# Bidirectional path tracing

15-468, 15-668, 15-868
Physics-based Rendering
Spring 2025, Lecture 13

http://graphics.cs.cmu.edu/courses/15-468

# Course announcements

- Materials from yesterday’s recitation posted on Slack.
- Yannis is traveling for SIGGRAPH next week.
  - Lectures by Sreekar Ranganathan (Tuesday) and Bailey Miller (Thursday).

# Overview of today’s lecture

- Types of light paths.
- Light tracing.
- Bidirectional path tracing.

# Slide credits

Most of these slides were directly adapted from:

- Wojciech Jarosz (Dartmouth).

# Light Paths

# Light Paths

Express light paths in terms of the surface interactions that have occurred

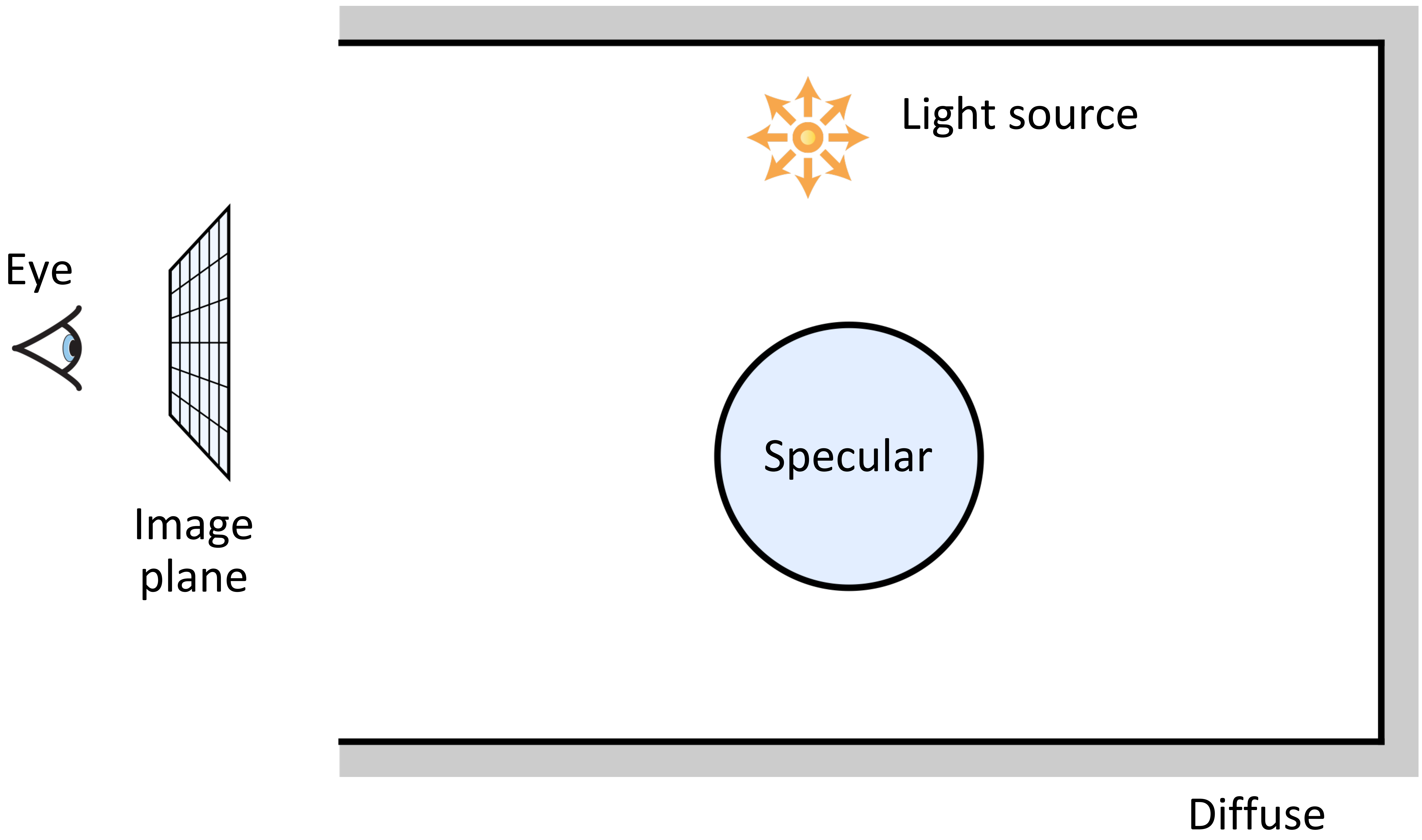
A light path is a chain of linear segments joined at event "vertices"

# Heckbert's Classification

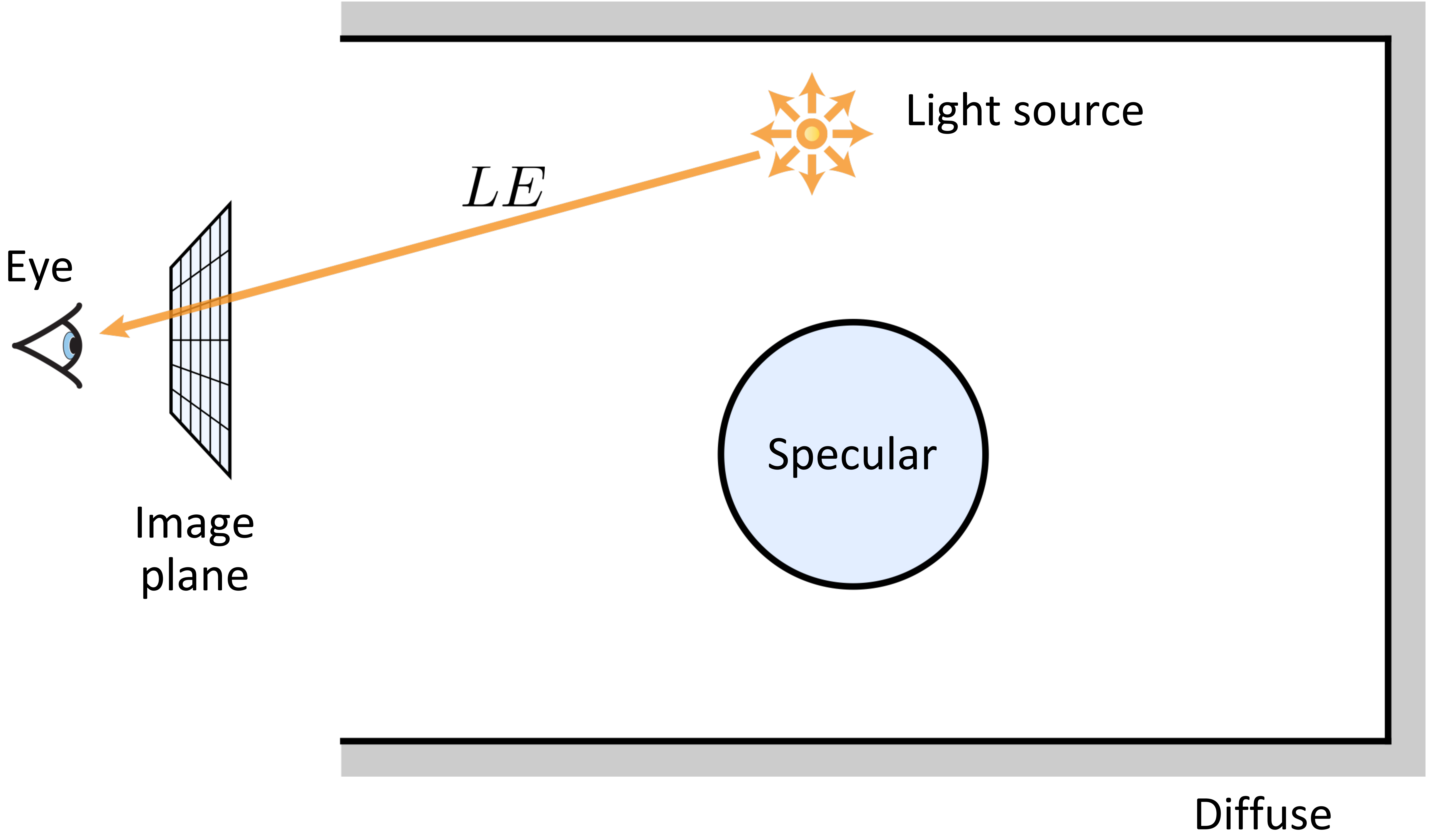
Classification of "vertices":

- $L$ : a light source
- $E$ : the eye
- $S$ : a specular reflection
- $D$ : a diffuse reflection

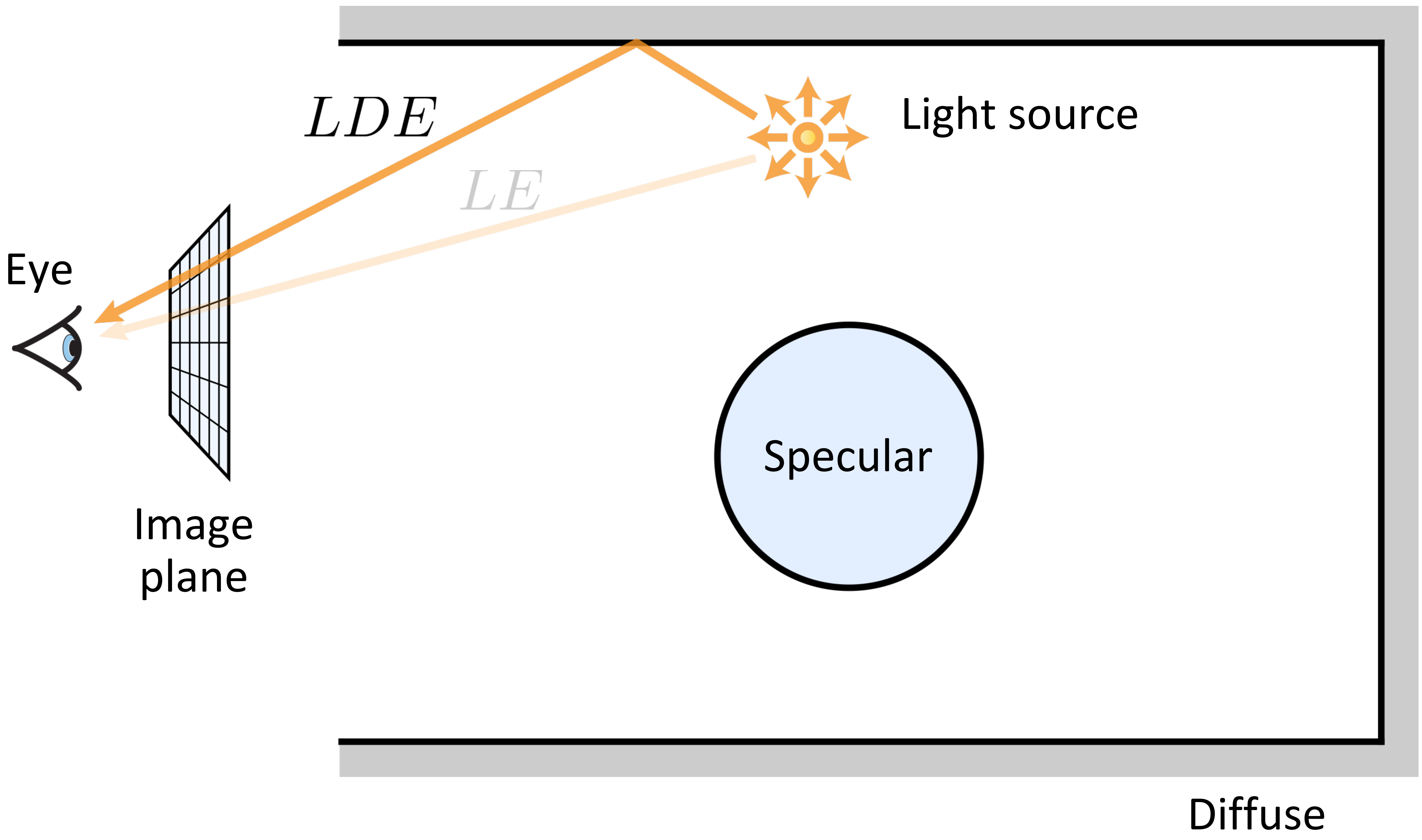
# Heckbert's Classification



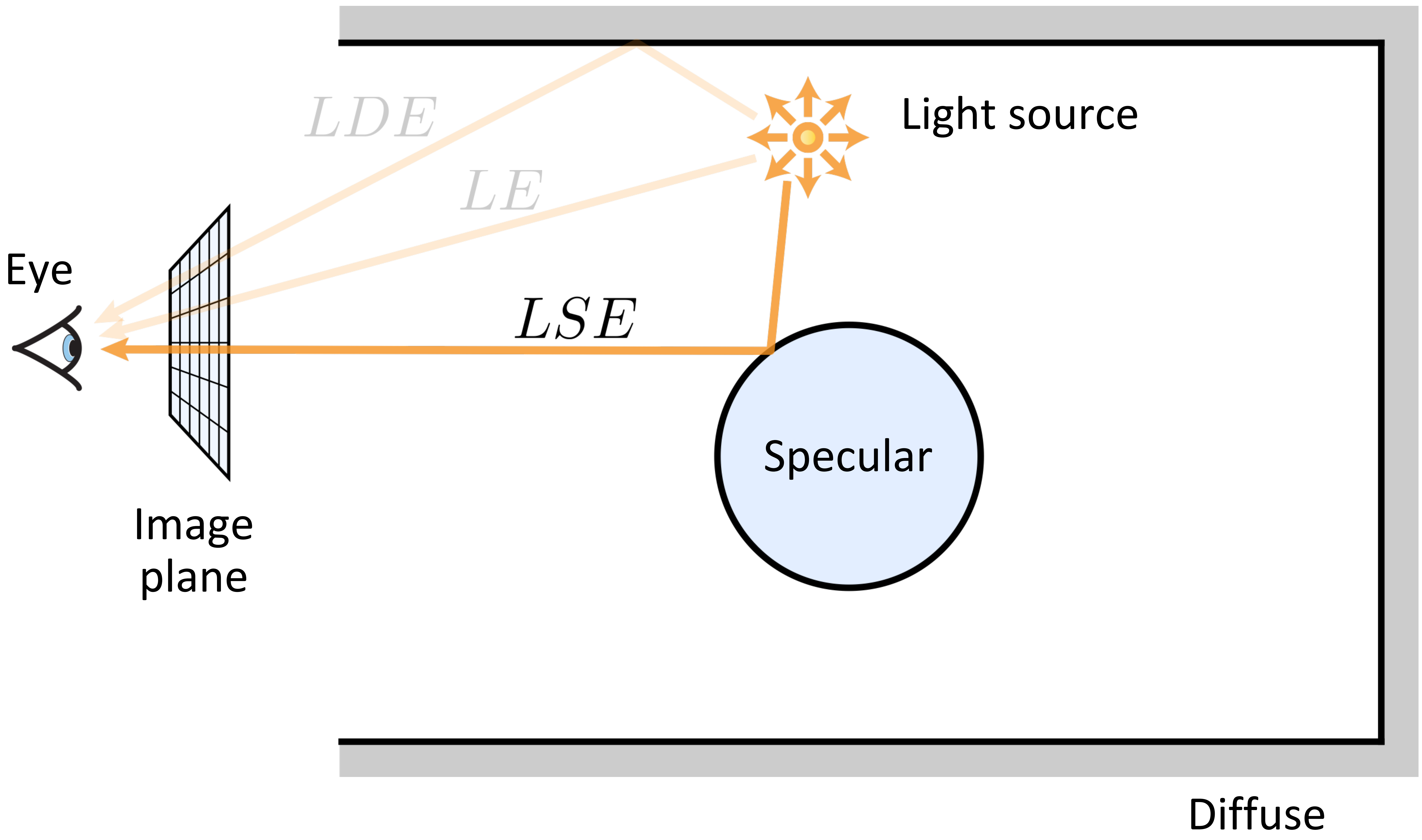
# Heckbert's Classification



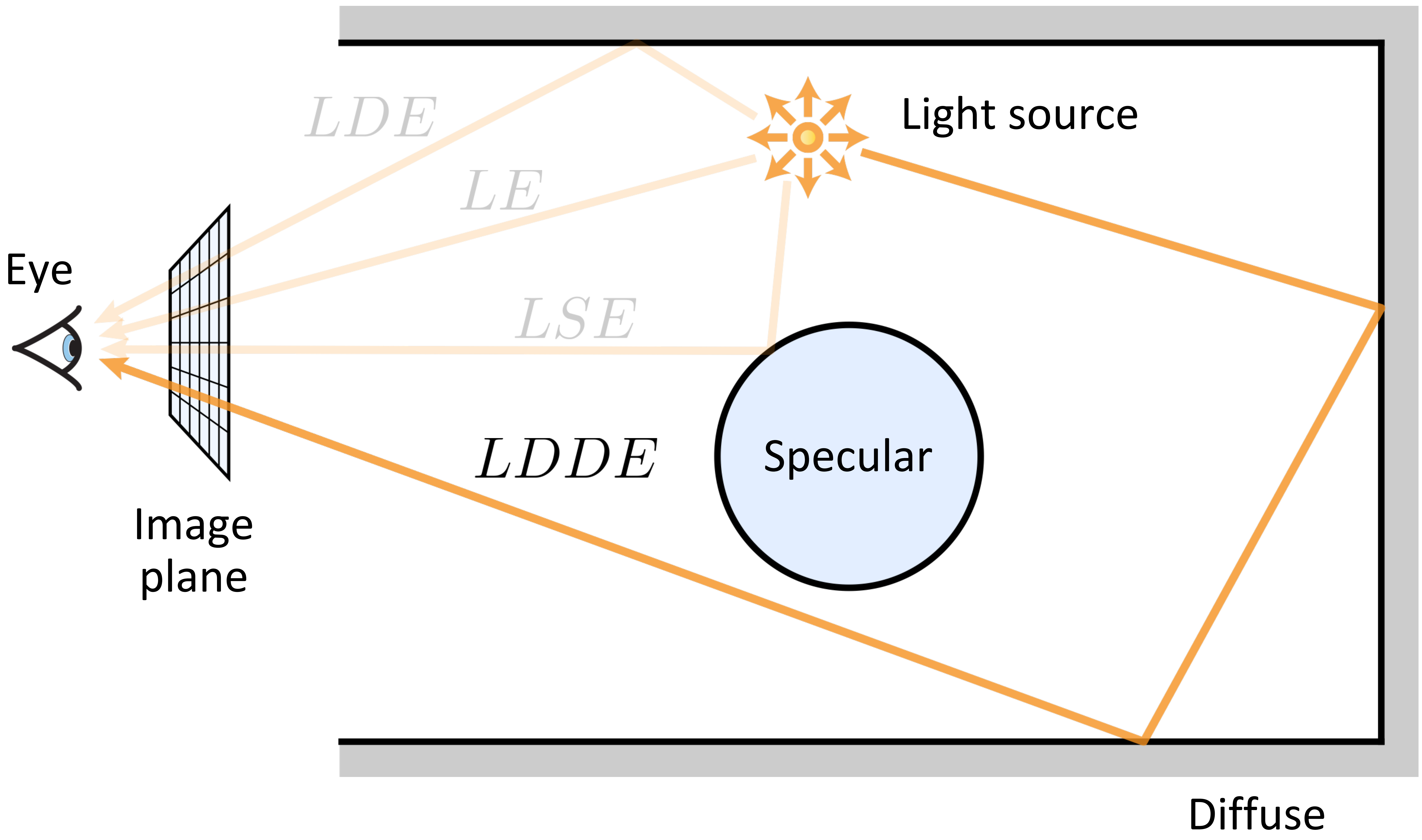
# Heckbert's Classification



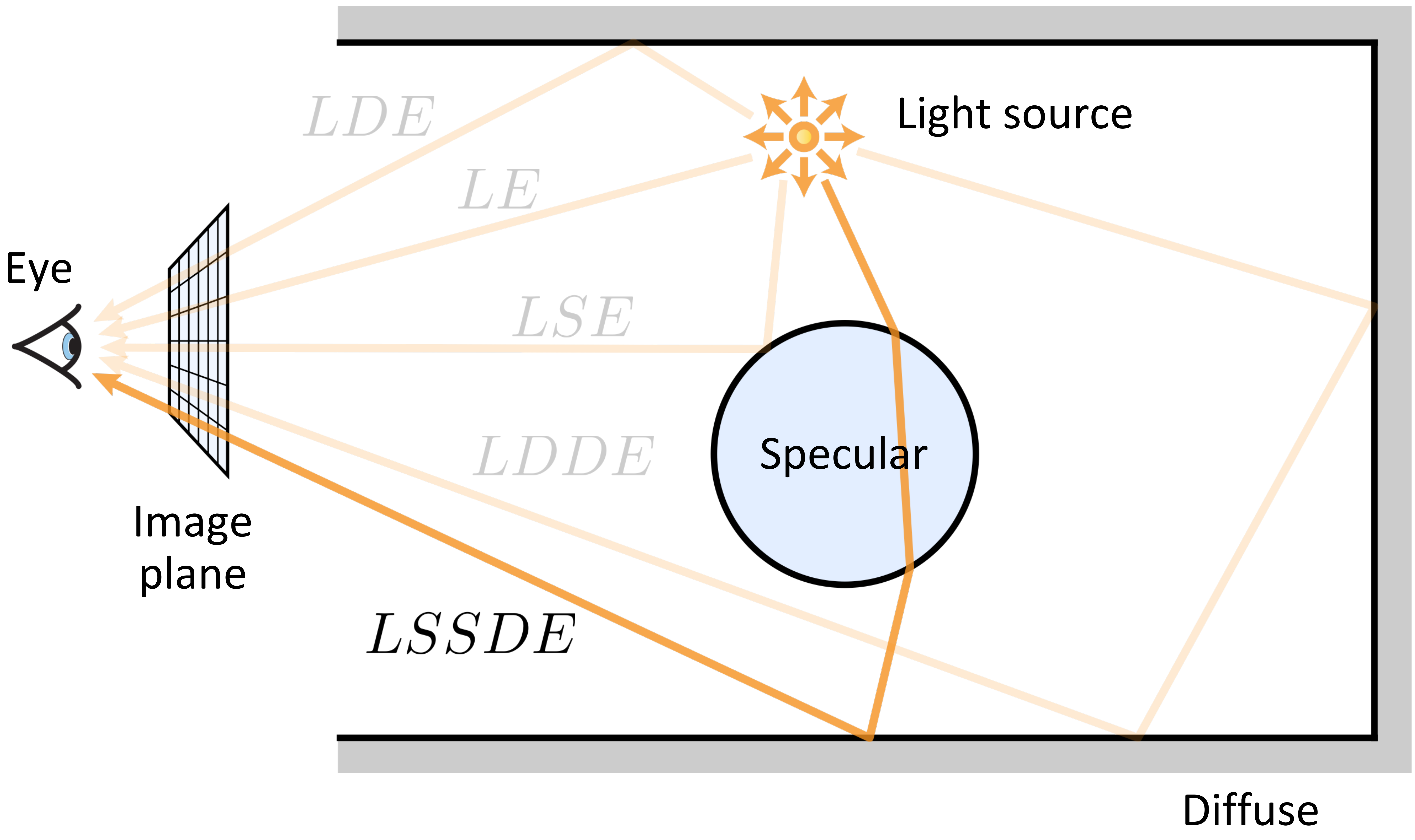
# Heckbert's Classification



# Heckbert's Classification



# Heckbert's Classification



# Heckbert's Classification

Can express arbitrary classes of paths using a regular expression type syntax:

- $k^+$ : one or more of event $k$
- $k^*$ : zero or more of event $k$
- $k?$ : zero or one $k$ events
- $(k|h)$ : a $k$ or $h$ event

# Heckbert's Classification

Direct illumination: $L(D|S)E$

Indirect illumination: $L(D|S)(D|S)^{+}E$

# Heckbert's Classification

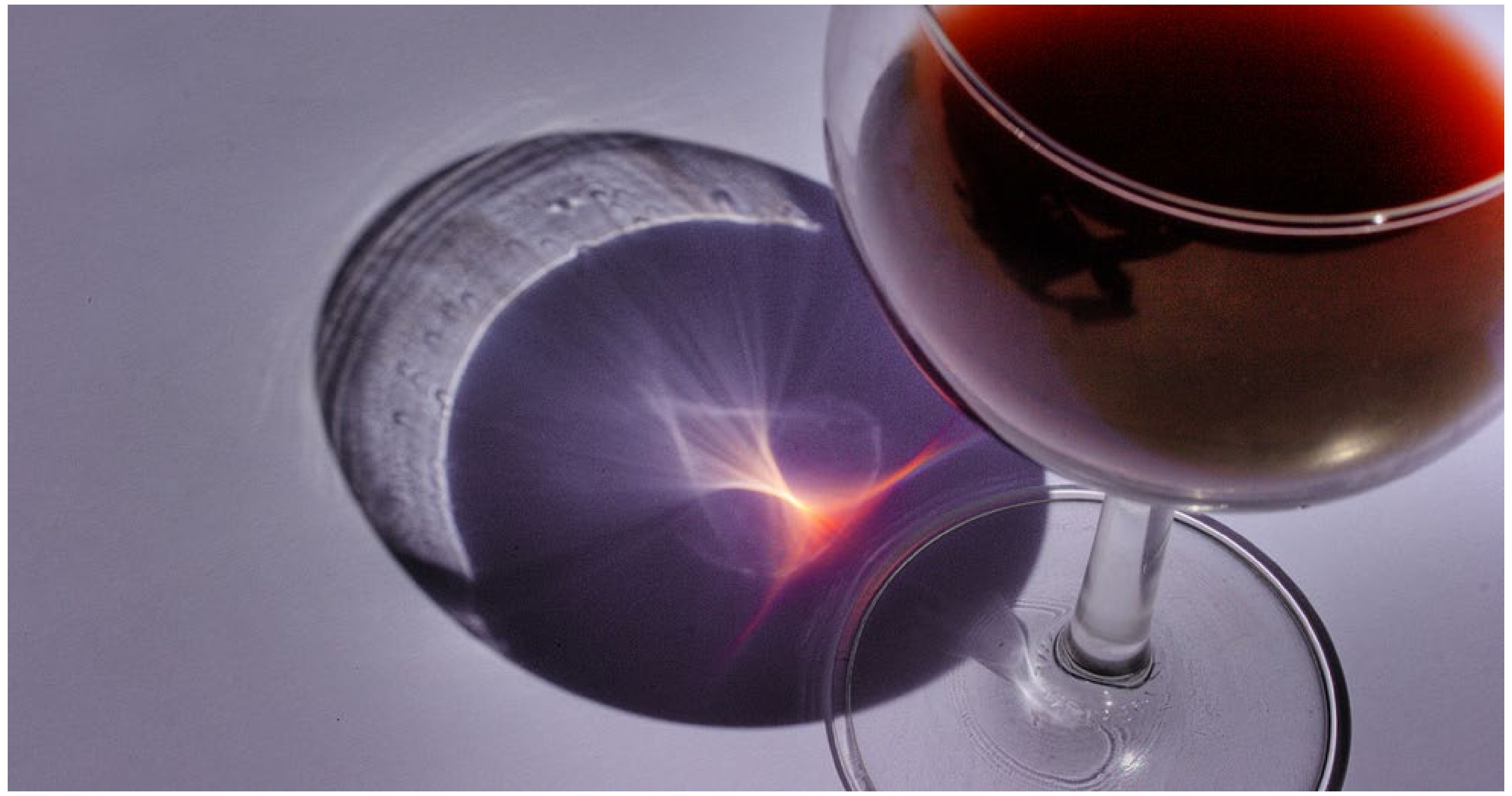
Direct illumination: $L(D|S)E$

Indirect illumination: $L(D|S)(D|S)^{+}E$

Full global illumination: $L(D|S)^{*}E$

# Diffuse inter-reflections: $LDD^{+}E$

# Caustics: $LS^+DE$

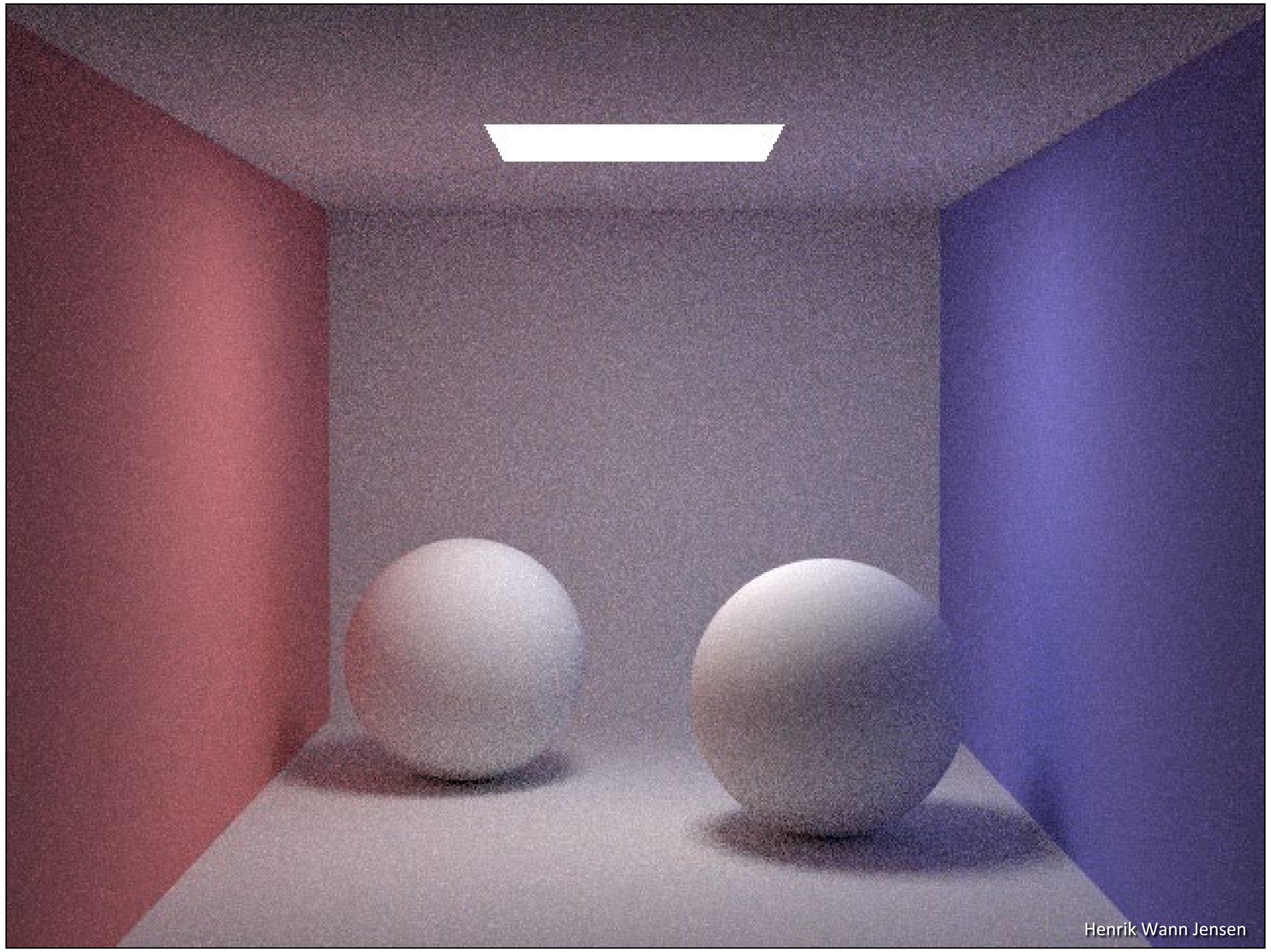



source: Flickr

# Subsurface Scattering

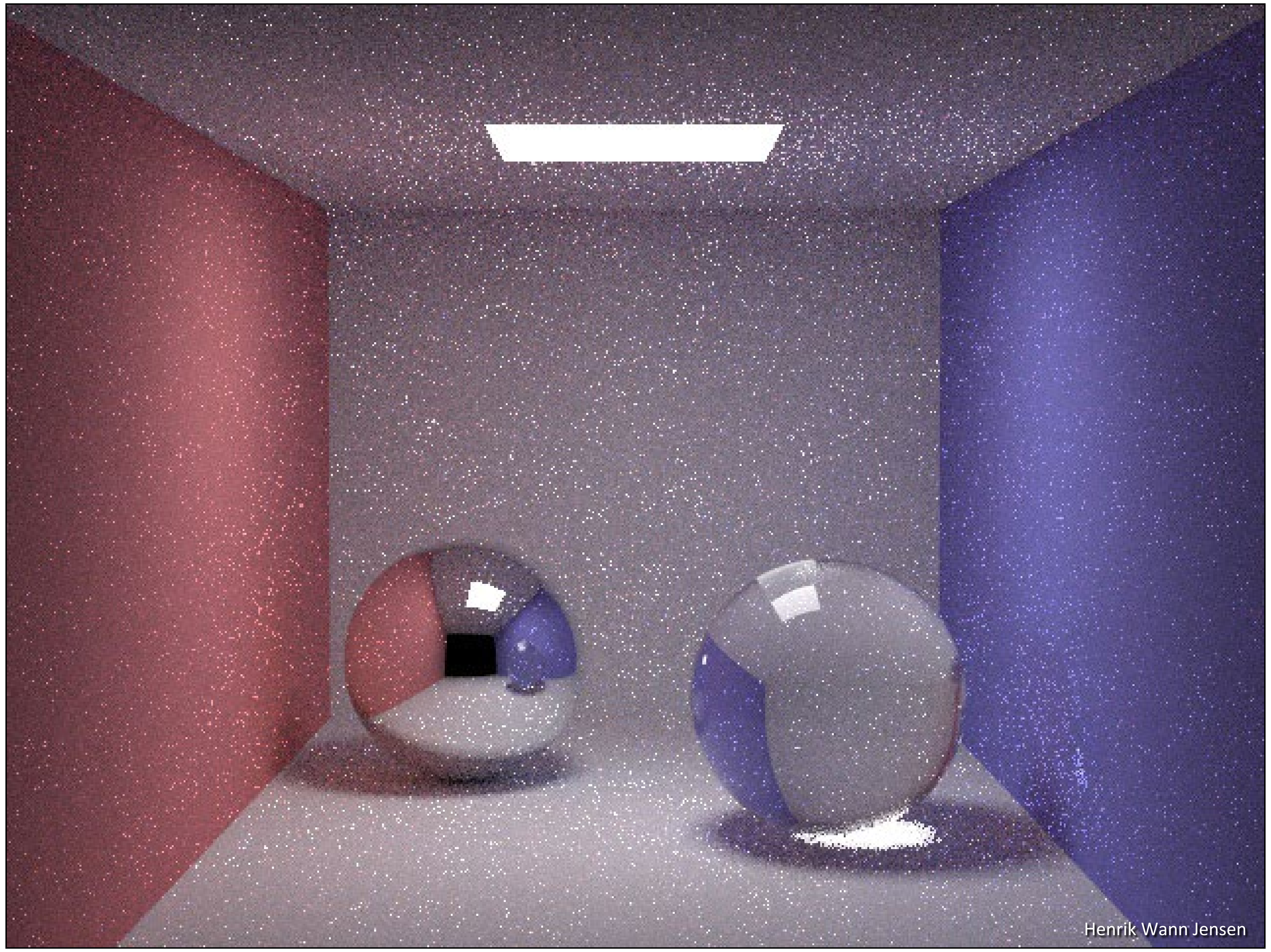


# A Simple Scene



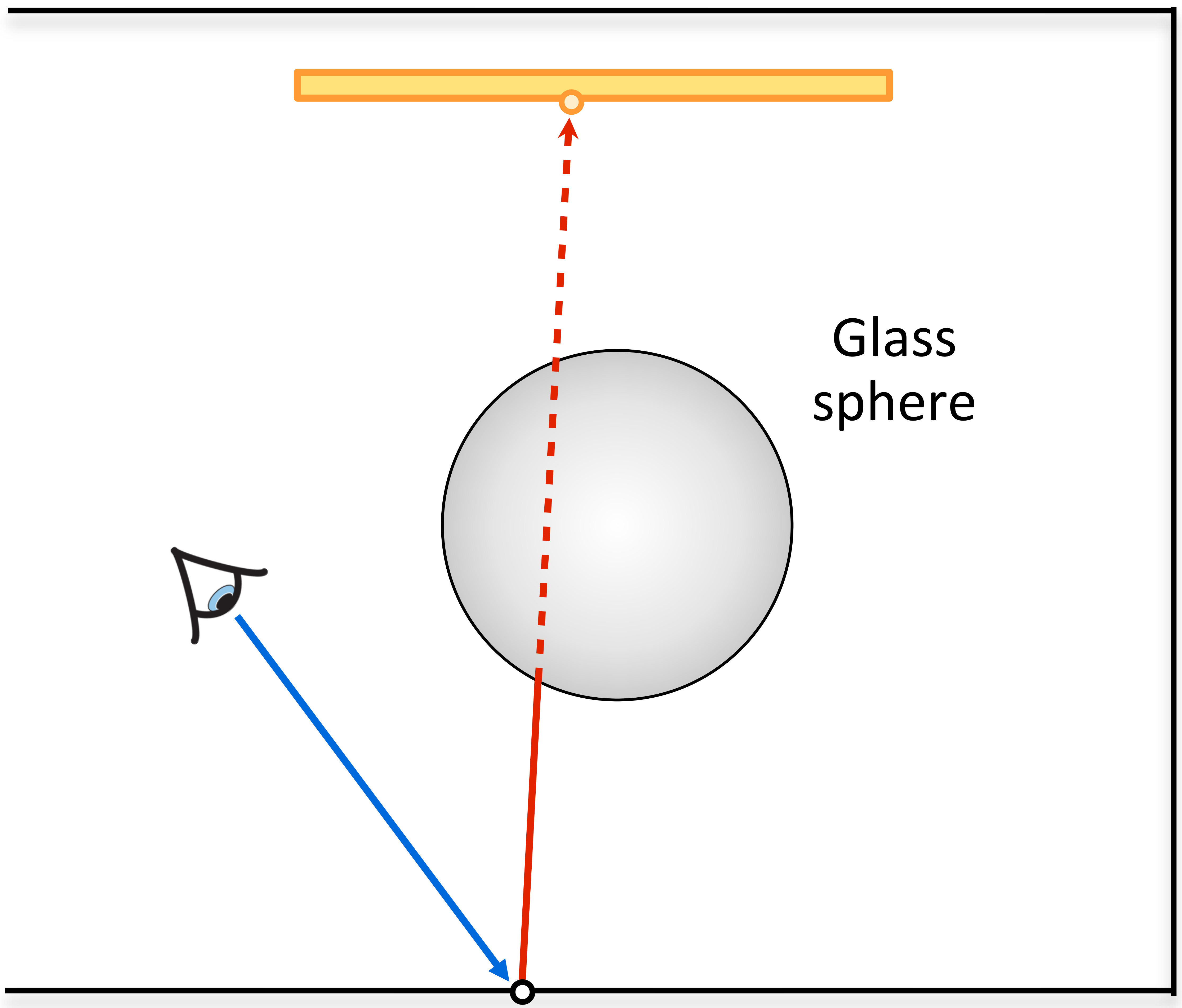

10 paths/pixel

# + Glass/Mirror Material

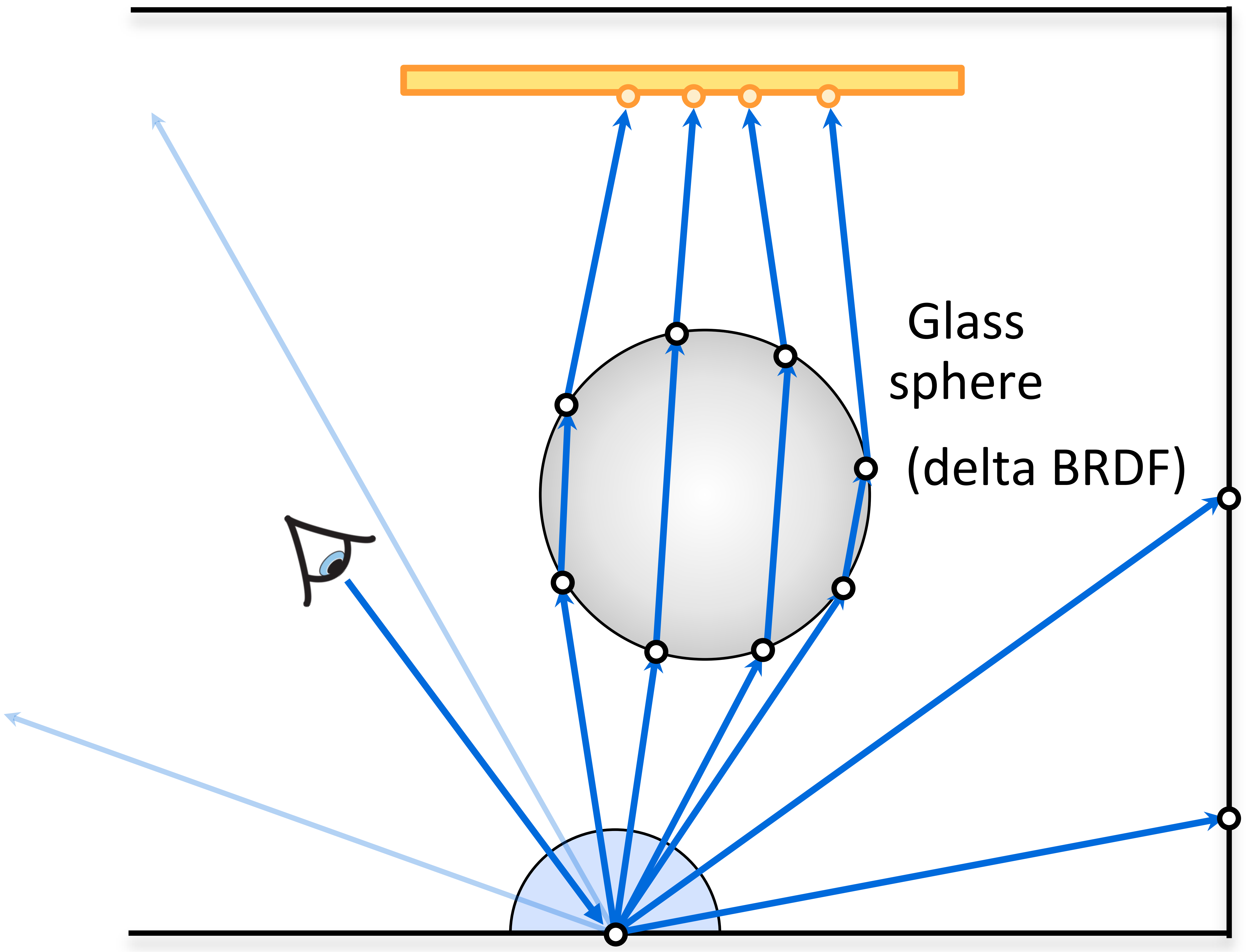



10 paths/pixel

# Path Tracing Caustics

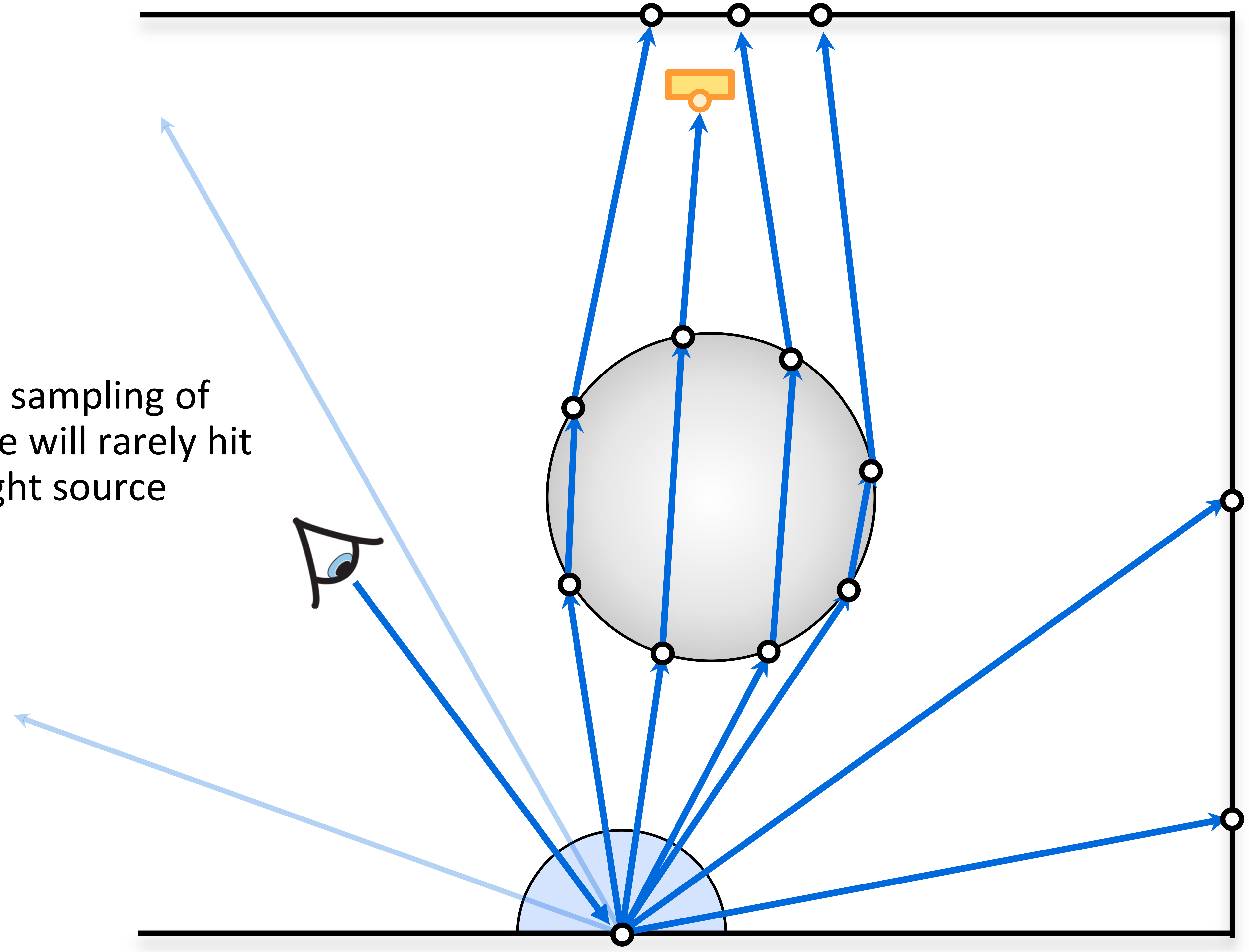


# Path Tracing Caustics

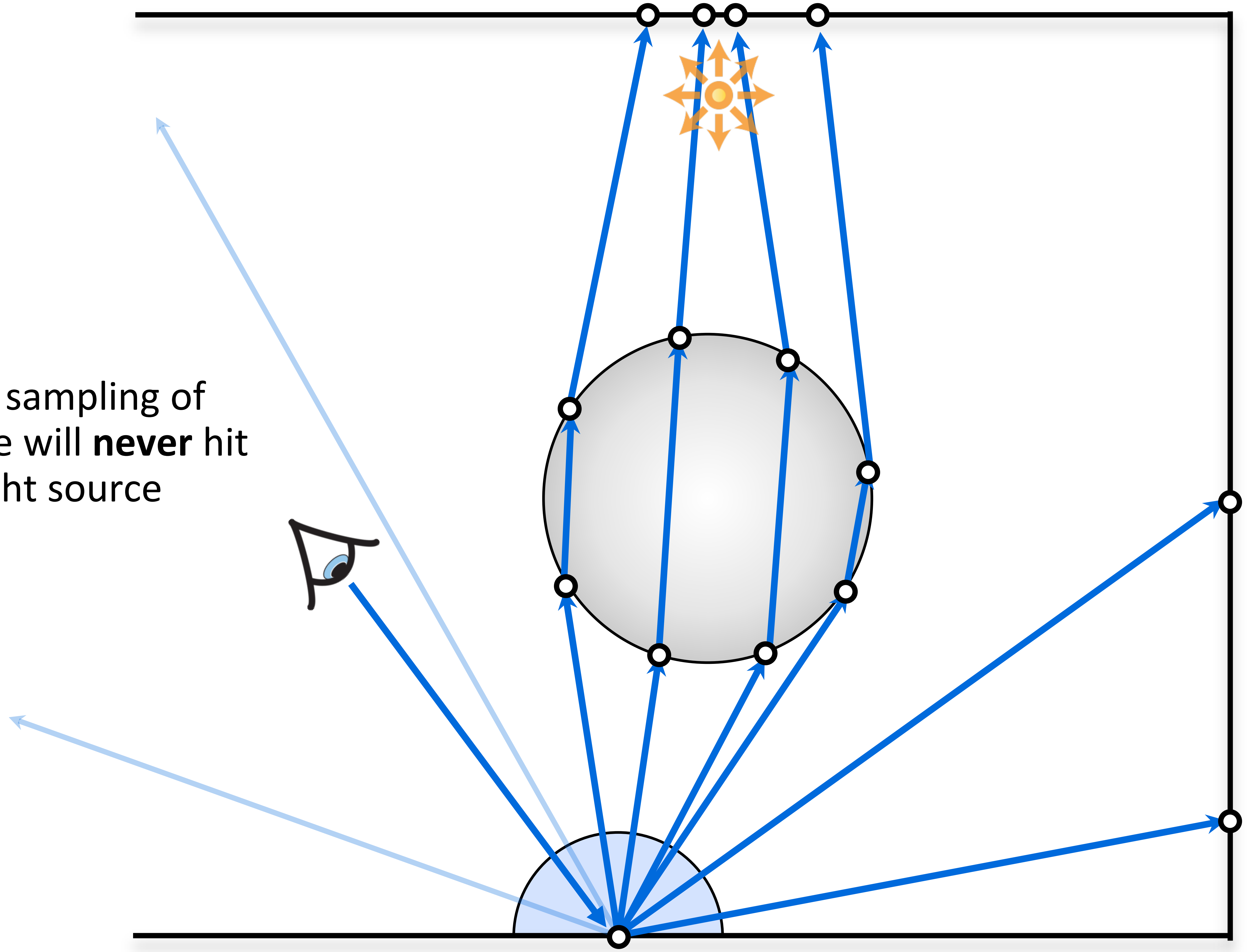


# Path Tracing Caustics

Random sampling of
hemisphere will rarely hit
the light source



# Path Tracing Caustics



Random sampling of hemisphere will **never** hit the light source

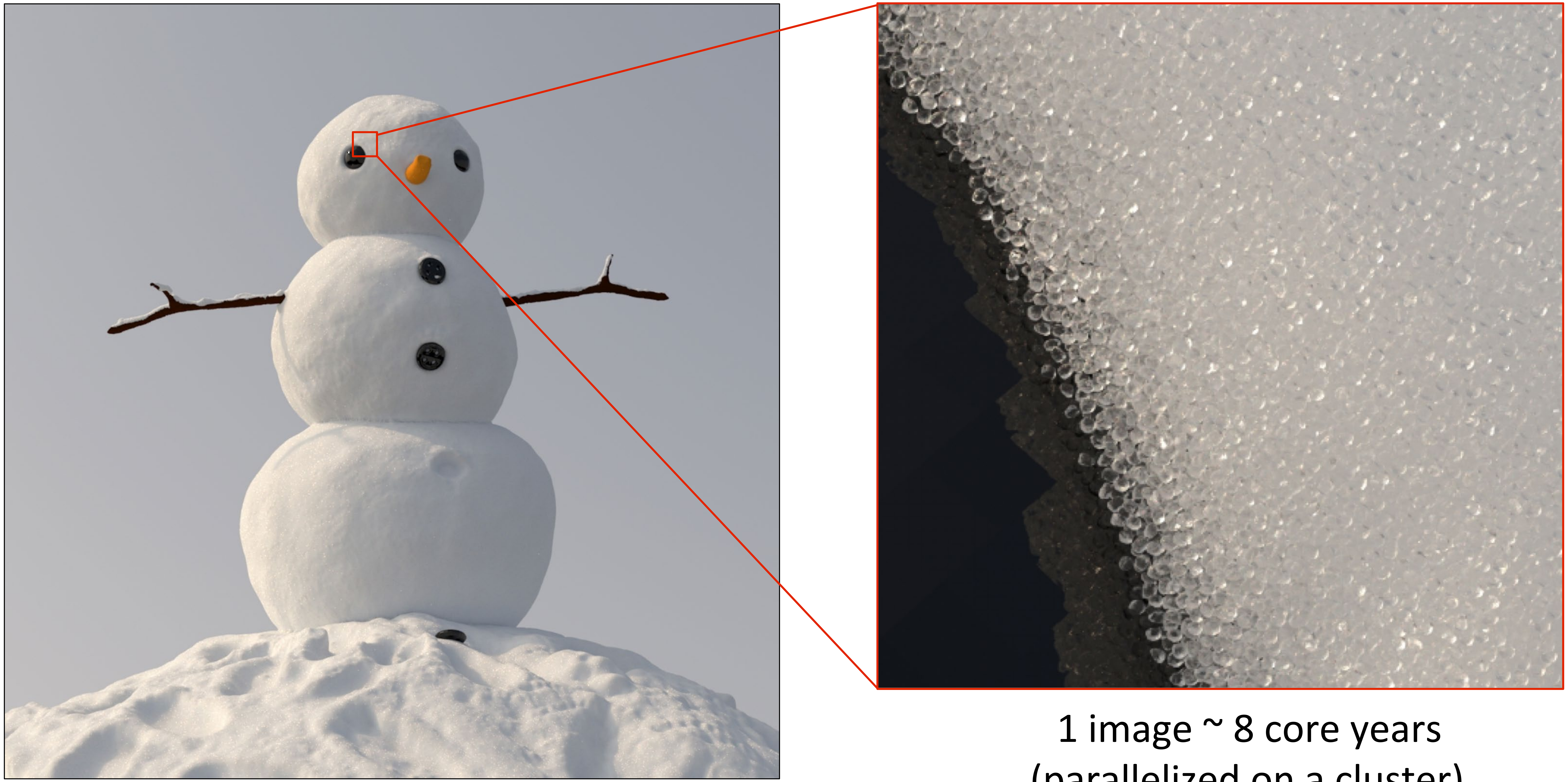
# Let’s just give it more time...

Nature ~ $2 \times 10^{33}$ / second

Fastest GPU ray tracer ~ $2 \times 10^{8}$ / second


Tim Webber, Gravity VFX supervisor

# Let’s just give it more time...



1 image ~ 8 core years
(parallelized on a cluster)

# Path Tracing - Summary

✓ Full solution to the rendering equation

✓ Simple to implement

✗ Slow convergence

- requires 4x more samples to half the error

✗ Robustness issues

- does not handle some light paths well (or not at all), e.g. caustics ($LS^+DE$)

✗ No reuse or caching of computation

✗ General sampling issue

- makes only locally good decisions

# Today's agenda

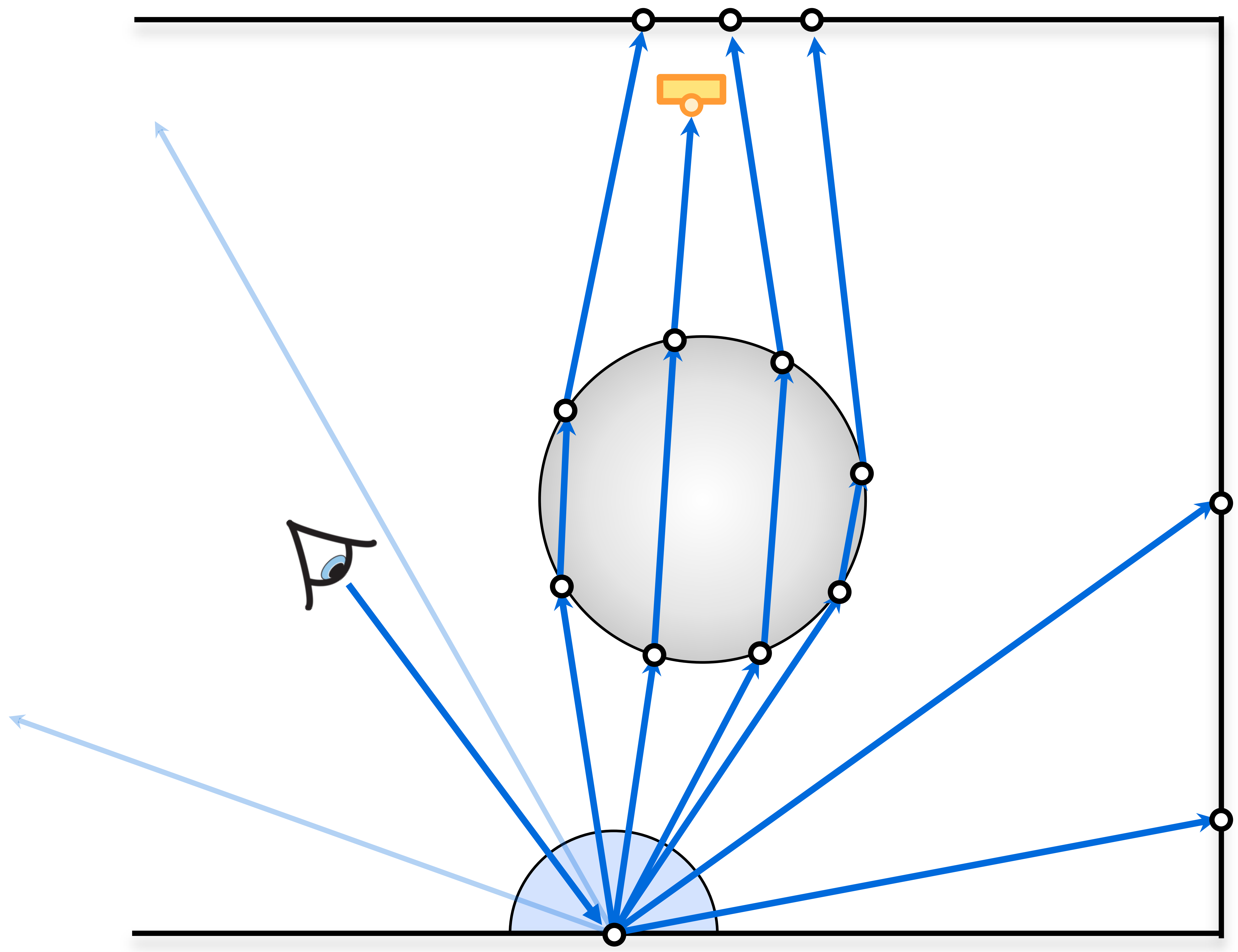
Measurement Equation

Path Integral Framework

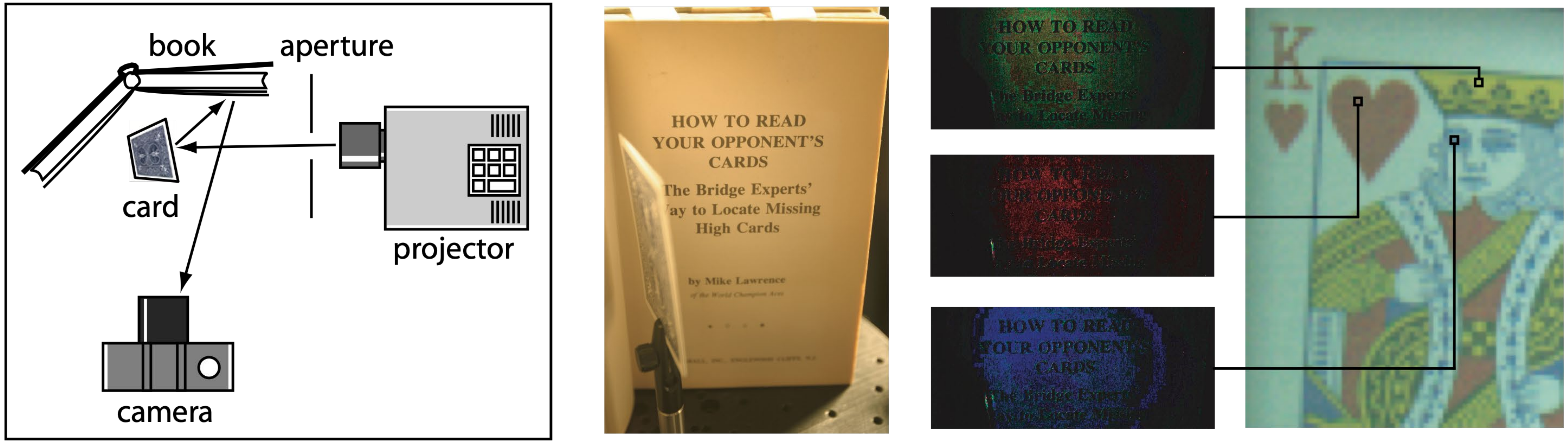
Solving the Rendering Equation

- Light tracing
- Bidirectional path tracing

# Can we simulate this better?



# Light transport is symmetric



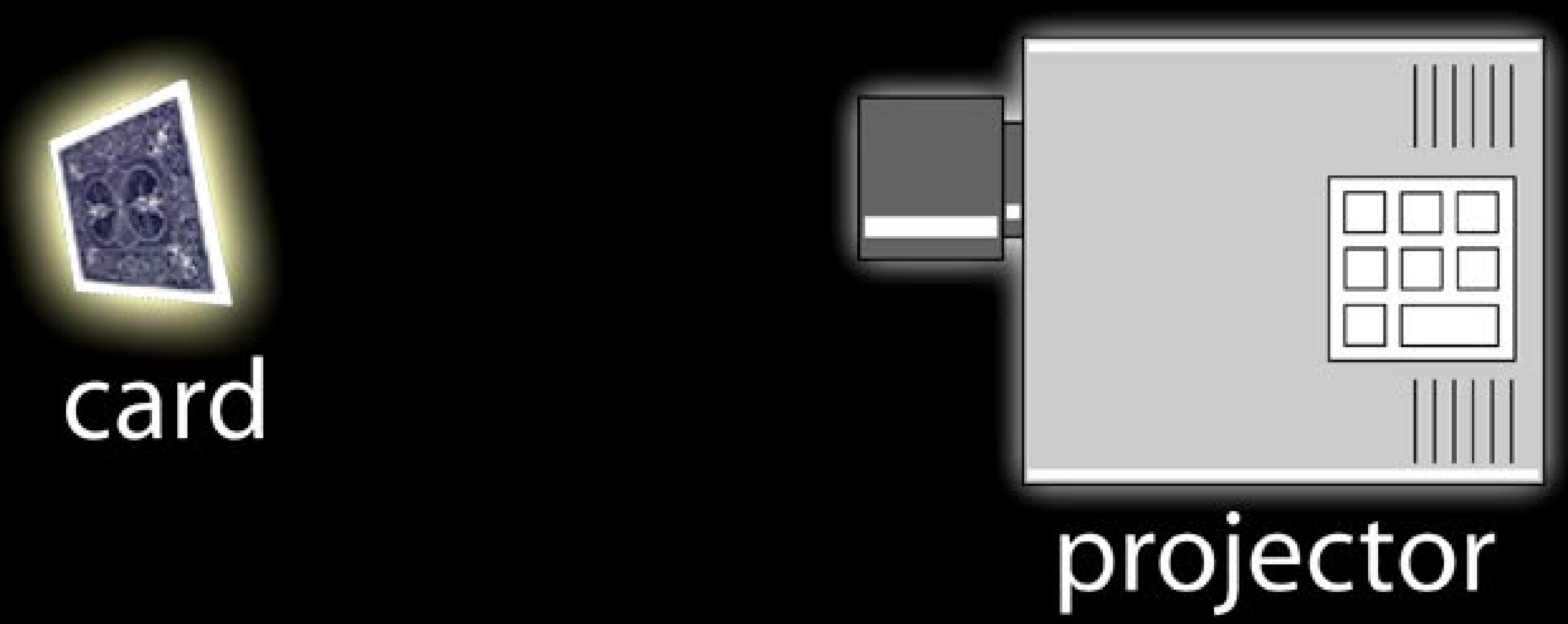

Dual Photography [Sen et al. 2005]

# Dual Photography

Pradeep Sen* Billy Chen* Gaurav Garg* Stephen R. Marschner†
Mark Horowitz* Marc Levoy* Hendrik P.A. Lensch*

*Stanford University †Cornell University

SIGGRAPH2005


card
projector

# Duality of Radiance and Importance

# Measurement Equation

Rendering equation describes radiative equilibrium at point $\mathbf{x}$:

$$L_o(\mathbf{x}, \vec{\omega}_o) = L_e(\mathbf{x}, \vec{\omega}_o) + \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_o) L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

We are interested in the total radiance contributing to pixel $j$:

$$I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x}$$

*response* of the sensor at film location $\mathbf{x}$
to radiance arriving from direction $\vec{\omega}$
(often referred to as *emitted importance*)

# Radiometry as Measurements

Weighted integral of 5D radiance function

$$\int_V \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L(\mathbf{x}, \vec{\omega}) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\mathbf{x}$$

Other radiometric quantities are measurements

- expressing *irradiance* in terms of radiance:

$$\int_{H^2} L(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} = E(\mathbf{x})$$

Integrate radiance over hemisphere

- expressing *flux/power* in terms of radiance:

$$\int_A \int_{H^2} L(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} dA(\mathbf{x}) = \Phi(A)$$

Integrate radiance over hemisphere and area

# Radiance vs. Importance

Radiance

- emitted from light sources
- describes *amount of light* traveling within a differential beam

Importance

- “emitted” from sensors
- describes the *response of the sensor* to radiance traveling within a differential beam

# Duality of Radiance & Importance

$$I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x}$$

# Duality of Radiance & Importance

$$I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x}$$

$$= \int_{A_{\text{film}}} \int_{A} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) L_o(\mathbf{y}, \mathbf{x}) \, d\mathbf{y} d\mathbf{x}$$

outgoing quantities

Let's expand $L_o$ and consider direct illumination only

# Duality of Radiance & Importance

$$
\begin{aligned}
I_j &= \int_{A_{\text{film}}}\int_{H^2} W_e(\mathbf{x},\vec{\omega})L_i(\mathbf{x},\vec{\omega})\cos\theta\, d\vec{\omega}d\mathbf{x} \\
&= \int_{A_{\text{film}}}\int_{A} W_e(\mathbf{x},\mathbf{y})G(\mathbf{x},\mathbf{y})L_o(\mathbf{y},\mathbf{x})\, d\mathbf{y}d\mathbf{x} \\
&= \int_{A_{\text{film}}}\int_{A}\int_{A_{\text{light}}} W_e(\mathbf{x},\mathbf{y})G(\mathbf{x},\mathbf{y})f(\mathbf{y},\mathbf{z},\mathbf{x})G(\mathbf{y},\mathbf{z})L_e(\mathbf{z},\mathbf{y})\, d\mathbf{z}d\mathbf{y}d\mathbf{x}
\end{aligned}
$$

emitted quantities with identical measure

Let's swap the inner and outer integral

# Duality of Radiance & Importance

$$
\begin{aligned}
I_j &= \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) L_o(\mathbf{y}, \mathbf{x}) \, d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} \int_{A_{\text{light}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}, \mathbf{z}, \mathbf{x}) G(\mathbf{y}, \mathbf{z}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{z} d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{light}}} \int_{A} \int_{A_{\text{film}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}, \mathbf{z}, \mathbf{x}) G(\mathbf{y}, \mathbf{z}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{x} d\mathbf{y} d\mathbf{z}
\end{aligned}
$$

symmetric functions

# Duality of Radiance & Importance

$$
\begin{aligned}
I_j &= \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) L_o(\mathbf{y}, \mathbf{x}) \, d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} \int_{A_{\text{light}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}, \mathbf{z}, \mathbf{x}) G(\mathbf{y}, \mathbf{z}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{z} d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{light}}} \int_{A} \int_{A_{\text{film}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{y}, \mathbf{x}) f(\mathbf{y}, \mathbf{x}, \mathbf{z}) G(\mathbf{z}, \mathbf{y}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{x} d\mathbf{y} d\mathbf{z}
\end{aligned}
$$

symmetric functions

# Duality of Radiance & Importance

$$
\begin{aligned}
I_j &= \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) L_o(\mathbf{y}, \mathbf{x}) \, d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} \int_{A_{\text{light}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}, \mathbf{z}, \mathbf{x}) G(\mathbf{y}, \mathbf{z}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{z} d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{light}}} \int_{A} \int_{A_{\text{film}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{y}, \mathbf{x}) f(\mathbf{y}, \mathbf{x}, \mathbf{z}) G(\mathbf{z}, \mathbf{y}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{x} d\mathbf{y} d\mathbf{z} \\
&= \int_{A_{\text{light}}} \int_{A} W_o(\mathbf{y}, \mathbf{z}) G(\mathbf{z}, \mathbf{y}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{y} d\mathbf{z}
\end{aligned}
$$

# Duality of Radiance & Importance

$$\begin{aligned}
I_j &= \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) L_o(\mathbf{y}, \mathbf{x}) \, d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} \int_{A_{\text{light}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}, \mathbf{z}, \mathbf{x}) G(\mathbf{y}, \mathbf{z}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{z} d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{light}}} \int_{A} \int_{A_{\text{film}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{y}, \mathbf{x}) f(\mathbf{y}, \mathbf{x}, \mathbf{z}) G(\mathbf{z}, \mathbf{y}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{x} d\mathbf{y} d\mathbf{z} \\
&= \int_{A_{\text{light}}} \int_{A} W_o(\mathbf{y}, \mathbf{z}) G(\mathbf{z}, \mathbf{y}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{y} d\mathbf{z} \\
&= \int_{A_{\text{light}}} \int_{H^2} W_i(\mathbf{z}, \vec{\omega}) L_e(\mathbf{z}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{z}
\end{aligned}$$

# Duality of Radiance & Importance

emitted *importance*

incident *radiance*

$$
\begin{aligned}
I_j &= \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x} \\
&= \int_{A_{\text{light}}} \int_{H^2} W_i(\mathbf{z}, \vec{\omega}) L_e(\mathbf{z}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{z}
\end{aligned}
$$

emitted *radiance*

incident *importance*

# Duality of Radiance & Importance

Path tracing

start from *film*, search for *radiance* at light

$$
\begin{aligned}
I_j &= \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x} \\
&= \int_{A_{\text{light}}} \int_{H^2} W_i(\mathbf{z}, \vec{\omega}) L_e(\mathbf{z}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{z}
\end{aligned}
$$

Light tracing

start from *light*, search for *importance* at sensor

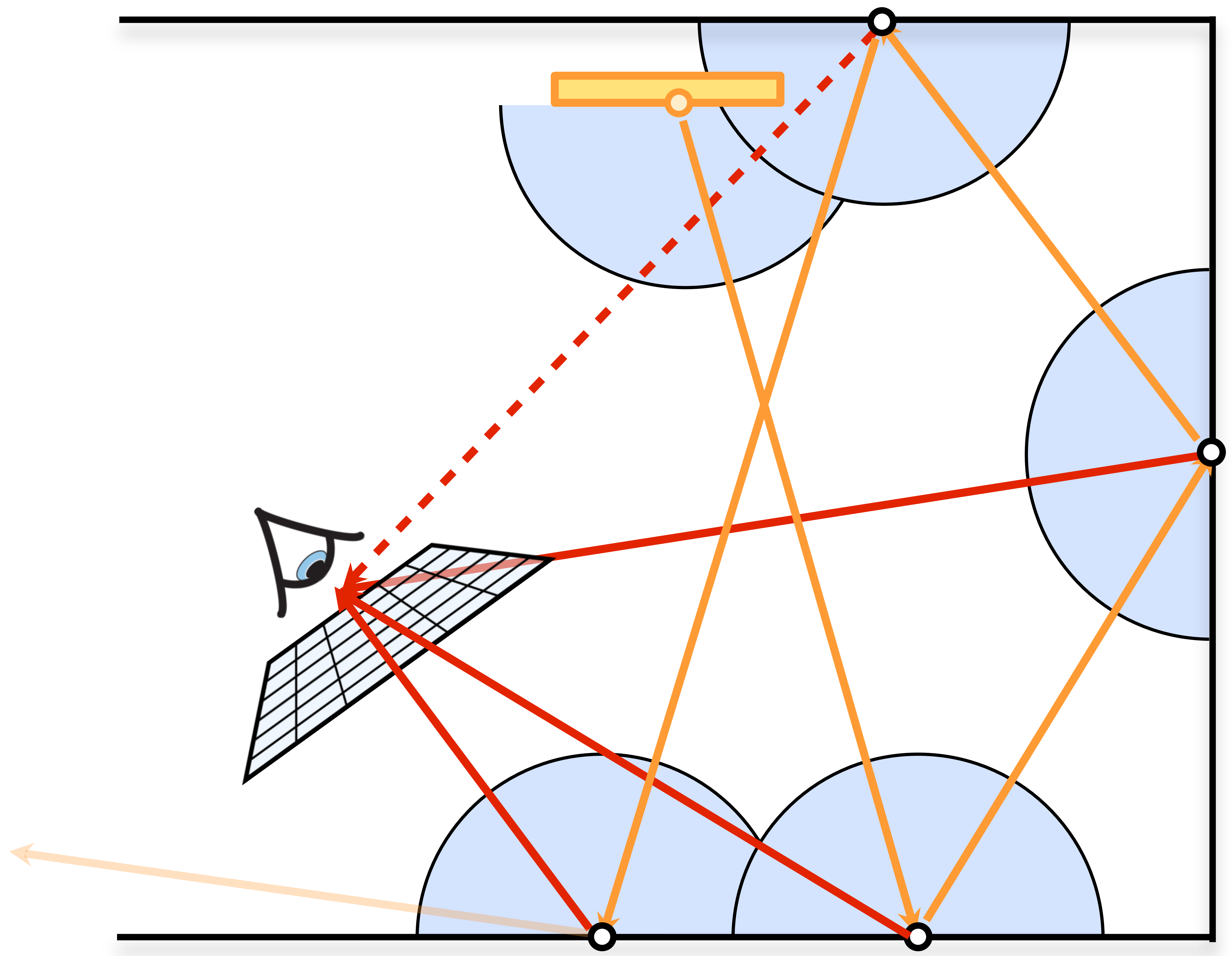
# Light Tracing

# Light Tracing

Shoot multiple paths from light sources hoping to randomly hit the sensor

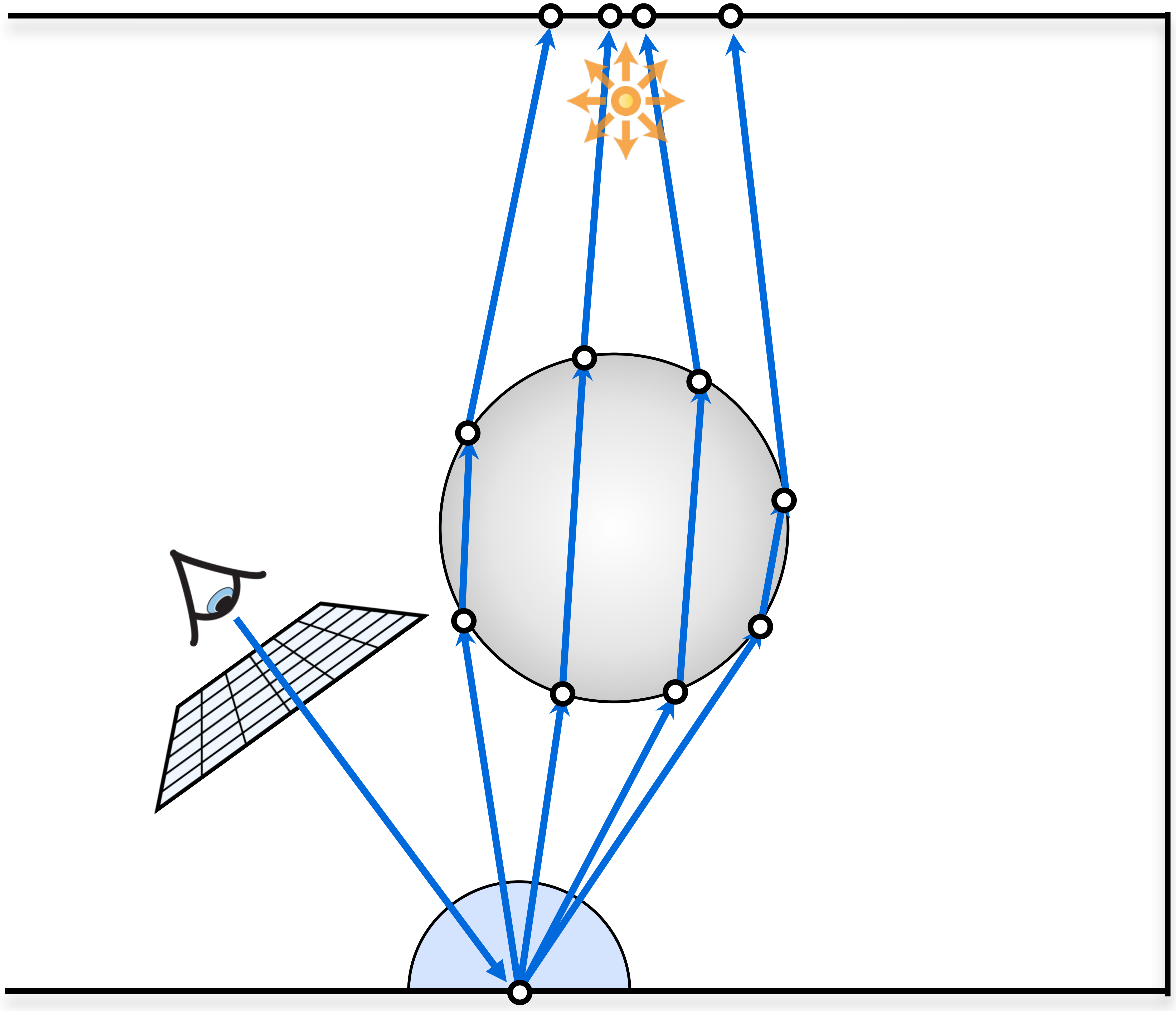
- Optionally: at each path vertex, connect to the image using next-event estimation (a.k.a. shadow rays in PT)

# Light Tracing with NEE

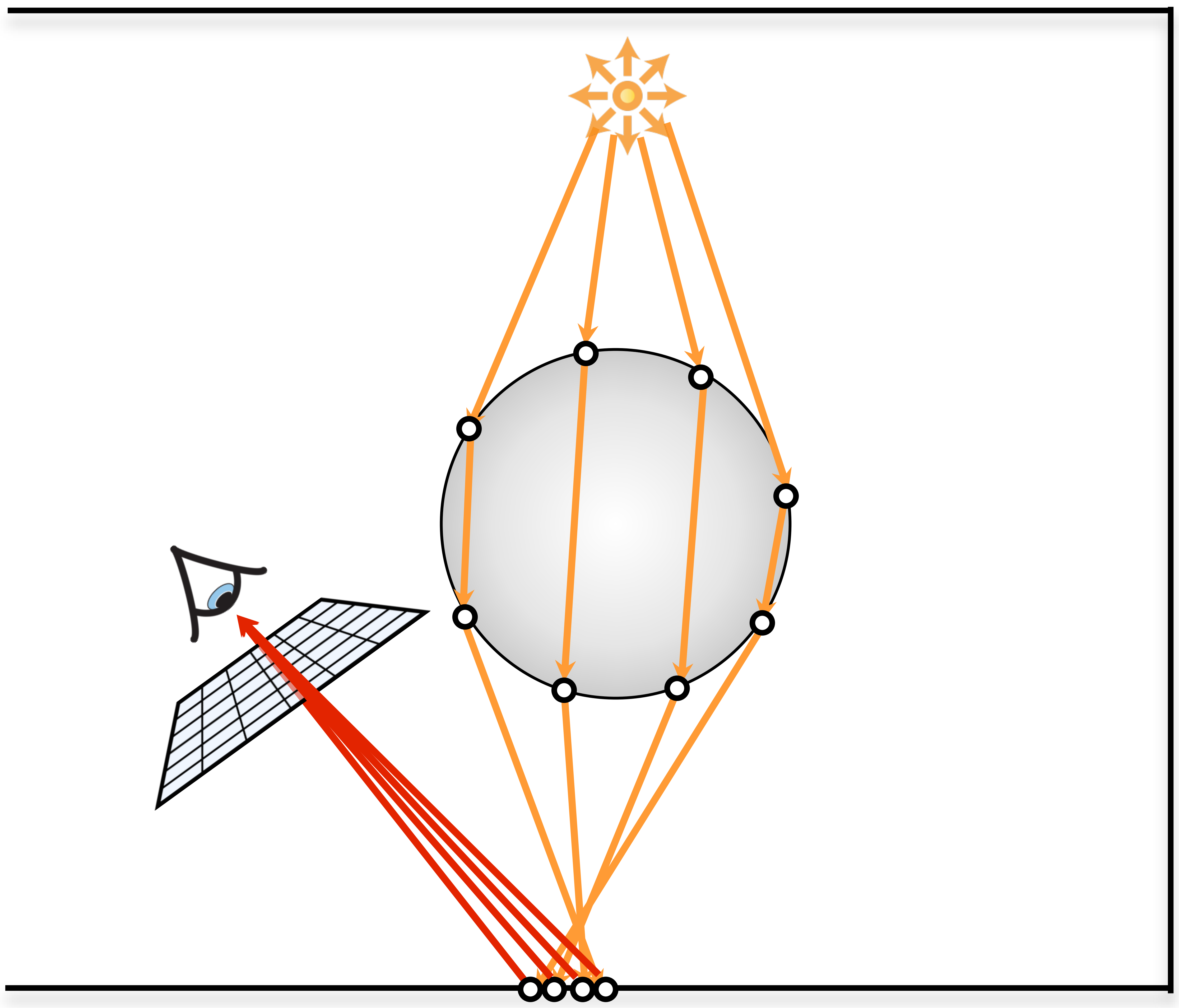


Splat to the image at each vertex

# Path Tracing Caustics

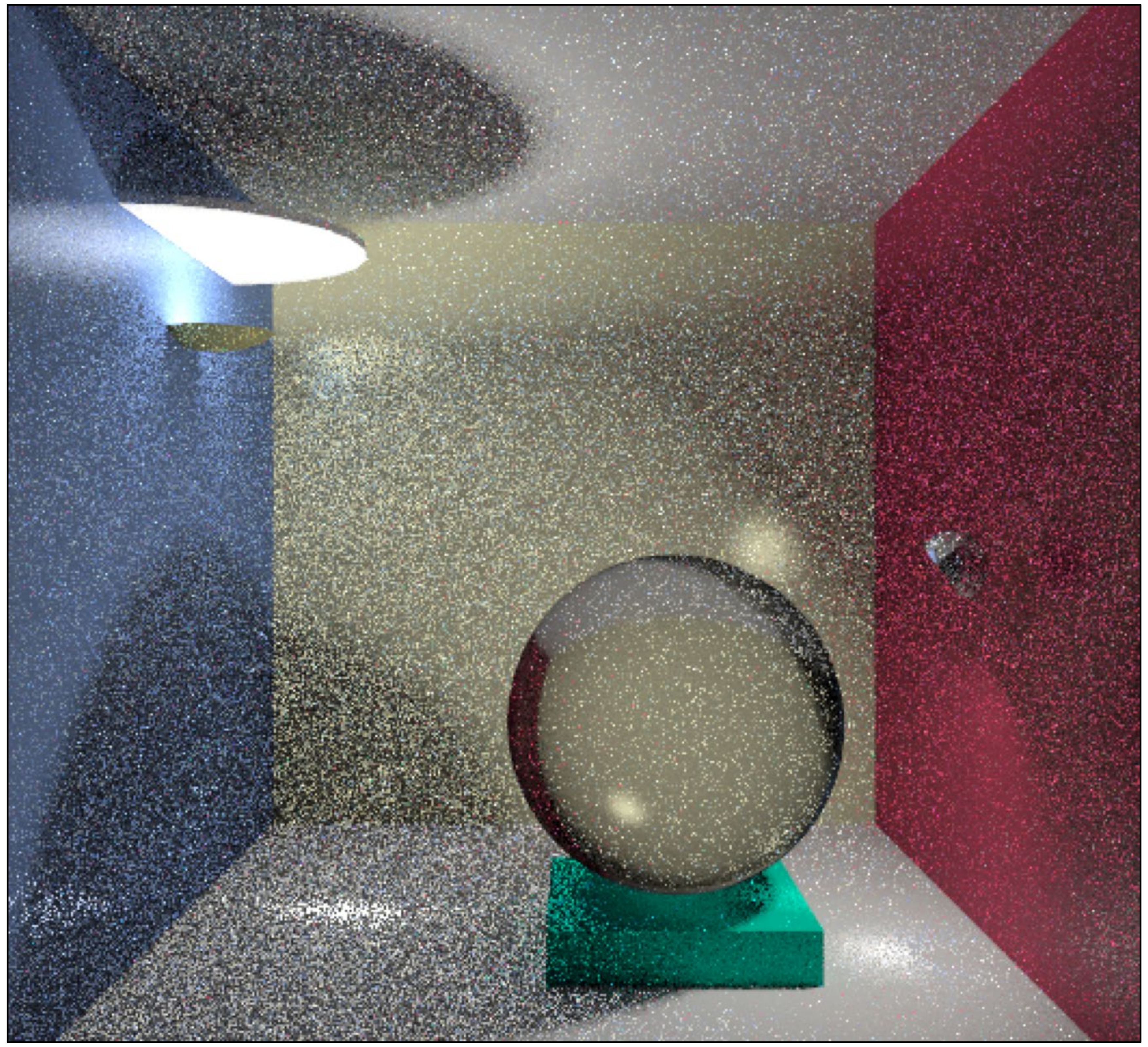


# Light Tracing Caustics

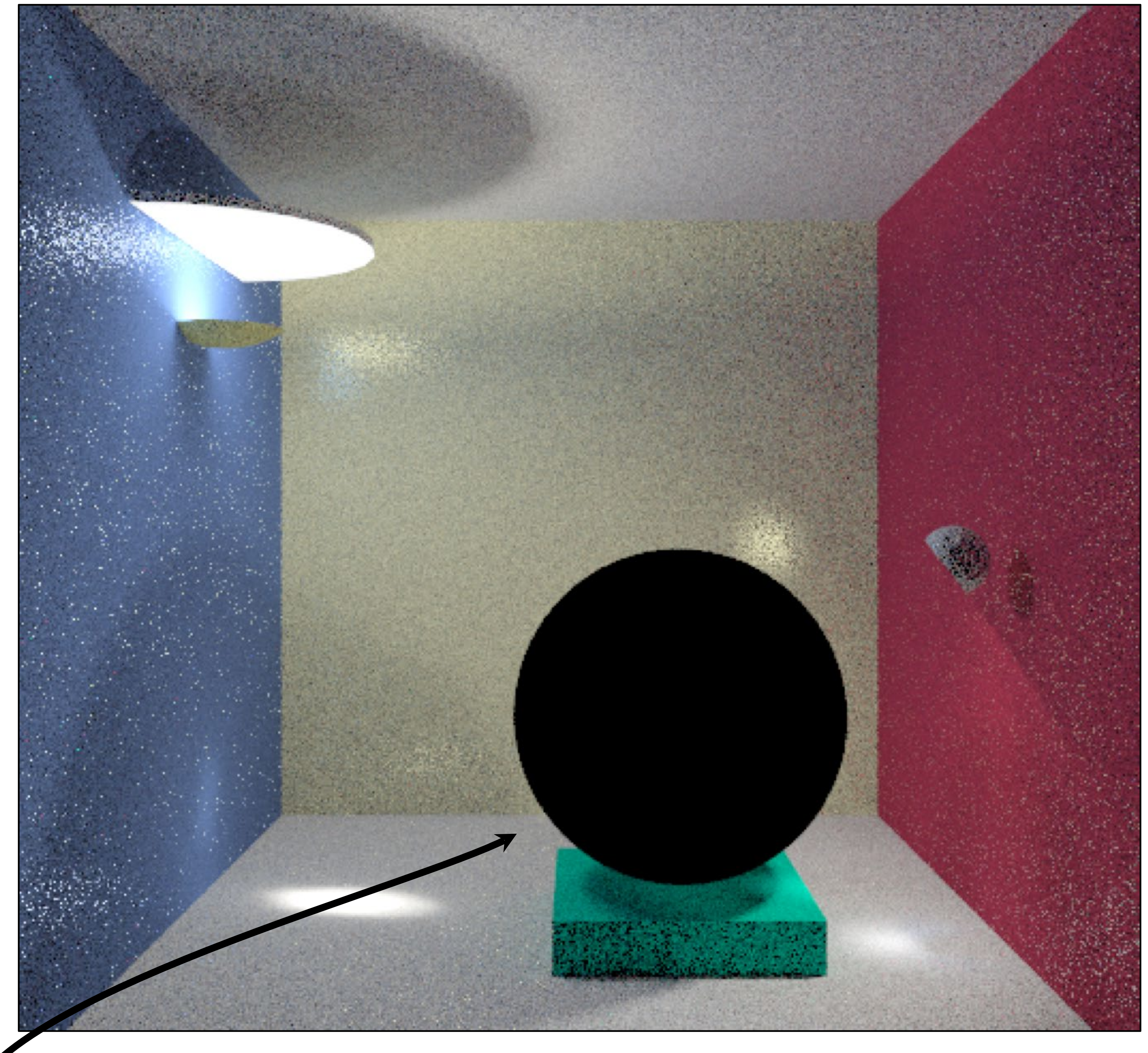


# Path vs. Light Tracing

Path tracing



Light tracing

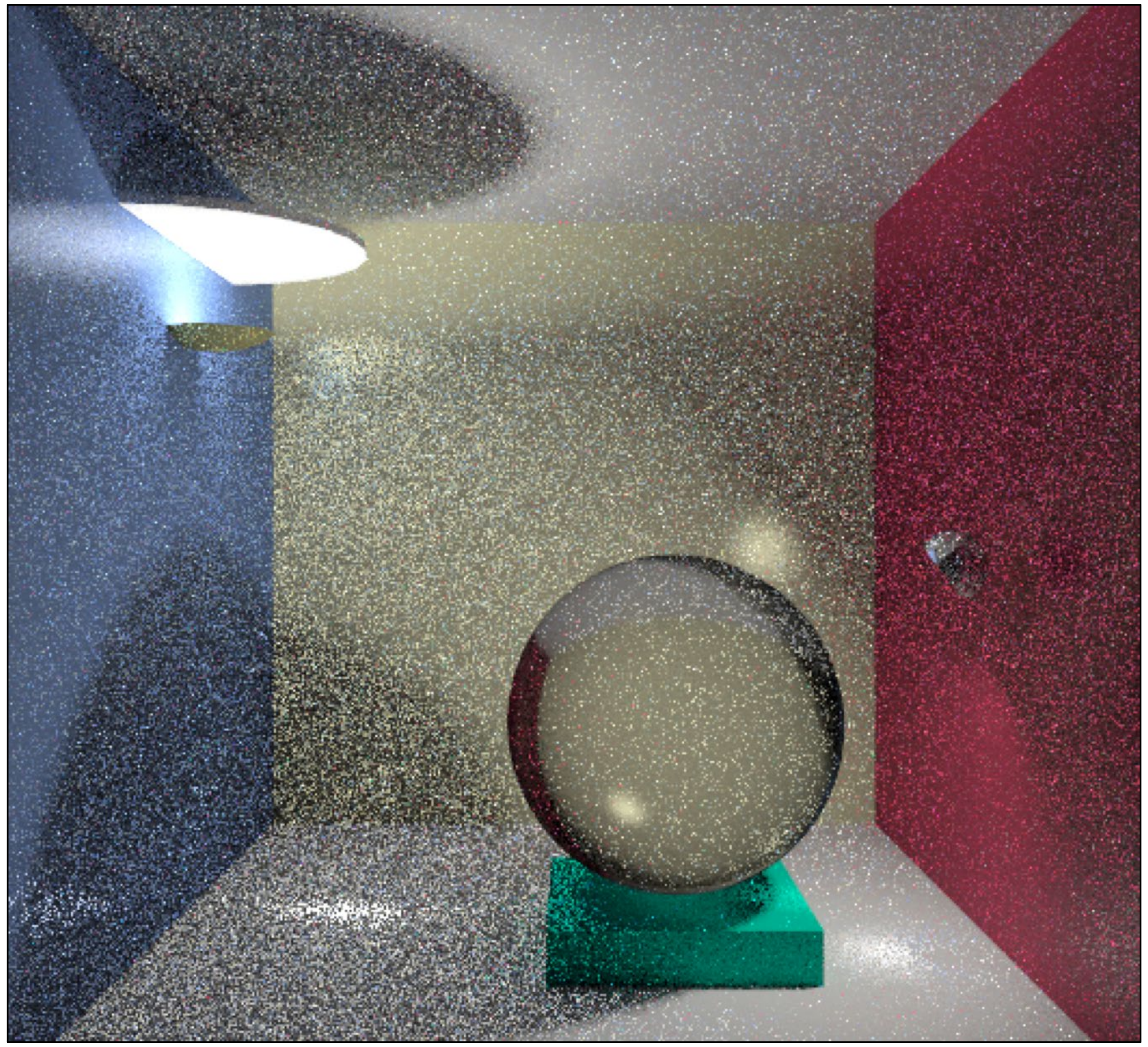


Images courtesy of F. Suykens

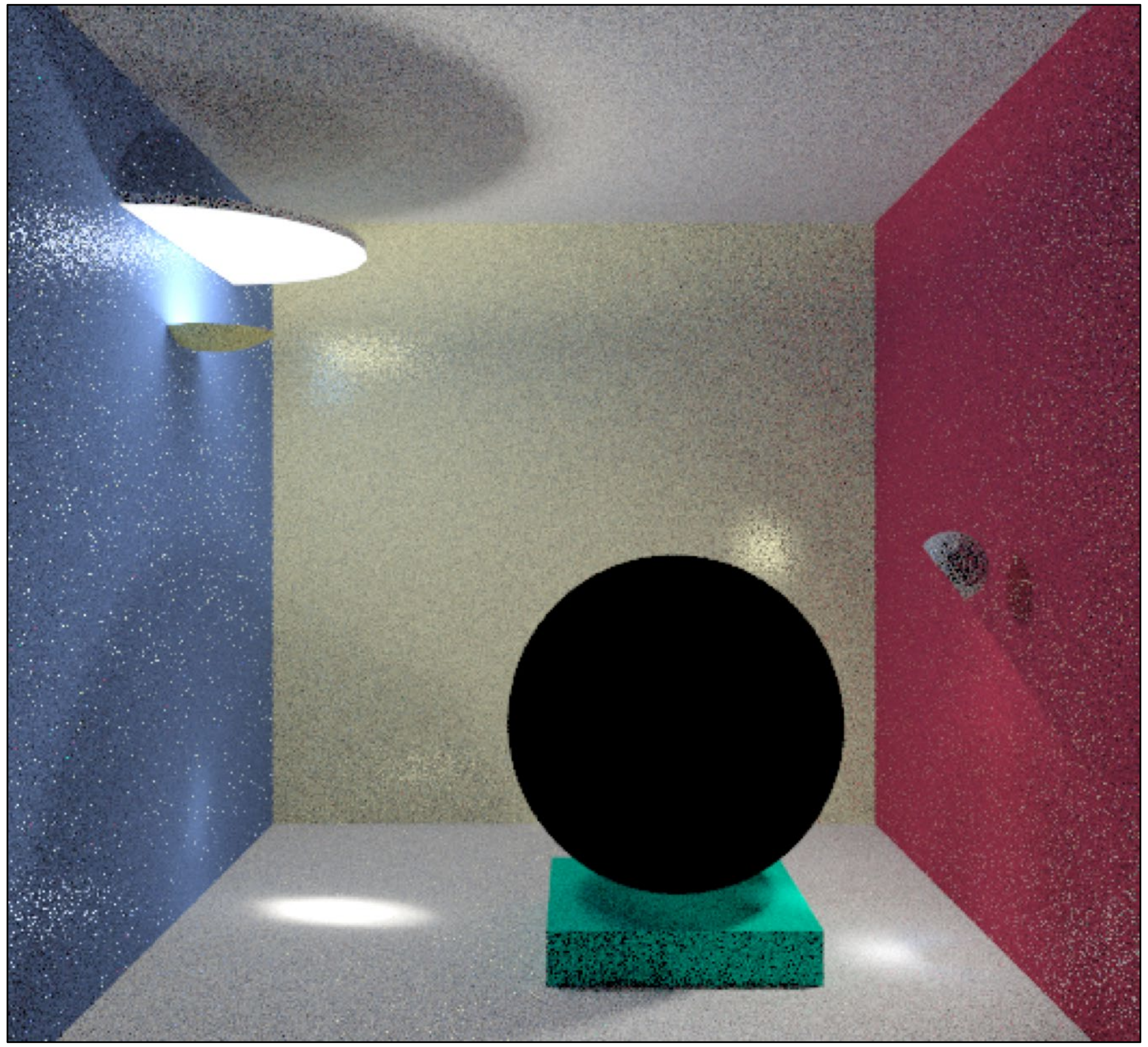
Why is this glass sphere black?

# Path vs. Light Tracing

Path tracing



Light tracing



Images courtesy of F. Suykens

Can we combine them?

# Path Integral Framework

# Measurement Equation

$$I_j = \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_o(\mathbf{x}_1, \mathbf{x}_0)\, d\mathbf{x}_1 d\mathbf{x}_0$$

$$= \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_1, \mathbf{x}_0) + \int_A f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_0) G(\mathbf{x}_1, \mathbf{x}_2) L_o(\mathbf{x}_2, \mathbf{x}_1) d\mathbf{x}_2 d\mathbf{x}_1 d\mathbf{x}_0$$

$$= \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_1, \mathbf{x}_0) + \int_A f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_0) G(\mathbf{x}_1, \mathbf{x}_2) L_e(\mathbf{x}_2, \mathbf{x}_1) + \int_A f(\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_1) G(\mathbf{x}_2, \mathbf{x}_3)\, L_e(\mathbf{x}_3, \mathbf{x}_2) + \int_A \cdots\ d\mathbf{x}_4 d\mathbf{x}_3 d\mathbf{x}_2 d\mathbf{x}_1 d\mathbf{x}_0$$

Hard to concisely express arbitrary light transport with all the nested integrals

Let's find a better way

# Path Integral Form of Measurement Eq.

$$I_j = \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_o(\mathbf{x}_1, \mathbf{x}_0) \, d\mathbf{x}_1 d\mathbf{x}_0$$

Emission

$$= \iint_A W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_1, \mathbf{x}_0) G(\mathbf{x}_0, \mathbf{x}_1) d\mathbf{x}_1 d\mathbf{x}_0$$

Direct illumination (3 vertices)

$$+ \iiint_A W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_2, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_0) G(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_2 d\mathbf{x}_1 d\mathbf{x}_0 + \cdots$$

(k-2)-bounce illumination (k vertices)

$$+ \int \cdots \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) G(\mathbf{x}_0, \mathbf{x}_1) \prod_{j=1}^{k-1} f(\mathbf{x}_j, \mathbf{x}_{j+1}, \mathbf{x}_{j-1}) G(\mathbf{x}_j, \mathbf{x}_{j+1}) \, d\mathbf{x}_k \cdots d\mathbf{x}_0 + \cdots$$

introduce: $\mathcal{P}_k = \{\bar{\mathbf{x}} = \mathbf{x}_0 \cdots \mathbf{x}_k ; \; \mathbf{x}_0 \cdots \mathbf{x}_k \in A\}$

space of all paths with $k$ segments

# Path Integral Form of Measurement Eq.

$$I_j = \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_o(\mathbf{x}_1, \mathbf{x}_0)\, d\mathbf{x}_1 d\mathbf{x}_0$$

Emission

$$= \int_{\mathcal{P}_1} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_1, \mathbf{x}_0) G(\mathbf{x}_0, \mathbf{x}_1) d\bar{\mathbf{x}}_1$$

Direct illumination (3 vertices)

$$+ \int_{\mathcal{P}_2} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_2, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_0) G(\mathbf{x}_1, \mathbf{x}_2) d\bar{\mathbf{x}}_2 + \cdots$$

(k-2)-bounce illumination (k vertices)

$$+ \int_{\mathcal{P}_k} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) G(\mathbf{x}_0, \mathbf{x}_1) \prod_{j=1}^{k-1} f(\mathbf{x}_j, \mathbf{x}_{j+1}, \mathbf{x}_{j-1}) G(\mathbf{x}_j, \mathbf{x}_{j+1})\, d\bar{\mathbf{x}}_k + \cdots$$

introduce: $$T(\bar{\mathbf{x}}_k) = G(\mathbf{x}_0, \mathbf{x}_1) \prod_{j=1}^{k-1} f(\mathbf{x}_j, \mathbf{x}_{j+1}, \mathbf{x}_{j-1}) G(\mathbf{x}_j, \mathbf{x}_{j+1})$$

*throughput* of path $\bar{\mathbf{x}}_k$

# Path Integral Form of Measurement Eq.

$$I_j = \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_o(\mathbf{x}_1, \mathbf{x}_0)\, d\mathbf{x}_1 d\mathbf{x}_0$$

Emission

$$= \int_{\mathcal{P}_1} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_1, \mathbf{x}_0) T(\bar{\mathbf{x}}_1) d\bar{\mathbf{x}}_1$$

Direct illumination (3 vertices)

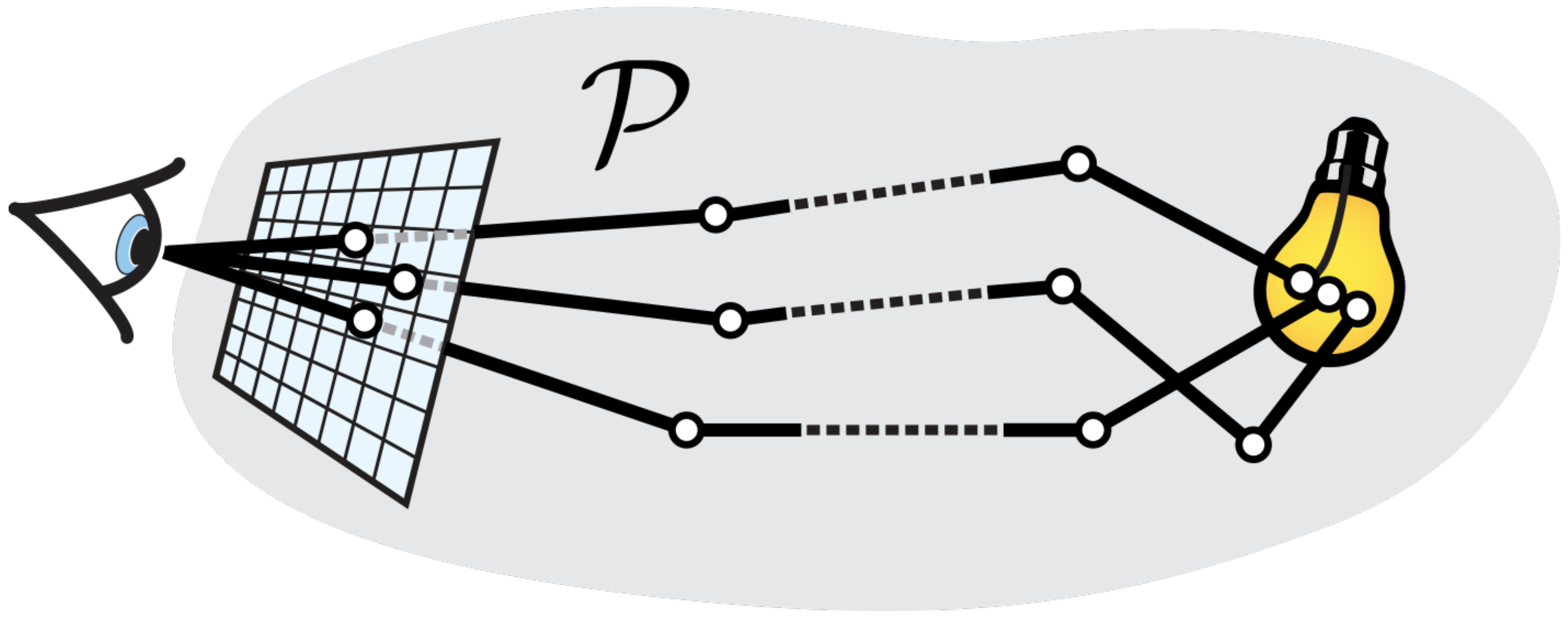
$$+ \int_{\mathcal{P}_2} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_2, \mathbf{x}_1) T(\bar{\mathbf{x}}_2)\, d\bar{\mathbf{x}}_2 + \cdots$$

(k-2)-bounce illumination (k vertices)

$$+ \int_{\mathcal{P}_k} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) T(\bar{\mathbf{x}}_k)\, d\bar{\mathbf{x}}_k + \cdots$$

introduce: $\mathcal{P} = \bigcup_{k=1}^{\infty} \mathcal{P}_k$

the *path space*, i.e. the space of all paths of all lengths



# Path Integral Form of Measurement Eq.

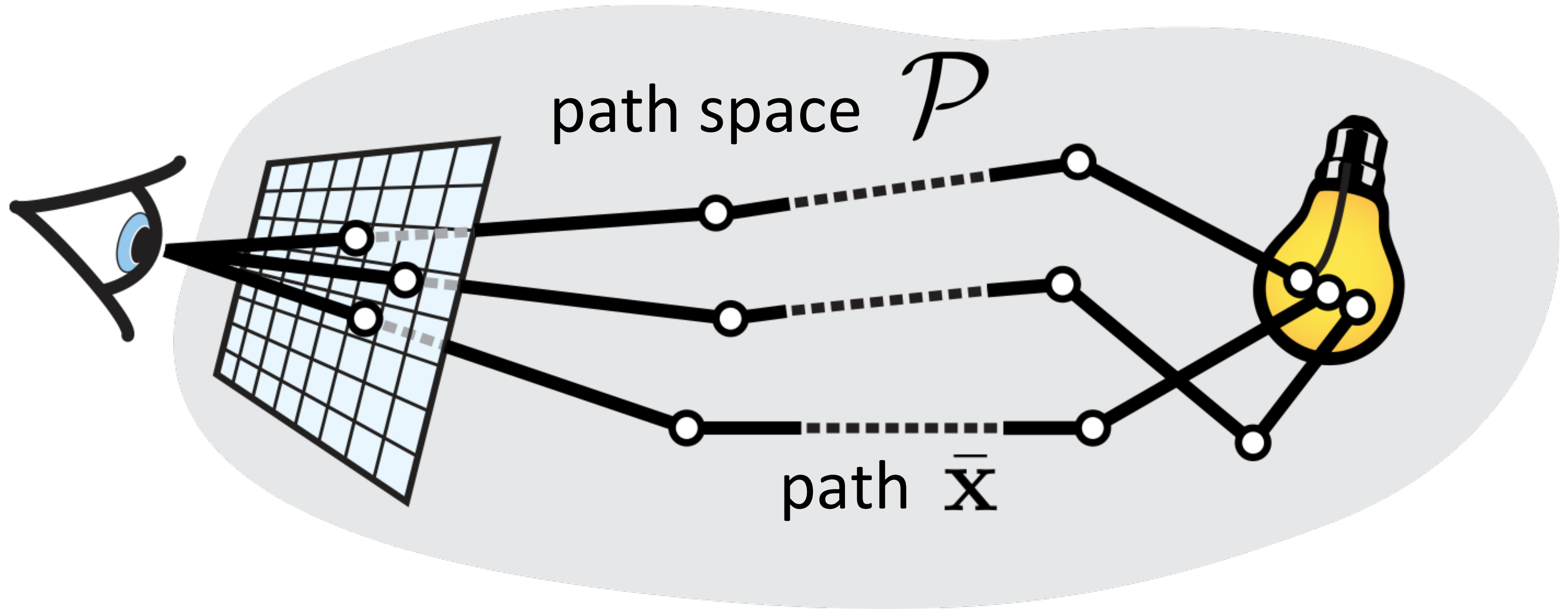
$$I_j = \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_o(\mathbf{x}_1, \mathbf{x}_0)\, d\mathbf{x}_1 d\mathbf{x}_0$$

global illumination (all paths of all lengths)

$$= \int_{\mathcal{P}} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) T(\bar{\mathbf{x}})\, d\bar{\mathbf{x}}$$

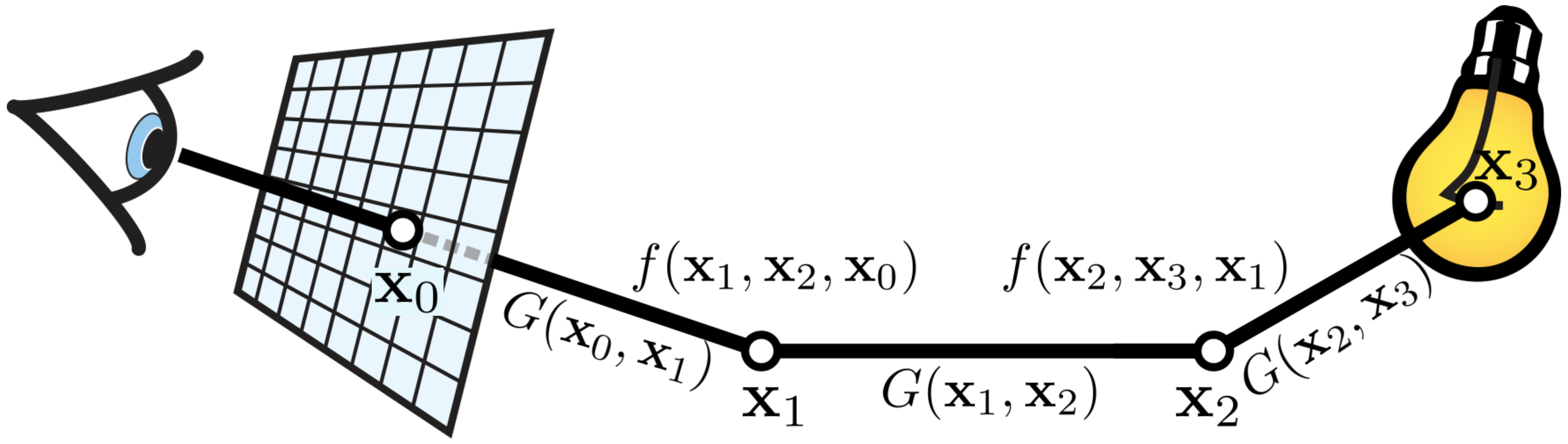
# Path Integral Form of Measurement Eq.

$$I_j = \int_{\mathcal{P}} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) T(\bar{\mathbf{x}})\, d\bar{\mathbf{x}}$$




path throughput

$$T(\bar{\mathbf{x}}) = G(\mathbf{x}_0, \mathbf{x}_1) \prod_{j=1}^{k-1} f(\mathbf{x}_j, \mathbf{x}_{j+1}, \mathbf{x}_{j-1}) G(\mathbf{x}_j, \mathbf{x}_{j+1})$$



# Path Integral Form of Measurement Eq.

$$I_j = \int_{\mathcal{P}} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) T(\bar{\mathbf{x}})\, d\bar{\mathbf{x}}$$

Advantages:

- no recursion, no “nasty” nested integrals
- emphasizes symmetry of light transport
- easy to relate different rendering algorithms
- focuses on path geometry, independent of strategy for constructing paths
- MC estimator on path space looks much simpler

# Path Integral Form of Measurement Eq.

$$I_j = \int_{\mathcal{P}} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) T(\bar{\mathbf{x}})\, d\bar{\mathbf{x}}$$

Monte Carlo estimator:

$$I_j \approx \frac{1}{N} \sum_{i=1}^{N} \frac{W_e(\mathbf{x}_{i,0}, \mathbf{x}_{i,1}) L_e(\mathbf{x}_{i,k}, \mathbf{x}_{i,k-1}) T(\bar{\mathbf{x}}_i)}{p(\bar{\mathbf{x}}_i)}$$

$$p(\bar{\mathbf{x}}) = \underbrace{p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k)}_{\text{joint PDF of path vertices}}$$

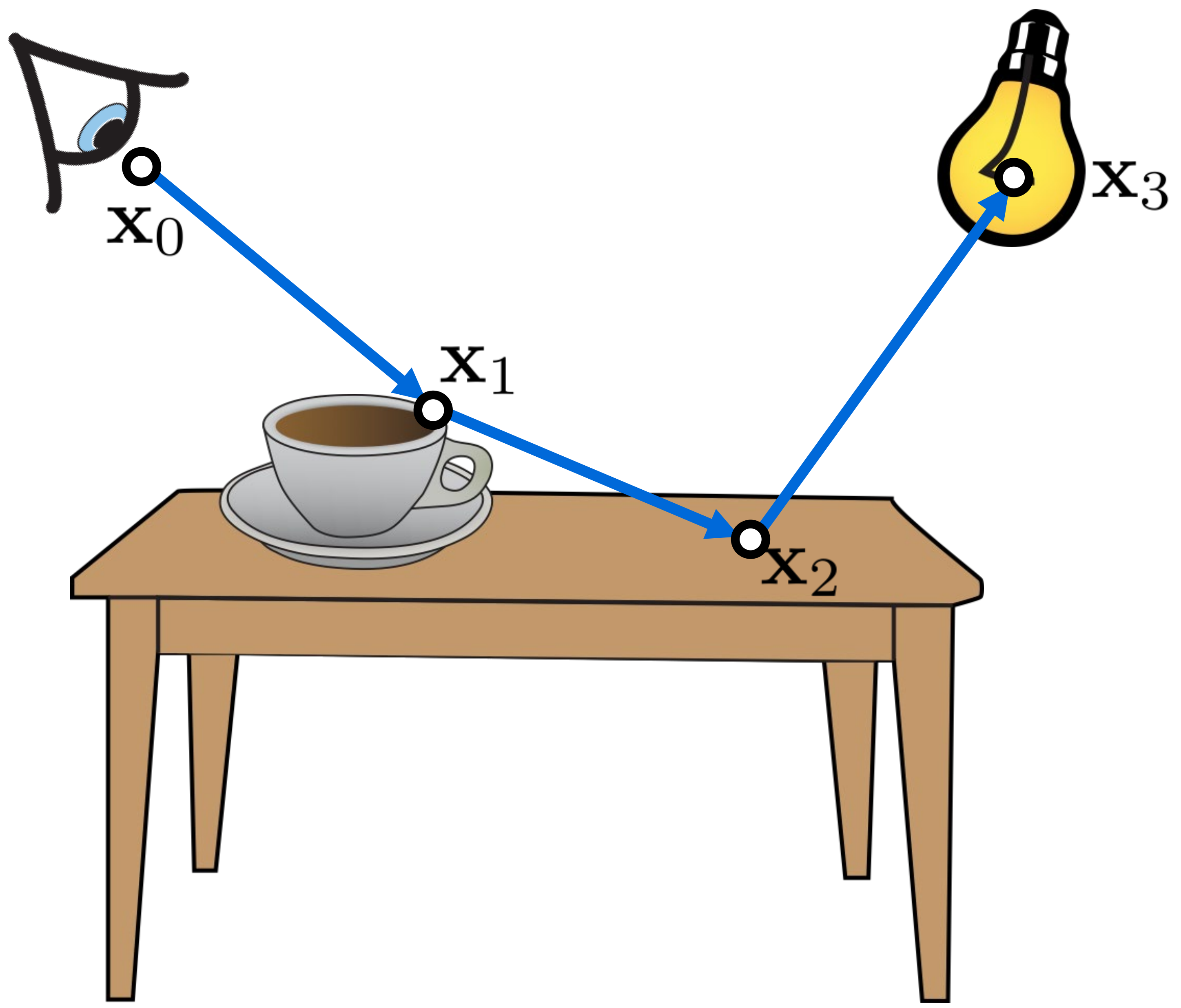
path PDF

joint PDF of path vertices

# Path Construction

$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k)$$

Path tracing w/o NEE

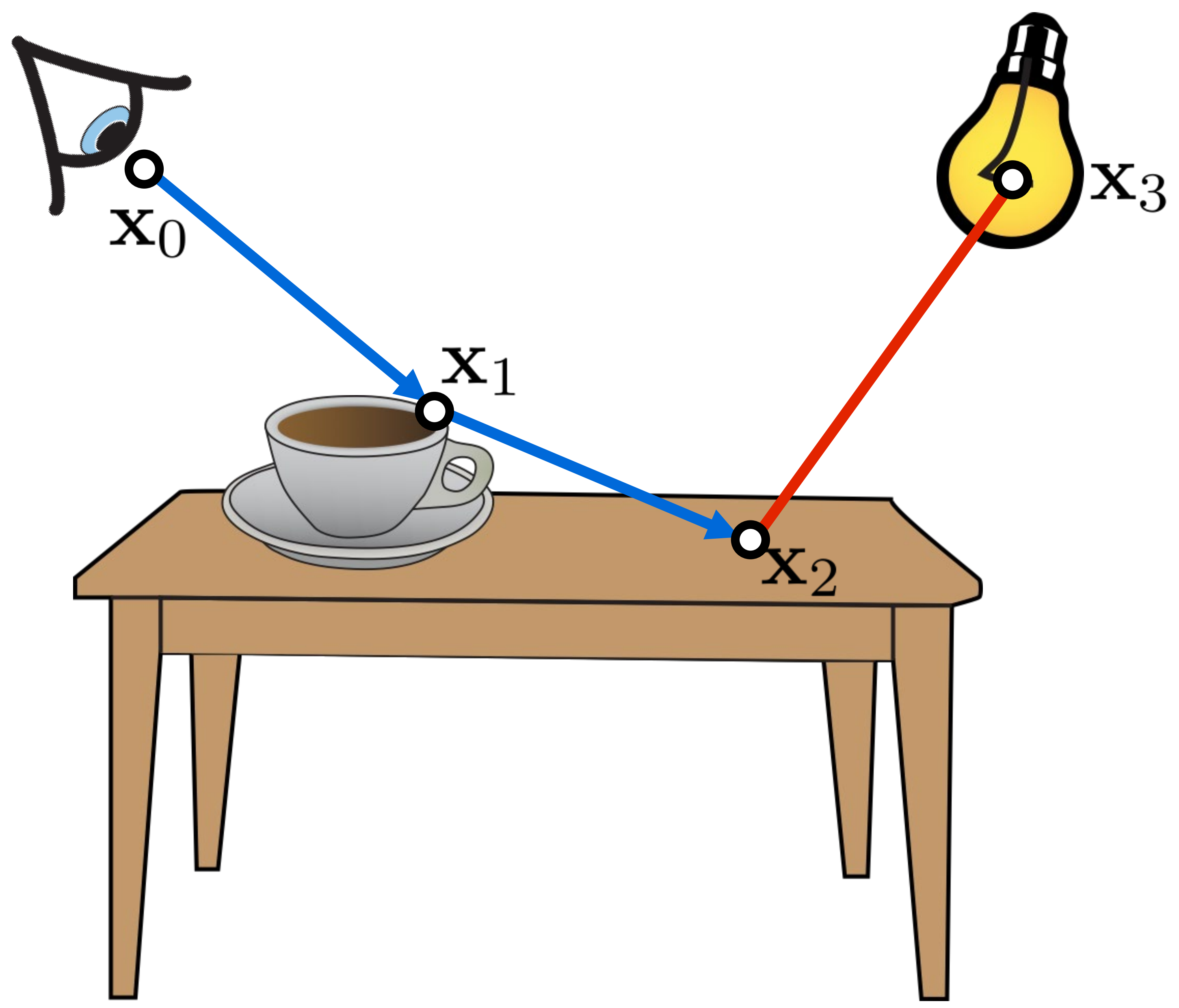



$$\begin{aligned} p(\bar{\mathbf{x}}) = & \, p(\mathbf{x}_0) \\ & \times p(\mathbf{x}_1 | \mathbf{x}_0) \\ & \times p(\mathbf{x}_2 | \mathbf{x}_0 \mathbf{x}_1) \\ & \times p(\mathbf{x}_3 | \mathbf{x}_0 \mathbf{x}_1 \mathbf{x}_2) \end{aligned}$$

# Path Construction

$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k)$$

Path tracing with NEE



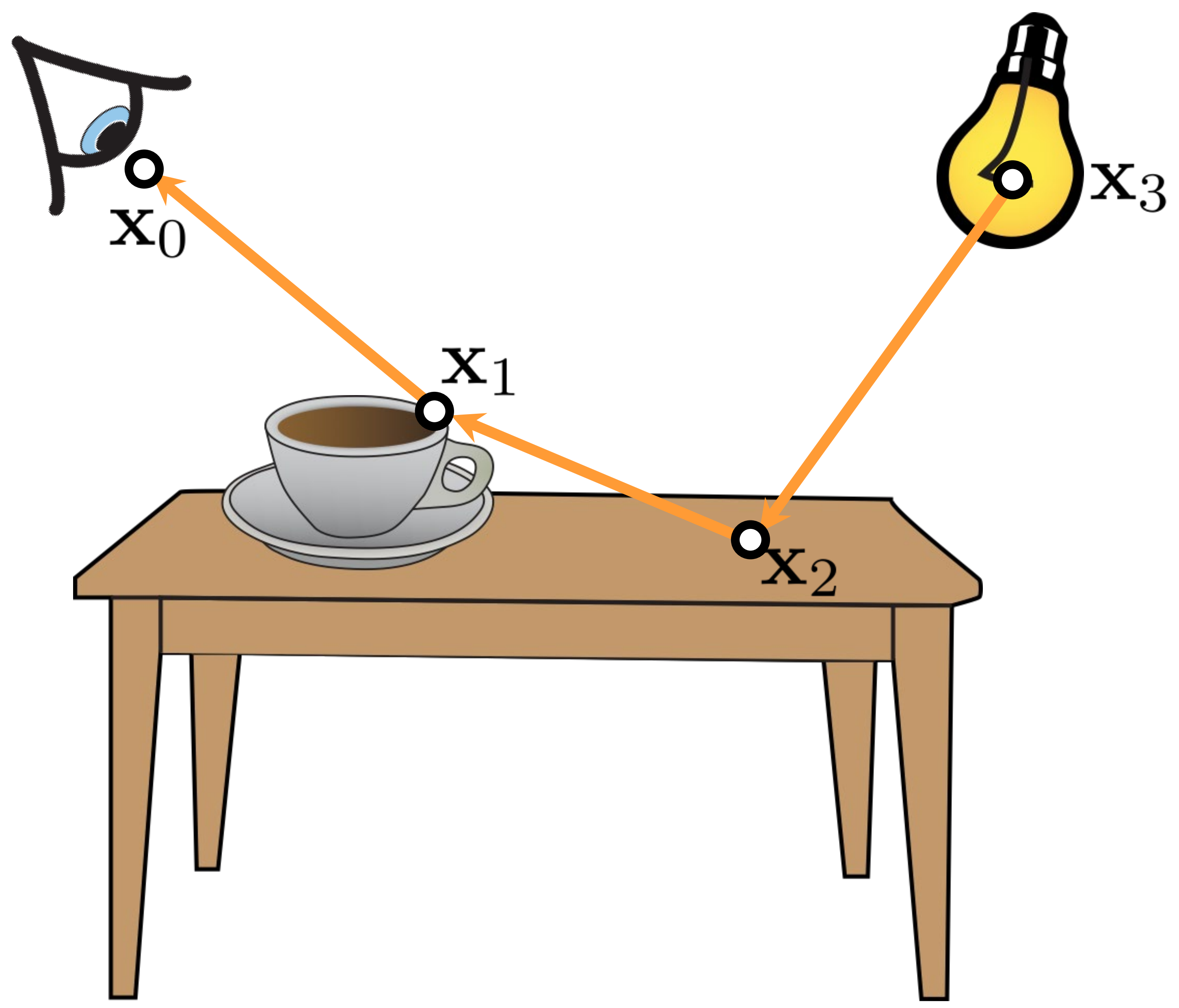

$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0) \times p(\mathbf{x}_1 | \mathbf{x}_0) \times p(\mathbf{x}_2 | \mathbf{x}_0 \mathbf{x}_1) \times p(\mathbf{x}_3)$$

assuming uniform area sampling

# Path Construction

$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k)$$

Light tracing

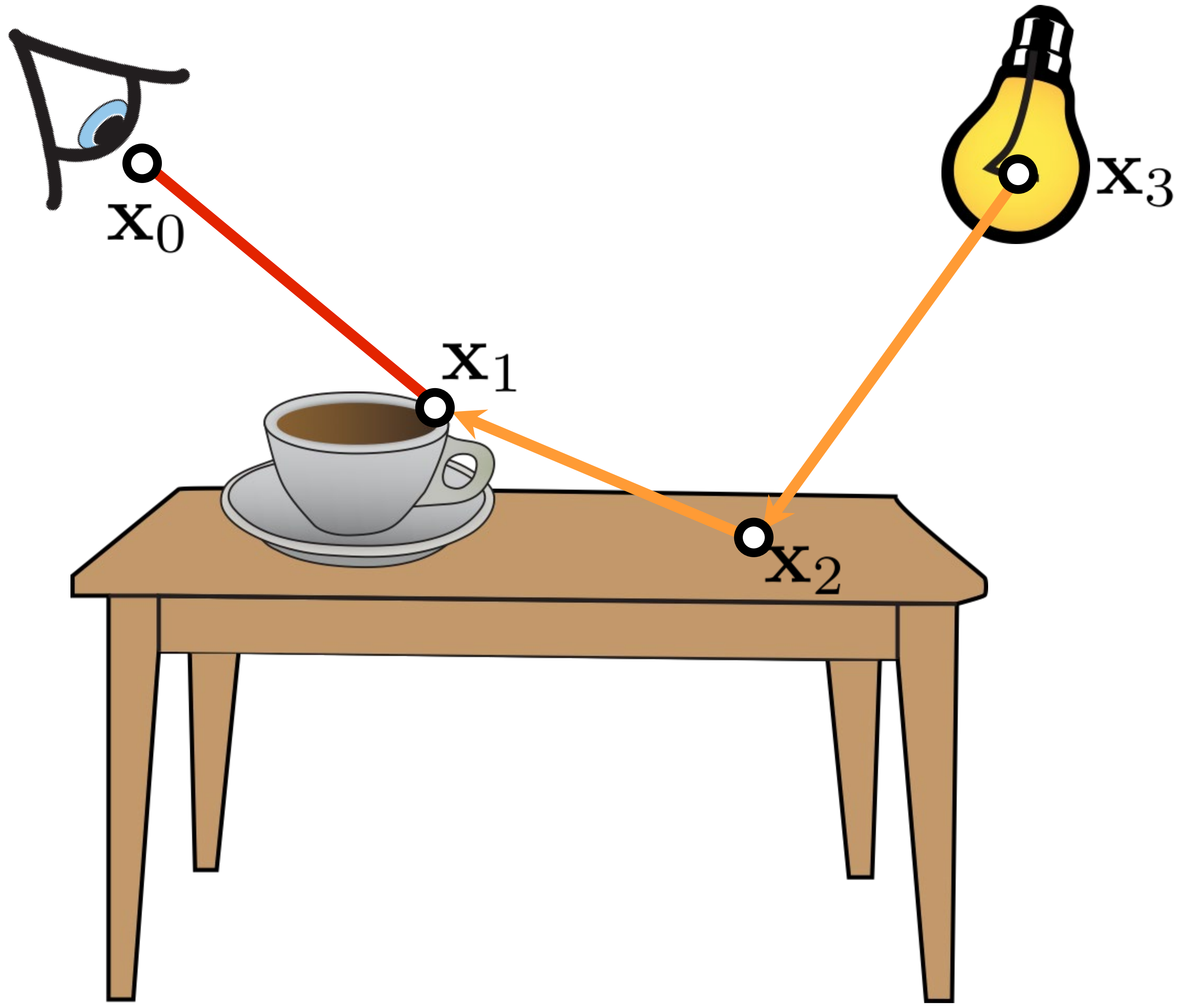



$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0 | \mathbf{x}_3 \mathbf{x}_2 \mathbf{x}_1) \times p(\mathbf{x}_1 | \mathbf{x}_3 \mathbf{x}_2) \times p(\mathbf{x}_2 | \mathbf{x}_3) \times p(\mathbf{x}_3)$$

# Path Construction

$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k)$$

Light tracing with NEE




assuming uniform aperture sampling

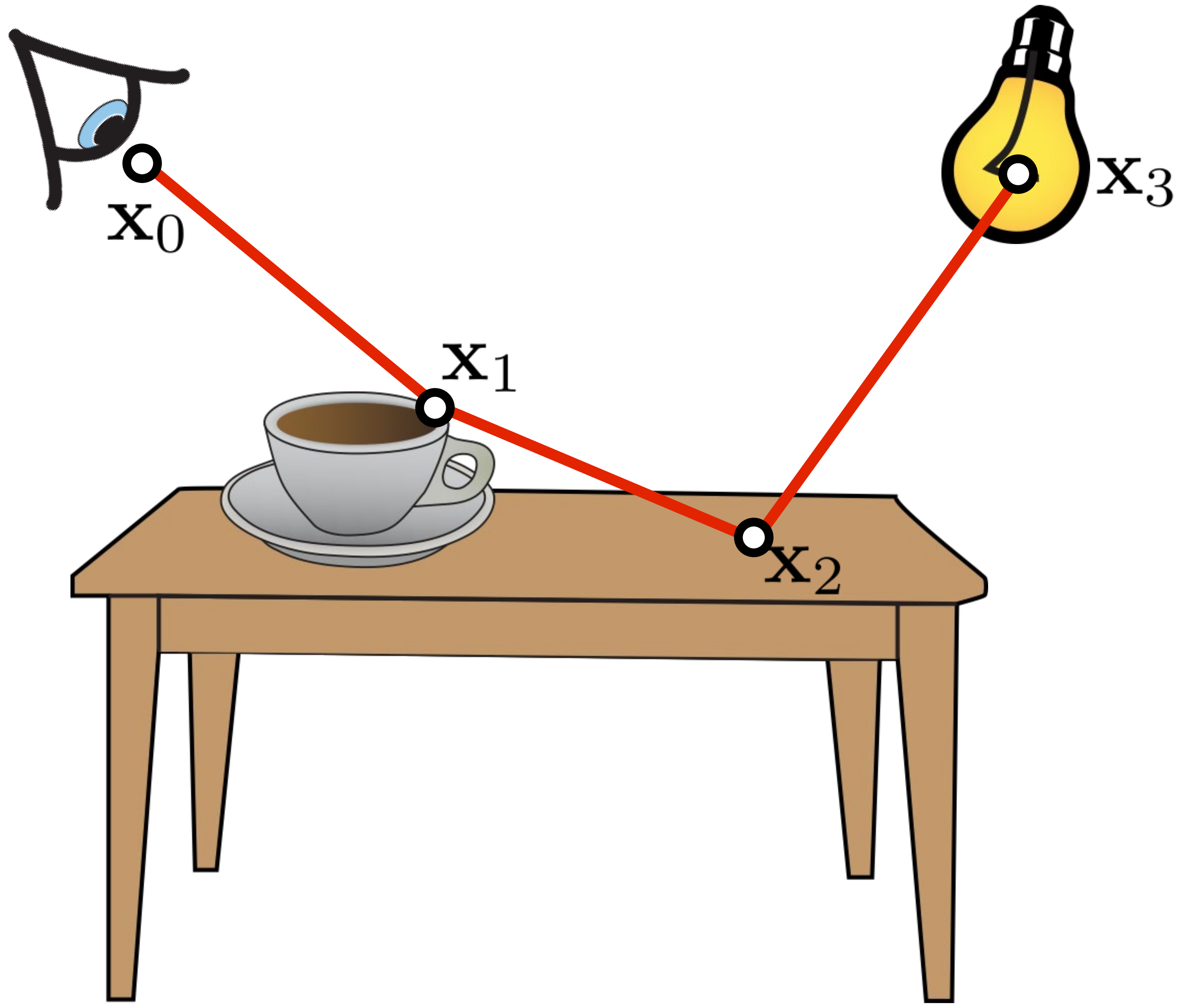
$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0) \times p(\mathbf{x}_1 | \mathbf{x}_3 \mathbf{x}_2) \times p(\mathbf{x}_2 | \mathbf{x}_3) \times p(\mathbf{x}_3)$$

# Path Construction

$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k)$$

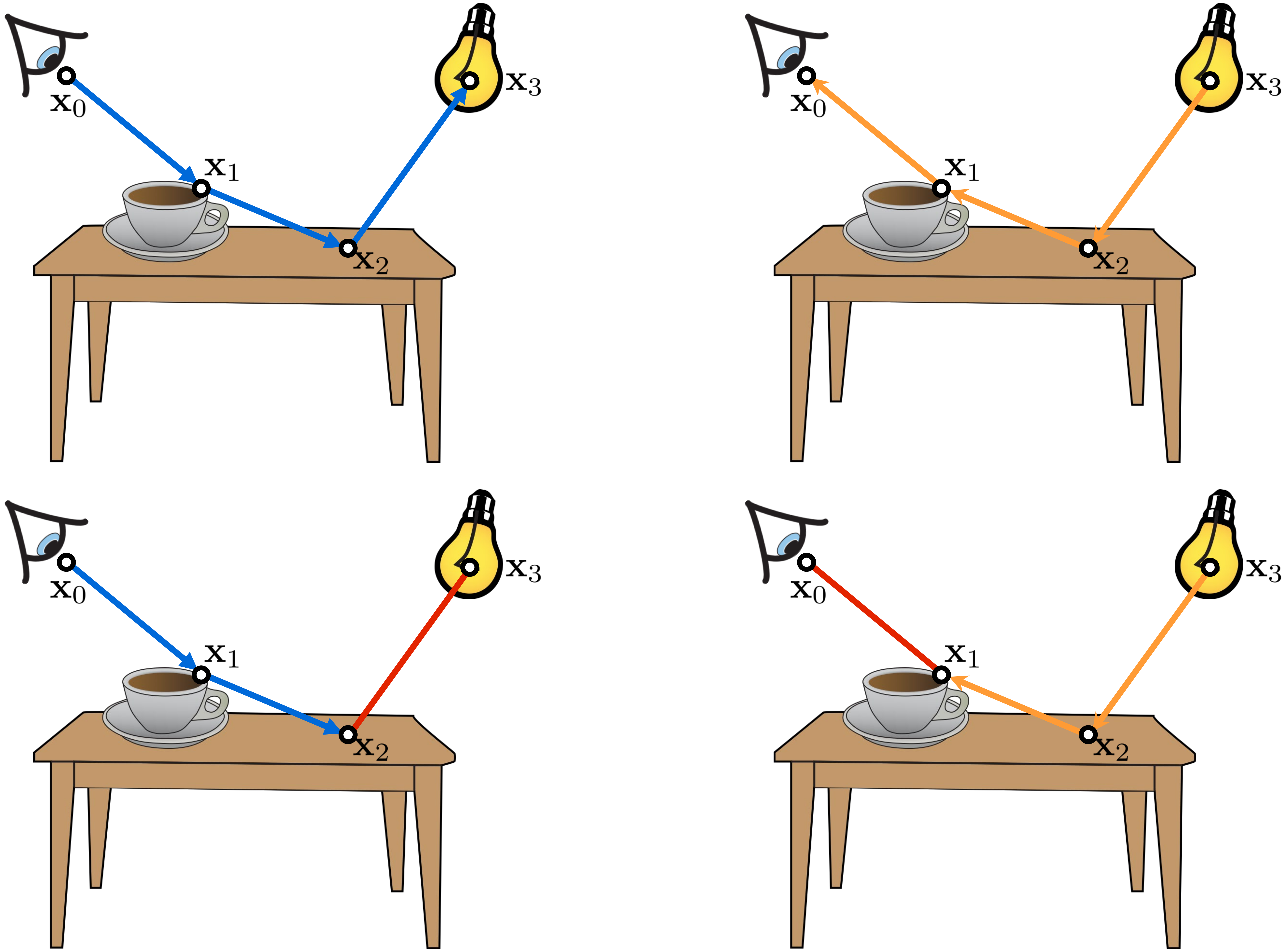
Independent sampling of path vertices

(not very practical though)




$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0) \times p(\mathbf{x}_1) \times p(\mathbf{x}_2) \times p(\mathbf{x}_3)$$

# Can we combine them?



# Bidirectional Path Tracing

# Bidirectional Path Tracing

[Lafortune and Willems 1993]
[Veach and Guibas 1994]

$t$ - # vertices on camera subpath
$s$ - # vertices on light subpath
$ts$ - # connections

# Bidirectional Path Tracing

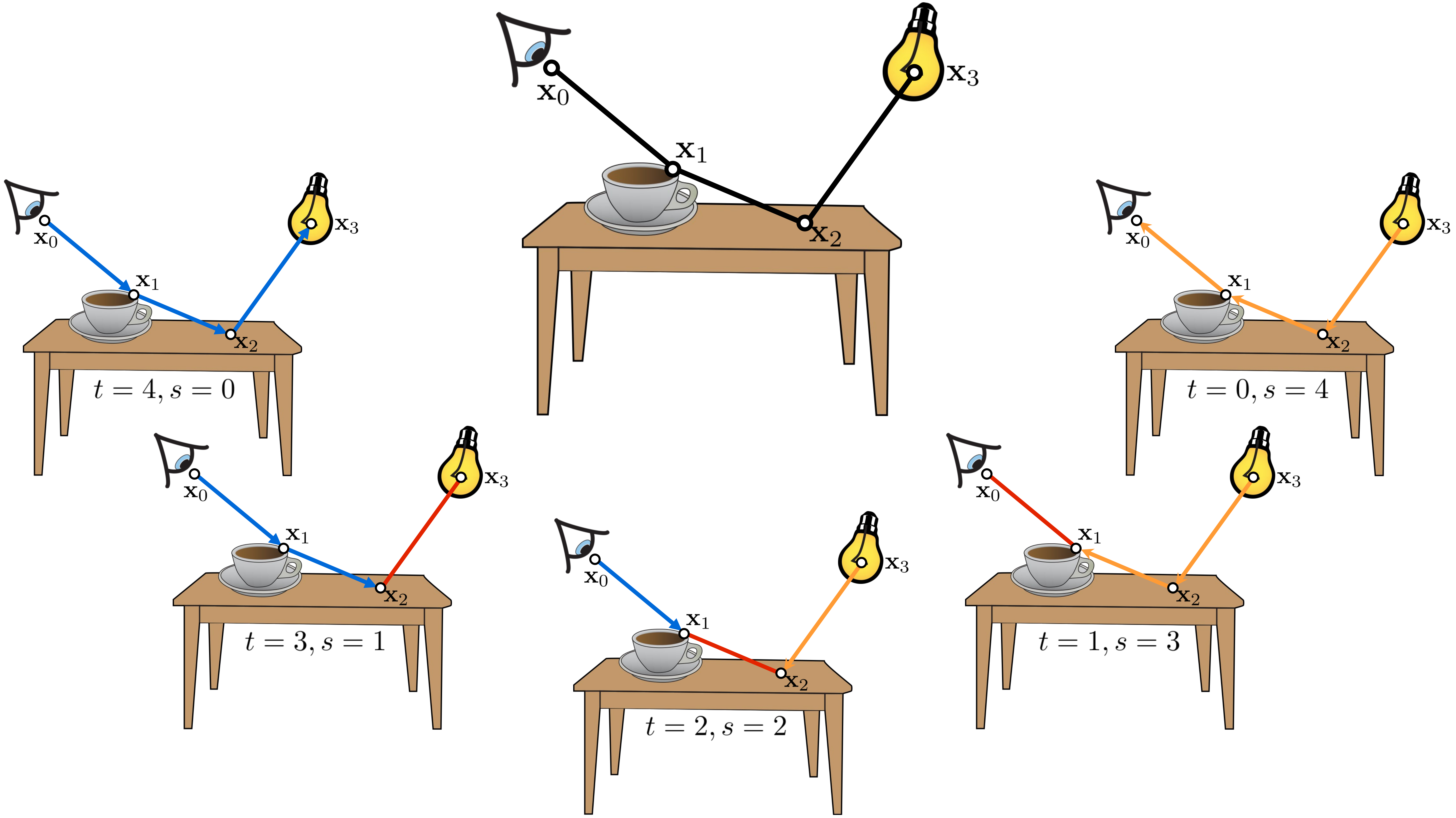
```
color estimate (point x)
{
   lp = sample light subpath
   cp = sample camera subpath for image point x

   for each vertex s in lp
      for each vertex t in cp
         fullPath = join(cp[0..s], lp[0..t])
         splat(fullPath.screenPos,
fullPath.contrib)
}
```

# Bidirectional Path Tracing

Key observations:

- Every path (formed by connecting camera sub-path to light sub-path) with $k$ vertices can be constructed using $k{+}1$ strategies
- For a particular path length, all strategies estimate the same integral
- Each strategy has a different PDF, i.e., each strategy has different strengths and weaknesses
- Let's combine them using MIS!

# Bidirectional Path Tracing



# Bidirectional Path Tracing


Images courtesy of W. Jakob

# Bidirectional Path Tracing (MIS)


Images courtesy of W. Jakob

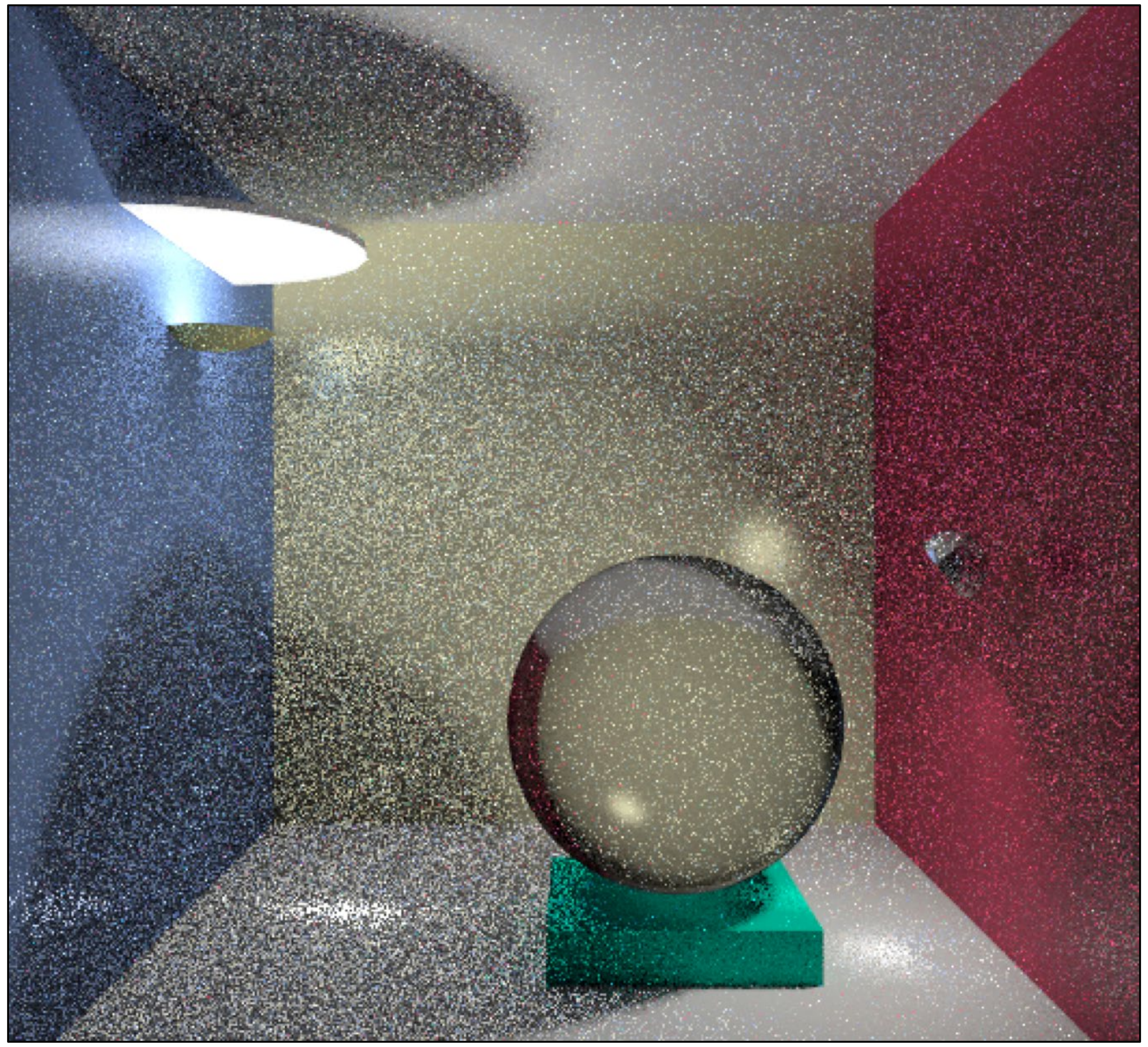
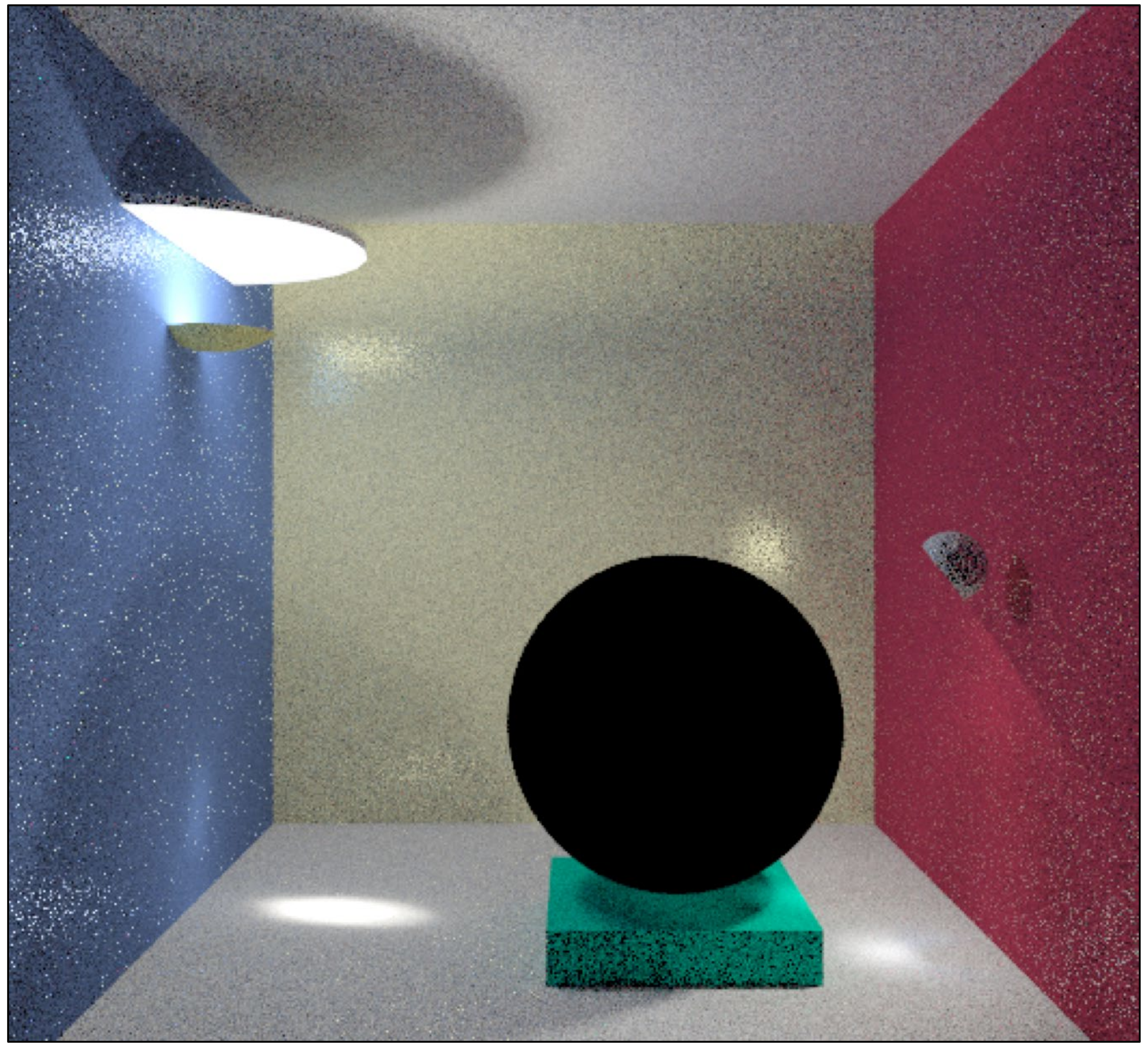
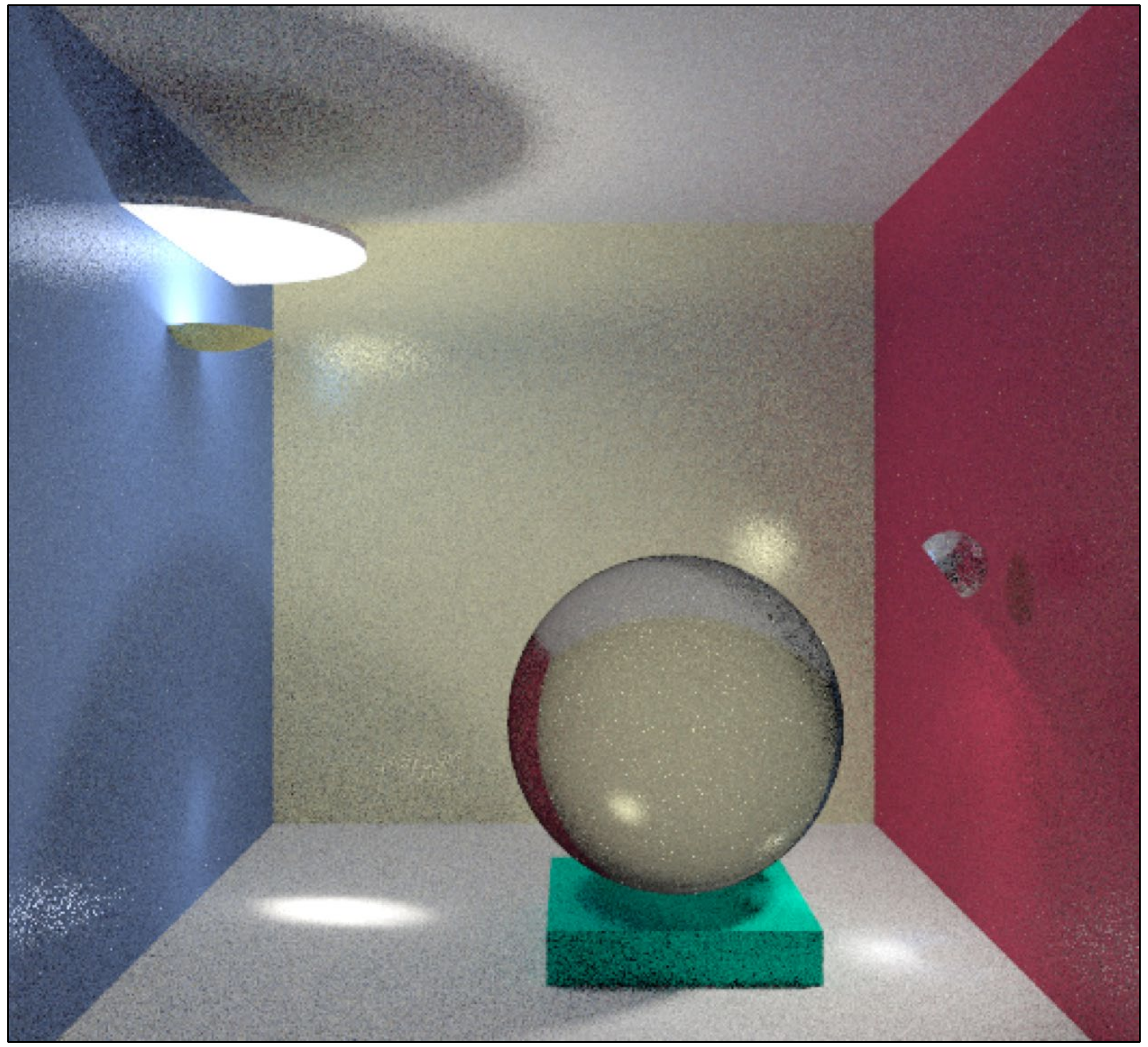
# Bidirectional Path Tracing

(Unidirectional) path tracing

Bidirectional path tracing

Images courtesy of W. Jakob

# Bidirectional Path Tracing

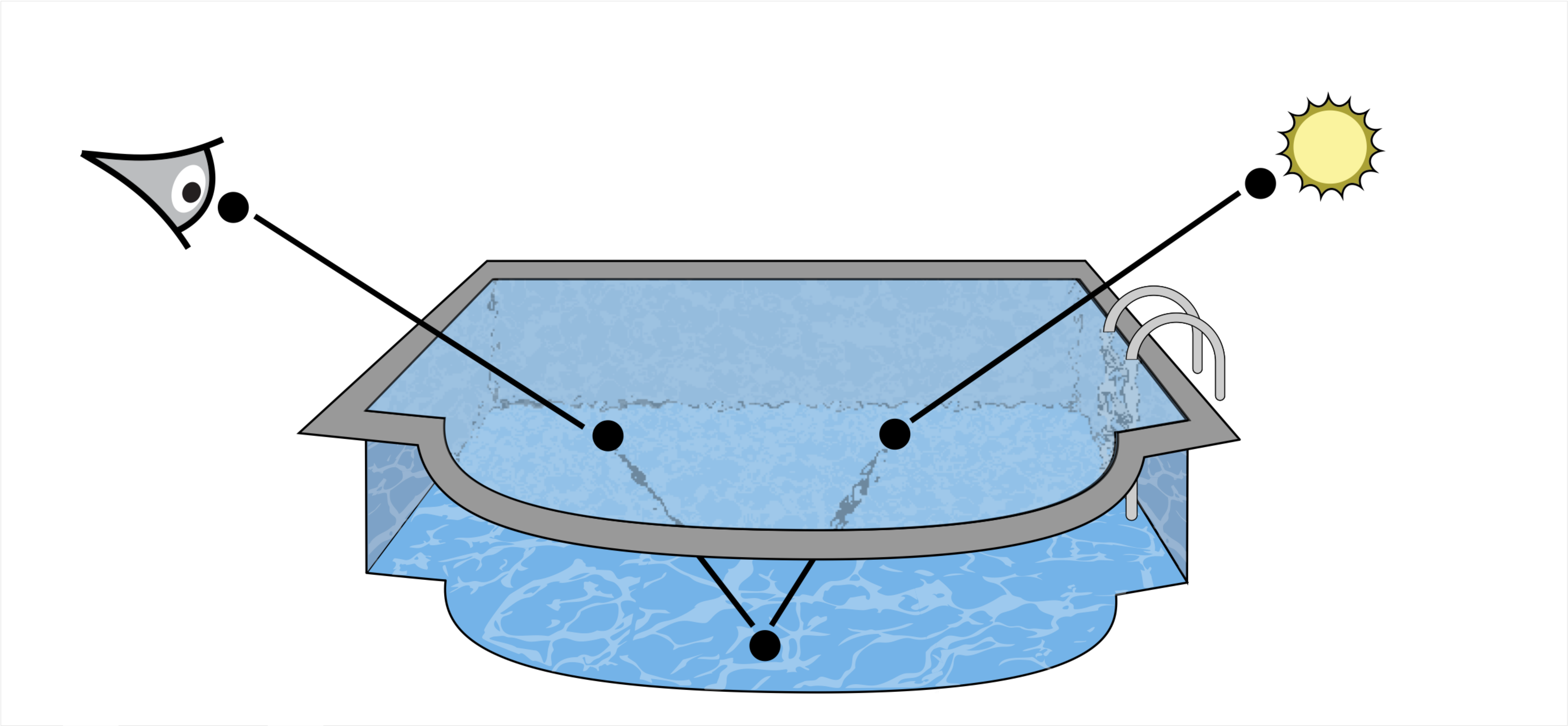
| Path tracing | Light tracing | Bidirectional PT |
|---|---|---|
|  |  |  |

Images courtesy of F. Suykens

# Still not robust enough...


Images courtesy of J. Křivánek

# Still not robust enough...



$LSDSE$ paths are difficult for any unbiased method

# Still not robust enough...

Extensions

- Combination with photon mapping
  - Unified Path Sampling [Hachisuka et al. 2012]
  - Vertex Connection Merging [Georgiev et al. 2012]
- Metropolis sampling (global PDF)
- Path-space regularization [Kaplanyan et al. 2013]
- Path guiding (learn global PDF)