_o(x_1, x_0) \, dx_1 dx_0 \]

\[ = \int_A \int_A W_e(x_0, x_1) G(x_0, x_1) L_e(x_1, x_0) + \int_A f(x_1, x_2, x_0) G(x_1, x_2) L_o(x_2, x_1) \, dx_2 dx_1 dx_0 \]

\[ = \int_A \int_A W_e(x_0, x_1) G(x_0, x_1) L_e(x_1, x_0) + \int_A f(x_1, x_2, x_0) G(x_1, x_2) L_e(x_2, x_1) + \int_A f(x_2, x_3, x_1) G(x_2, x_3) L_e(x_3, x_2) + \int_A \cdots dx_4 dx_3 dx_2 dx_1 dx_0 \]

Hard to concisely express arbitrary light transport with all the nested integrals

Let’s find a better way
Path Integral Form of Measurement Eq.

\[
I_j = \int_A \int_A W_e(x_0, x_1) G(x_0, x_1) L_o(x_1, x_0) \, dx_1 dx_0
\]

\[
= \int_A \int_A W_e(x_0, x_1) L_e(x_1, x_0) G(x_0, x_1) \, dx_1 dx_0
\]

**Emission**

**Direct illumination (3 vertices)**

\[
+ \int_A \int_A \int_A W_e(x_0, x_1) L_e(x_2, x_1) G(x_0, x_1) f(x_1, x_2, x_0) G(x_1, x_2) \, dx_2 dx_1 dx_0 + \cdots
\]

**(k-2)-bounce illumination (k vertices)**

\[
+ \int_A \cdots \int_A W_e(x_0, x_1) L_e(x_k, x_{k-1}) G(x_0, x_1) \prod_{j=1}^{k-1} f(x_j, x_{j+1}, x_{j-1}) G(x_j, x_{j+1}) \, dx_k \cdots dx_0 + \cdots
\]

introduce: \( \mathcal{P}_k = \{ \vec{x} = x_0 \cdots x_k; \ x_0 \cdots x_k \in A \} \)

space of all paths with \( k \) segments
Path Integral Form of Measurement Eq.

\[ I_j = \int_A \int_A W_e(x_0, x_1) G(x_0, x_1) L_o(x_1, x_0) \, dx_1 \, dx_0 \]

\[ = \int_{P_1} W_e(x_0, x_1) L_e(x_1, x_0) G(x_0, x_1) \, d\bar{x}_1 \]

\[ + \int_{P_2} W_e(x_0, x_1) L_e(x_2, x_1) G(x_0, x_1) f(x_1, x_2, x_0) G(x_1, x_2) \, d\bar{x}_2 + \cdots \]

\[ + \int_{P_k} W_e(x_0, x_1) L_e(x_k, x_{k-1}) G(x_0, x_1) \prod_{j=1}^{k-1} f(x_j, x_{j+1}, x_{j-1}) G(x_j, x_{j+1}) \, d\bar{x}_k + \cdots \]

introduce: \[ T(\bar{x}_k) = G(x_0, x_1) \prod_{j=1}^{k-1} f(x_j, x_{j+1}, x_{j-1}) G(x_j, x_{j+1}) \]

throughput of path \( \bar{x}_k \)
Path Integral Form of Measurement Eq.

\[ I_j = \int_{A} \int_{A} W_e(x_0, x_1) G(x_0, x_1) L_o(x_1, x_0) \, dx_1 \, dx_0 \]

\[ = \int_{P_1} W_e(x_0, x_1) L_e(x_1, x_0) T(\tilde{x}_1) \, d\tilde{x}_1 \]

\[ + \int_{P_2} W_e(x_0, x_1) L_e(x_2, x_1) T(\tilde{x}_2) \, d\tilde{x}_2 \]

\[ + \cdots \]

\[ + \int_{P_k} W_e(x_0, x_1) L_e(x_k, x_{k-1}) T(\tilde{x}_k) \, d\tilde{x}_k \]

\[ + \cdots \]

introduce: \( P = \bigcup_{k=1}^{\infty} P_k \)

the path space, i.e. the space of all paths of all lengths
Path Integral Form of Measurement Eq.

\[ I_j = \int_A \int_A W_e(x_0, x_1) G(x_0, x_1) L_o(x_1, x_0) \, dx_1 \, dx_0 \]

\[ = \int_P W_e(x_0, x_1) L_e(x_k, x_{k-1}) T(\tilde{x}) \, d\tilde{x} \]
Path Integral Form of Measurement Eq.

\[ I_j = \int_{\mathcal{P}} W_e(x_0, x_1)L_e(x_k, x_{k-1})T(\bar{x}) \, d\bar{x} \]

path space \( \mathcal{P} \)

path \( \bar{x} \)

path throughput

\[ T(\bar{x}) = G(x_0, x_1) \prod_{j=1}^{k-1} f(x_j, x_{j+1}, x_{j-1})G(x_j, x_{j+1}) \]
Path Integral Form of Measurement Eq.

\[ I_j = \int_{\mathcal{P}} W_e(x_0, x_1) L_e(x_k, x_{k-1}) T(\bar{x}) \, d\bar{x} \]

Advantages:

- no recursion, no “nasty” nested integrals
- emphasizes symmetry of light transport
- easy to relate different rendering algorithms
- focuses on path geometry, independent of strategy for constructing paths
- MC estimator on path space looks much simpler
Path Integral Form of Measurement Eq.

\[ I_j = \int_{\mathcal{P}} W_e(x_0, x_1)L_e(x_k, x_{k-1})T(\bar{x}) \, d\bar{x} \]

Monte Carlo estimator:

\[ I_j \approx \frac{1}{N} \sum_{i=1}^{N} \frac{W_e(x_{i,0}, x_{i,1})L_e(x_{i,k}, x_{i,k-1})T(\bar{x}_i)}{p(\bar{x}_i)} \]

\[ p(\bar{x}) = p(x_0, x_1, \cdots, x_{k-1}, x_k) \]  
\[ \text{path PDF} \quad \text{joint PDF of path vertices} \]
Path Construction

\[ p(\bar{x}) = p(x_0, x_1, \ldots, x_{k-1}, x_k) \]

Path tracing w/o NEE

\[ p(\bar{x}) = p(x_0) \times p(x_1|x_0) \times p(x_2|x_0 x_1) \times p(x_3|x_0 x_1 x_2) \]
Path Construction

\[ p(\bar{x}) = p(x_0, x_1, \cdots, x_{k-1}, x_k) \]

Path tracing with NEE

\[ p(\bar{x}) = p(x_0) \times p(x_1 | x_0) \times p(x_2 | x_0 x_1) \times p(x_3) \]

assuming uniform area sampling
Path Construction

\[ p(\mathbf{x}) = p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k) \]

Light tracing

\[ p(\mathbf{\bar{x}}) = p(\mathbf{x}_0 | \mathbf{x}_3 \mathbf{x}_2 \mathbf{x}_1) \]
\[ \times p(\mathbf{x}_1 | \mathbf{x}_3 \mathbf{x}_2) \]
\[ \times p(\mathbf{x}_2 | \mathbf{x}_3) \]
\[ \times p(\mathbf{x}_3) \]
Path Construction

\[ p(\bar{x}) = p(x_0, x_1, \cdots, x_{k-1}, x_k) \]

Light tracing with NEE

\[ p(\bar{x}) = p(x_0) \]
\[ \times p(x_1 | x_3 x_2) \]
\[ \times p(x_2 | x_3) \]
\[ \times p(x_3) \]

assuming uniform aperture sampling
Path Construction

\[ p(\bar{x}) = p(x_0, x_1, \cdots, x_{k-1}, x_k) \]

Independent sampling of path vertices
(not very practical though)

\[ p(\bar{x}) = p(x_0) \times p(x_1) \times p(x_2) \times p(x_3) \]
Can we combine them?
Bidirectional Path Tracing
Bidirectional Path Tracing

- # vertices on camera subpath
- # vertices on light subpath
- # connections

[Lafortune and Willems 1993]
[Veach and Guibas 1994]
Bidirectional Path Tracing

color estimate (point x)
{
  lp = sample light subpath
  cp = sample camera subpath for image point x
  for each vertex s in lp
    for each vertex t in cp
      fullPath = join(cp[0..s], lp[0..t])
      splat(fullPath.screenPos, fullPath.contrib)
}
Bidirectional Path Tracing

Key observations:

- Every path (formed by connecting camera sub-path to light sub-path) with $k$ vertices can be constructed using $k+1$ strategies.
- For a particular path length, all strategies estimate the same integral.
- Each strategy has a different PDF, i.e., each strategy has different strengths and weaknesses.
- Let’s combine them using MIS!
Bidirectional Path Tracing
Bidirectional Path Tracing
Bidirectional Path Tracing (MIS)
Bidirectional Path Tracing

(Unidirectional) path tracing

Bidirectional path tracing

Images courtesy of W. Jakob
Bidirectional Path Tracing

Path tracing

Light tracing

Bidirectional PT

Images courtesy of F. Suykens
Still not robust enough...

Reference  
Bidirectional PT

Images courtesy of J. Křivánek
Still not robust enough...

paths are difficult for any unbiased method
Still not robust enough...

Extensions

- Combination with photon mapping
  - Unified Path Sampling [Hachisuka et al. 2012]
  - Vertex Connection Merging [Georgiev et al. 2012]
- Metropolis sampling (global PDF)
- Path-space regularization [Kaplanyan et al. 2013]
- Path guiding (learn global PDF)