Participating media



http://graphics.cs.cmu.edu/courses/15-468

15-468, 15-668, 15-868 Physics-based Rendering Spring 2025, Lecture 12

Course announcements

• Vote for PA2 competition!

Overview of today's lecture

- Participating media. •
- Scattering material characterization.
- Volume rendering equation.
- Ray marching. \bullet
- Volumetric path tracing. •
- Delta tracking. \bullet

Slide credits

Most of these slides were directly adapted from:

• Wojciech Jarosz (Dartmouth).



Fog





Clouds & Crepuscular rays





Fire





Underwater





Surface or Volume?







Antelope Canyon, Az.





Aerial (Atmospheric) Perspective



Leonardo da Vinci (1480)





Thus, if one is to be five times as distant, make it five times bluer.

—Treatise on Painting, Leonardo Da Vinci, pp 295, circa 1480.



Nebula





T.A.Rector (NOAO/AURA/NSF) and the Hubble Heritage Team (STScI/AURA/NASA)



Emission





Absorption





Scattering





Defining Participating Media

Typically, we do not model particles of a medium explicitly

The properties are described statistically using various coefficients and densities

- Conceptually similar idea as microfacet models

- (wouldn't fit in memory, completely impractical to ray trace)



Defining Participating Media

Homogeneous:

- Infinite or bounded by a surface or simple shape







Defining Participating Media

Heterogeneous (spatially varying coefficients):

- Procedurally, e.g., using a noise function
- Simulation + volume discretization, e.g., a voxel grid





Radiance

Previously: radiance remains constant along rays between surfaces

$$L_i(\mathbf{x}, \vec{\omega}) = L_o(\mathbf{y}, -\vec{\omega})$$
$$\mathbf{y} = r(\mathbf{x}, \vec{\omega})$$

The main quantity we are interested in for rendering is radiance







Radiance

The main quantity we are interested in for rendering is radiance

Now: radiance may *change* along rays between surfaces

 $L_i(\mathbf{x},\vec{\omega}) \neq L_o(\mathbf{y},-\vec{\omega})$ $\mathbf{y} = r(\mathbf{x},\vec{\omega})$





Participating Media





Differential Beam



How much light is *lost/gained* along the differential beam due to interactions of light with the medium?



Differential Beam Segment





Absorption



$dL(\mathbf{x},\vec{\omega}) = -\sigma_a$

$$(\mathbf{x})L(\mathbf{x},\vec{\omega})dz$$

 $\sigma_a(\mathbf{x})$: absorption coefficient $[m^{-1}]$



Out-scattering



 $[m^{-1}]$



In-scattering



 $L_s(\mathbf{x}, \vec{\omega})$: in-scattered radiance

 $[m^{-1}]$



Emission

dz $\vec{\omega}$ X $dL(\mathbf{x},\vec{\omega}) = \sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})dz$ $\sigma_a(\mathbf{x}):$ absorption coefficient *Sometimes modeled without the absorption coefficient just $L_e(\mathbf{x}, \vec{\omega})$: emitted radiance by specifying a "source" term



 $[m^{-1}]$



Radiative Transfer Equation (RTE)



In-scattering



Losses (Extinction)

Absorption

$$dL(\mathbf{x}, \vec{\omega}) = -\sigma_a(\mathbf{x})L(\mathbf{x})$$
$$= -\sigma_t(\mathbf{x})L(\mathbf{x})$$



What about a beam with a finite length?

Out-scattering $(\mathbf{x}, \vec{\omega})dz - \sigma_s(\mathbf{x})L(\mathbf{x}, \vec{\omega})dz$ $(z, \vec{\omega}) dz$

 $\sigma_t(\mathbf{x})$: extinction coefficient $[m^{-1}]$: total loss of light per unit distance



Extinction Along a Finite Beam

$$dL(\mathbf{x}, \vec{\omega}) = -\sigma_t(\mathbf{x})L(\mathbf{x}, \vec{\omega})$$
$$\frac{dL(\mathbf{x}, \vec{\omega})}{L(\mathbf{x}, \vec{\omega})} = -\sigma_t dz \quad // \text{ Integendent}$$
$$\ln(L_z) - \ln(L_0) = -\sigma_t z$$
$$\ln\left(\frac{L_z}{L_0}\right) = -\sigma_t z \quad // \text{ Exp}$$

$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

 $(z,ec{\omega})dz$ // Assume constant $\sigma_t(\mathbf{x})$, reorganize

egrate along beam from 0 to z

onentiate



Beer-Lambert Law

through a medium with constant extinction coefficient

The fraction is referred to as the *transmittance*

Think of this as fractional visibility between points



Expresses the remaining radiance after traveling a finite distance



Transmittance

Homogeneous volume:

 $T_r(\mathbf{x},\mathbf{y}) = \epsilon$

Heterogeneous volume (spatially varying σ_t):

 $T_r(\mathbf{x}, \mathbf{y}) =$



$$e^{-\sigma_t \|\mathbf{x}-\mathbf{y}\|}$$

$$e^{-\int_{0}^{\|\mathbf{x}-\mathbf{y}\|} \sigma_{t}(t)dt}$$

 \int Optical thickness



Transmittance

Homogeneous volume:

 $T_r(\mathbf{x},\mathbf{y}) = \epsilon$

Heterogeneous volume (spatially varying σ_t):

 $T_r(\mathbf{x},\mathbf{y}) = \epsilon$

Transmittance is multiplicative:

 $T_r(\mathbf{x}, \mathbf{z}) = T_r(\mathbf{x}, \mathbf{z})$

$$e^{-\sigma_t \|\mathbf{x}-\mathbf{y}\|}$$

$$e^{-\int_{0}^{\|\mathbf{x}-\mathbf{y}\|} \sigma_{t}(t)dt}$$

 \int Optical thickness

$$T_r(\mathbf{x},\mathbf{y})T_r(\mathbf{y},\mathbf{z})$$





Radiative Transfer Equation (RTE)

Absorption





Emission

Out-scattering

 $dL(\mathbf{x},\vec{\omega}) = -\sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega})dz \quad -\sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega})dz$ $+ \sigma_a(\mathbf{x}) L_e(\mathbf{x}, \vec{\omega}) dz + \sigma_s(\mathbf{x}) L_s(\mathbf{x}, \vec{\omega}) dz$



In-scattering


$L(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$





Reduced (background) surface radiance



 $L(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$ + $\int_{0}^{z} T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt$ $L_e(\mathbf{x}_t, \vec{\omega})$ \mathbf{x}_t $T_r(\mathbf{x}, \mathbf{x}_t)$

Accumulated emitted radiance





 $L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$ + $\int_{0}^{z} T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt$ $+ \int_{0}^{z} T_{r}(\mathbf{x}, \mathbf{x}_{t}) \sigma_{s}(\mathbf{x}_{t}) L_{s}(\mathbf{x}_{t}, \vec{\omega}) dt$ $L_s(\mathbf{x}_t, \vec{\omega})$ \mathbf{x}_t $T_r(\mathbf{x}, \mathbf{x}_t)$





 $L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$ + $\int_{0}^{z} T_{r}(\mathbf{x}, \mathbf{x}_{t}) \sigma_{a}(\mathbf{x}_{t}) L_{e}(\mathbf{x}_{t}, \vec{\omega}) dt$

 $T_r(\mathbf{x}, \mathbf{x}_t)$





$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a$$
$$+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s$$

 $\vec{\omega}$

 $(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt$

 $S(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}' dt$



Scattering in Media

Phase Function f_p

Describes distribution of scattered light

Analog of BRDF but for scattering in media

Integrates to unity (unlike BRDF)

 $\int_{\mathbb{C}^2} f_p(\mathbf{x}, \vec{\omega}, \vec{\omega}) d\vec{\omega}' = 1$

*We will use the same convention that phase function direction vectors always point away from the shading point x. Many publications, however, use a different convention for phase functions, in which direction vectors "follow" the light, i.e. one direction points towards **x** and the other away from **x**. When reading papers, be sure to clarify the meaning of the vectors to avoid misinterpretation.

Why do we have this property?





Isotropic Scattering

Uniform scattering, analogous to Lambertian BRDF

$$f_p(\vec{\omega}',\vec{\omega}) = \frac{1}{4\pi}$$

Where does this value come from?







Anisotropic Scattering

Quantifying anisotropy (Q, "average cosine"):

$$g = \int_{S^2} f_p($$

where:

 $\cos\theta = -\vec{\omega}\cdot\vec{\omega}'$

g = 0 : isotropic scattering (on average) g > 0: forward scattering g < 0 : backward scattering

 $(\mathbf{x}, \vec{\omega}, \vec{\omega}) \cos \theta \, d\vec{\omega}'$





Anisotropic scattering

 $f_{p\rm HG}(\theta) = \frac{1}{4\pi} \frac{1}{(1+g^2)^2}$



$$\frac{1-g^2}{-2g\cos\theta}^{3/2}$$



Anisotropic scattering







$$\frac{1-g^2}{-2g\cos\theta)^{3/2}}$$



-0.99



g



Empirical phase function

- Introduced for intergalactic dust
- Very popular in graphics and other fields



Schlick's Phase Function

Empirical phase function Faster approximation of HG

 $f_{p\mathrm{Schlick}}(\theta) =$

k =

$$\frac{1}{4\pi} \frac{1 - k^2}{(1 - k\cos\theta)^2}$$
$$1.55g - 0.55g^3$$



Schlick's Phase Function

Empirical phase function Faster approximation of HG





$$= 0 \quad k = 0 \qquad \qquad g = 0.8 \quad k = 0.96$$



Lorenz-Mie Scattering

If the diameter of scatterers is on the order of light wavelength, we cannot neglect the wave nature of light

Solution to Maxwell's equations for scattering from any spherical dielectric particle

Explains many phenomena

Complicated:

- Solution is an infinite analytic series



Lorenz-Mie Phase Function



Data obtained from http://www.philiplaven.com/mieplot.htm



Rainbows





Lorenz-Mie Approximations







Lorenz-Mie Approximations

Hazy atmosphere

$$f_{p \text{ hazy}}(\theta) = \frac{1}{4\pi} \left(5 + \left(\frac{1 + \cos \theta}{2} \right)^8 \right)$$



Murky atmosphere

$$f_{p \,\mathrm{murky}}(\theta) = \frac{1}{4\pi} \left(17 + \left(\frac{1 + \cos\theta}{2}\right)^{32} \right)$$





Rayleigh Scattering

Approximation of Lorenz-Mie for tiny scatterers that are typically smaller than 1/10th the wavelength of visible light

Used for atmospheric scattering, gasses, transparent solids

Highly wavelength dependent



Rayleigh Phase Function



$f_{p\text{Rayleigh}}(\theta) = \frac{3}{16\pi} (1 + \cos^2 \theta)$

Scattering at right angles is half as likely as scattering forward or backward



Rayleigh Scattering



Diameter of scatterers



Rayleigh Scattering



 $\sigma_{s\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \, \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2}\right)^2$







Steam



Forward scattering

Smoke

Backward scattering





Isotropic scattering







Forward scattering



Why is the Sky Blue?





Why is the Sunset Red?





Rayleigh Scattering







Media Properties (Recap)

Given:

- Absorption coefficient
- Scattering coefficient
- Phase function

Derived:

- Extinction coefficient
- Albedo
- Mean-free path
- Transmittance

 $\sigma_a(\mathbf{x})$ $\sigma_s(\mathbf{x})$ $f_p(\mathbf{x}, \vec{\omega}, \vec{\omega})$

 $1/\sigma_t(\mathbf{x})$





Homogeneous Isotropic Medium

Given:

- Absorption coefficient σ_a - Scattering coefficient σ_s 1 - Phase function $\overline{4\pi}$ Derived: - Extinction coefficient - Albedo $1/\sigma_t$ - Mean-free path - Transmittance





What is this?





Crepuscular Rays





Anti-Crepuscular Rays




Crepuscular rays from space





Solving the Volume Rendering Equation

Complexity Progression

homogeneous vs. heterogeneous scattering

- none —
- fake ambient
- single
- multiple



Volume Rendering Equation

$$\mathcal{L}(\mathbf{x}, \vec{\omega}) = \frac{T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z)}{+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_c} + \frac{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_c}{+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s}$$

background radiance

Accumulated in-scattered radiance



Purely absorbing media

 $L(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$



Attenuated background radiance



Fog





Participating Media



$$\mathbf{x}_{s}$$



Fog



 $L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t,\vec{\omega}) dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$



Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant:

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} \Big|$$

 $\frac{L_i(\mathbf{x}_t,\vec{\omega})}{dt} + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$



Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant:

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(\mathbf{x}_t, \vec{\omega}) dt + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$
$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \frac{L_i}{\int_0^s e^{-t\sigma_t} dt} + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$
$$L(\mathbf{x}, \vec{\omega}) = \sigma_s L_i \frac{1 - e^{-s\sigma_t}}{\sigma_t} + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$
$$L(\mathbf{x}, \vec{\omega}) = \operatorname{lerp}\left(\frac{\sigma_s}{\sigma_t} L_i, \ L(\mathbf{x}_s, \vec{\omega}), \ e^{-s\sigma_t}\right)$$



OpenGL Fog





OpenGL Clear Day





Fog









http://anordinarymom.wordpress.com





Andreas Levers



Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z) + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_o dt + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_o dt$$

 $,ec{\omega})$

 $_{a}(\mathbf{x}_{t})L_{e}(\mathbf{x}_{t},\vec{\omega})dt$

 $_{s}(\mathbf{x}_{t})L_{s}(\mathbf{x}_{t},\vec{\omega})dt$

Accumulated in-scattered radiance



In-scattered Radiance

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r($$

$$L_s(\mathbf{x}_t, \vec{\omega}) = \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}) dt dt$$

Single scattering

- L_i arrives directly from a light source (direct illum.) i.e.:
- Multiple scattering
- L_i arrives through multiple bounces (indirect illum.)

 $(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt$

 $\mathbf{x}_t, \vec{\omega}, \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}'$

$L_i(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},r(\mathbf{x},\vec{\omega})) L_e(r(\mathbf{x},\vec{\omega}),-\vec{\omega})$



 $L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$ $L_e(\mathbf{x}, -\vec{\omega}') \mathbf{S}_{\mathbf{X}_e}$ $\vec{\omega}$ \mathbf{X} \mathbf{x}_t $T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t)$ $f_p(\mathbf{x}_t, \vec{\omega}, \vec{\omega})$





 $L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t,\vec{\omega},\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$ $L_e(\mathbf{x}, -\vec{\omega}') \boldsymbol{\delta}_{\mathbf{X}_e}$ X \mathbf{x}_t





 $L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t,\vec{\omega},\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$ $L_e(\mathbf{x}, -\vec{\omega}') \mathbf{S}_{\mathbf{x}_e}$ **X** $\vec{\omega}$ \mathbf{X}_t





 $L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$ $L_e(\mathbf{x}, -\vec{\omega}') \mathbf{S}_{\mathbf{x}_e}$ X





$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_S^z T_r(\mathbf{x},\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_S^z T_r(\mathbf{x},\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x},\mathbf{x}_t)$$

(Semi-)analytic solutions:

- Sun et al. [2005]
- Pegoraro et al. [2009, 2010]

Numerical solutions:

- Ray-marching
- Equiangular sampling

 $\int_{\mathbb{C}^2} f_p(\mathbf{x}_t, \vec{\omega}, \vec{\omega}, \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$



$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_S^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_S^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_S^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \sigma_s(\mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_S^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \sigma_s(\mathbf{x}_t) \sigma_s(\mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_S^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \sigma_s(\mathbf{x}_t)$$

Assumptions:

- Homogeneous medium
- Point or spot light
- Relatively simple phase function
- No occlusion

$$L(\mathbf{x},\vec{\omega}) = \frac{\Phi}{4\pi} \frac{1}{4\pi} \sigma_s \int_0^z e^{-\sigma_t \|\mathbf{x},\mathbf{x}_t\|} \frac{e^{-\sigma_t \|\mathbf{x}_t,\mathbf{x}_p\|}}{\|\mathbf{x}_t,\mathbf{x}_p\|^2} \mathrm{d}t$$

 $f_p(\mathbf{x}_t, \vec{\omega}, \vec{\omega}, \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$



OpenGL Fog

















$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} 2 \sum_{n=0}^{N-1} c(n) \sum_{k=0}^{2n} d(n, k) \int_{\nu_a}^{\nu_b} \frac{e^{-H\nu}}{(\nu^2 + 1)^{n+1}} \nu^k d\nu$$

$$\begin{split} \int \frac{e^{av}}{(v^2+1)^m} v^n \mathrm{d}v &= \frac{1}{2^{m-1}} \sum_{l=0}^{m-1} \frac{1}{2^l} \binom{m-1+l}{m-1} \binom{\min\{m-1-l,n\}}{k=0} \binom{n}{k} \binom{a^{m-1-l-k}}{(m-1-l-k)!} E(a,v,m-n-l+k) \\ &- e^{av} \sum_{j=1}^{m-1-l-k} \frac{(j-1)!}{(m-1-l-k)!} \frac{a^{m-1-l-k-j}}{(v^2+1)^j} \sum_{i=(m-n-l+k-j) \bmod 2}^{\leq j} (-1)^{\frac{m-n-l+k-j+i}{2}} \binom{j}{i} v^i \end{pmatrix} \\ &+ \frac{e^{av}}{a} \sum_{k=0}^{n-m+l} \binom{n}{k} \sum_{j=0}^{n-m+l-k} \frac{(n-m+l-k)!}{j!} \frac{1}{(-a)^{n-m+l-k-j}} \sum_{i=(-m+l+k-j) \bmod 2}^{\leq j} (-1)^{\frac{-m+l+k-j+i}{2}} \binom{j}{i} v^i \end{pmatrix}$$

No shadows, implementation nightmare, computationally intensive... Let's try brute force!







N $L(\mathbf{x}, \vec{\omega}) \approx \sum T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$ i=1





Ni=1







 $L(\mathbf{x}, \vec{\omega}) \approx \sum T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$ $\mathbf{x}_{t,i}$

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 $L(\mathbf{x},\vec{\omega}) \approx \sum T_r(\mathbf{x},\mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i},\vec{\omega}) \Delta t$ Heterogeneous volume: $T_r(\mathbf{x}, \mathbf{x}_{t,i}) = T_r(\mathbf{x}, \mathbf{x}_{t,i-1}) e^{-\sigma_t(\mathbf{x}_{t,i})\Delta t}$ $\mathbf{x}_{t,i}$



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N $L(\mathbf{x}, \vec{\omega}) \approx \sum T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) \frac{L_s(\mathbf{x}_{t,i}, \vec{\omega})}{\Delta t}$ i=1 $\mathbf{x}_{t,i}$ $L_s(\mathbf{x}_{t,i}, \vec{\omega})$




Ray-Marching





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Ray-Marching





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Ray-Marching

 \mathbf{X}

Single scattering:





Ray-Marching in Heterogeneous Media

Marching towards the light source

- Connections are expensive, many, and uniformly distributed along the primary ray





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- 1. Ray-march and cache transmittance
- variations



- Choose step-size w.r.t. frequency content to accurately capture



- 2. Estimate in-scattering using MC integration
- Distribute samples \propto (part of) the integrand





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- 2. Estimate in-scattering using MC integration
- Distribute samples \propto (part of) the integrand





Ray-marching



Equiangular sampling

Images courtesy of Kulla and Fajardo



Multiple Bounces

Same concept as in recursive Monte Carlo ray tracing, but taking into account volumetric scattering

Exponential growth:







Visual Break







Volumetric Path Tracing

Volumetric Path Tracing

Motivation:

- Paths can:
- Reflect/refract off surfaces
- Scatter inside a volume

- Same as with standard path tracing: avoid the exponential growth



Volume Rendering Equation

/ Accumulated emitted radiance

in-scattered radiance

 \mathbf{O}



Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = \int_{0}^{z} T_{r}(\mathbf{x}, \mathbf{x}_{t}) \left[\sigma_{a}(\mathbf{x}_{t}) L_{e}(\mathbf{x}_{t}, \vec{\omega}) + \sigma_{s}(\mathbf{x}_{t}) L_{s}(\mathbf{x}_{t}, \vec{\omega}) \right] dt$$
$$+ \frac{T_{r}(\mathbf{x}, \mathbf{x}_{z}) L(\mathbf{x}_{z}, \vec{\omega})}{\int_{0}^{t} \text{Attenuated background radiance}}$$



Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \left[\sigma_a(\mathbf{x}_t) + T_r(\mathbf{x}, \mathbf{x}_t) L(\mathbf{x}_t, \vec{\omega}) \right]$$

$(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega}) dt$



1-Sample Monte Carlo Estimator

$$\langle L(\mathbf{x}, \vec{\omega}) \rangle = \frac{T_r(\mathbf{x}, \mathbf{x}_t)}{p(t)} \left[\sigma_a(\mathbf{x}_t) + \frac{T_r(\mathbf{x}, \mathbf{x}_t)}{P(t)} L(\mathbf{x}_t) \right]$$

p(t) - probabilitP(z) - probabilit

 $(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})$

 $ec{\omega})$

p(t) - probability *density* of distance t

P(z)-probability of exceeding distance z



1-Sample Monte Carlo Estimator

$$\langle L(\mathbf{x}, \vec{\omega}) \rangle = \frac{T_r(\mathbf{x}, \mathbf{x}_t)}{p(t)} \left[\sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) + \sigma_s(\mathbf{x}_t) \frac{f_p(\vec{\omega}, \vec{\omega}_i) L(\mathbf{x}_t, \vec{\omega}_i)}{p(\vec{\omega}_i)} \right]$$

$$+ \frac{T_r(\mathbf{x}, \mathbf{x}_z)}{P(z)} L(\mathbf{x}_z, \vec{\omega})$$

p(t) - probability *density* of distance t

P(z)-probability of exceeding distance z

 $p(\vec{\omega}_i)$ - probability *density* of direction $\vec{\omega}_i$



Volumetric Path Tracing

- 1. Sample distance to next interaction
- 2. Scatter in the volume or bounce off a surface





Volumetric Path Tracing with NEE





Sampling the Phase Function

Isotropic:

- Uniform sphere sampling
- Henyey-Greenstein:
- Using the inversion method we can derive

$$\cos \theta = \frac{1}{2g} \left(1 + g^2 - \left(\frac{1 - g^2}{1 - g + 2g\xi_1} \right)^2 \right)$$

 $\phi = 2\pi\xi_2$

PDF is the value of the HG phase function



Free-path (or free-flight distance):

- Distance to the next interaction within the medium
- Dense media (e.g. milk): short mean-free path
- Thin media (e.g. atmosphere): long mean-free path
- Ideally, we want to sample proportional to (part of) integrand, e.g. transmittance:
 - $p(\mathbf{x}_t | (\mathbf{x}, \vec{\omega})) \propto T_r(\mathbf{x}, \mathbf{x}_t)$ $p(t) \propto T_r(t)$

*)*simplified notation for brevity



Free-path Sampling Homogeneous media: $T_r(t)$ - PDF: $p(t) \propto e^{-\sigma_t t}$ $p(t) = \frac{e^{-\sigma_t t}}{\int_0^\infty e^{-\sigma_t s} ds} = \sigma_t e^{-\sigma_t t}$ - CDF: $P(t) = \int_0^t \sigma_t e^{-\sigma_t s} ds = 1 - D_0^{t} \sigma_t e^{-\sigma_t s} ds$

- Inverted CDF: $P^{-1}(\xi) = -\frac{\ln(1-\xi)}{\sigma_{t}}$

$$t) = e^{-\sigma_t t}$$

$$-e^{-\sigma_t t}$$

$$(1-\xi)$$



Homogeneous media: T_{T} Recipe:

- Generate random number
- Sample distance $t = -\frac{\ln(1-\xi)}{\sigma_t}$
- **Compute PDF** $p(t) = \sigma_t e^{-\sigma_t t}$



 ξ





Homogeneous media: Recipe:

- Generate random number
- Compute PDF







Volumetric PT for Homogeneous Volumes

Color <u>vPT</u>(**x**, ω) $tmax = nearestSurface(\mathbf{x}, \omega)$ $t = -\log(1 - randf()) / \sigma_t // Sample free path$ **if** t < tmax: // Volume interaction $\mathbf{x} += t \star \boldsymbol{\omega}$ pdf $t = \sigma_t * \exp(-\sigma_t * t)$ $(\omega', pdf_{\omega'}) = samplePF(\omega)$ return Tr(t) / $pdf_t * (\sigma_a * L_e(\mathbf{x}, \omega) + \sigma_s * PF(\omega, \omega') * vPT(\mathbf{x}, \omega') / pdf \omega')$ else: // Surface interaction \mathbf{x} += tmax * $\boldsymbol{\omega}$ $Pr_tmax = exp(-\sigma_t * tmax)$ $(\omega', pdf_{\omega'}) = sampleBRDF(\mathbf{n}, \omega)$ return Tr(*tmax*) / Pr_tmax * ($L_e(\mathbf{x}, \omega)$ + BRDF(ω, ω') * vPT(\mathbf{x}, ω') / pdf_ ω')

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)} \left[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)\frac{f_p(\vec{\omega},\vec{\omega}_i)L(\mathbf{x}_t,\vec{\omega}_i)}{p(\vec{\omega}_i)} \right] + \frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z)$$





Volumetric PT for Homogeneous Volumes

Color <u>vPT</u>(**x**, ω) $tmax = nearestSurface(\mathbf{x}, \omega)$ $t = -\log(1 - randf()) / \sigma_t // Sample free path$ **if** t < tmax: // Volume interaction $\mathbf{X} += t \star \boldsymbol{\omega}$ $pdf_t = \sigma_t * exp(-\sigma_t * t)$ $(\omega', pdf_{\omega'}) = samplePF(\omega)$ // Note: transmittance and PF cancel out with PDFs except for a constant factor $1/\sigma_t$ return Tr(t) / $pdf_t * (\sigma_a * L_e(\mathbf{x}, \omega) + \sigma_s * PF(\omega, \omega') * vPT(\mathbf{x}, \omega') / <math>pdf_\omega')$ else: // Surface interaction \mathbf{x} += tmax * $\boldsymbol{\omega}$ $Pr_tmax = exp(-\sigma_t * tmax)$ $(\omega', pdf_{\omega'}) = sampleBRDF(\mathbf{n}, \omega)$ // Note: transmittance and prob of sampling the distance cancel out return Tr(*tmax*) / Pr tmax * ($L_e(\mathbf{x}, \omega)$ + BRDF(ω, ω') * vPT(\mathbf{x}, ω') / pdf ω')

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)} \left[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)\frac{f_p(\vec{\omega},\vec{\omega}_i)L(\mathbf{x}_t,\vec{\omega}_i)}{p(\vec{\omega}_i)} \right] + \frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z)$$





Volumetric PT for Homogeneous Volumes

Color <u>vPT</u>(**x**, ω) $tmax = nearestSurface(\mathbf{x}, \omega)$ $t = -\log(1 - randf()) / \sigma_t // Sample free path$ if t < tmax: // Volume interaction $\mathbf{X} += t \star \boldsymbol{\omega}$ $pdf_t = \sigma_t * exp(-\sigma_t * t)$ $(\omega', pdf_{\omega'}) = samplePF(\omega)$ // Note: transmittance and PF cancel out with PDFs except for a constant factor $1/\sigma_t$ return $\sigma_a/\sigma_t * L_e(\mathbf{x}, \omega) + \sigma_s/\sigma_t * vPT(\mathbf{x}, \omega')$ else: // Surface interaction \mathbf{x} += tmax * $\boldsymbol{\omega}$ $Pr_tmax = exp(-\sigma_t * tmax)$ $(\omega', pdf_{\omega'}) = sampleBRDF(\mathbf{n}, \omega)$ // Note: transmittance and prob of sampling the distance cancel out return $L_e(\mathbf{x}, \omega) + BRDF(\omega, \omega') * vPT(\mathbf{x}, \omega') / pdf_{\omega'}$

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)} \left[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \right]$$

 $+\sigma_s(\mathbf{x}_t)\frac{f_p(\vec{\omega},\vec{\omega}_i)L(\mathbf{x}_t,\vec{\omega}_i)}{p(\vec{\omega}_i)}\Big]+\frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z,\vec{\omega})$





What about heterogeneous media?



- Heterogeneous media: $T_r(t)$
- Closed-form solutions exist only for simple media
 - e.g. linearly or exponentially varying extinction
- Other solutions:
 - Regular tracking (3D DDA)
 - Ray marching
 - Delta tracking

$$t) = e^{\int_0^t -\sigma_t(s)ds}$$

y for simple media y varying extinction



How to sample the flight distance to the next interaction?

$$T(t) = e^{-\int_0^t \sigma_t(s) \, ds} = P(X > t)$$

$$P(X \le t) = I$$
Partition of unit



ndom variable representing flight distance





Cumulative distribution function (CDF)

$$F(t) = 1 - T(t) = 1 - e^{-t}$$

Probability density function (**PDF**)

$$p(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left(1 - \mathrm{e}^{-\tau(t)}\right) = \sigma_{\mathrm{t}}(t)\mathrm{e}^{-\tau(t)}$$

Inverted cumulative distr. function (**CDF**⁻¹) $\xi = 1 - e^{-\tau(t)}$

Solve for t

$$\int_0^t \sigma_{\mathsf{t}}(s) \, \mathrm{d}s = -\ln(1-\xi)$$

 $-\tau(t)$

Approaches for finding t: 1) ANALYTIC (closed-form CDF⁻¹) 2) SEMI-ANALYTIC (regular tracking) 3) **APPROXIMATE** (ray marching)





Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

$$\int_0^t \sigma_{\mathsf{t}}(s) \, \mathrm{d}s = -\ln(1-\xi)$$
$$\sum_{i=1}^k \sigma_{\mathsf{t},i} \, \Delta_i = -\ln(1-\xi)$$

Regular tracking:

1) Draw a random number ξ 2) While LHS < RHS

move to the next intersection

3) Find the exact location

in the last segment analytically




Ray Marching

Find the collision distance approximately

$$\int_{0}^{t} \sigma_{t}(s) \, ds = -\ln(1-\xi)$$

$$\underset{i=1}{\overset{k}{\longrightarrow}} \sigma_{t,i} \Delta = -\ln(1-\xi)$$
Constant step

Ray marching:

1) Draw a random number ξ 2) While LHS < RHS

- make a (fixed-size) step
- 3) Find the exact location
 - in the last segment analytically





Ray Marching

Find the collision distance approximately

$$\int_{0}^{t} \sigma_{t}(s) \, ds = -\ln(1-\xi)$$

$$\underset{i=1}{\overset{k}{\longrightarrow}} \sigma_{t,i} \Delta = -\ln(1-\xi)$$
Constant step

Ray marching:

1) Draw a random number ξ 2) While LHS < RHS

- make a (fixed-size) step
- 3) Find the exact location
 - in the last segment analytically









Ray Marching

Find the collision distance approximately

$$\int_{0}^{t} \sigma_{t}(s) \, ds = -\ln(1-\xi)$$

$$\underset{i=1}{\overset{k}{\longrightarrow}} \sigma_{t,i} \Delta = -\ln(1-\xi)$$
Constant step

Ray marching:

1) Draw a random number ξ 2) While LHS < RHS

- make a (fixed-size) step
- 3) Find the exact location
 - in the last segment analytically

General volume







Free-path Sampling

ANALYTIC CDF⁻¹

REGULAR TRACKING

- Efficient & simple, limited to few volumes
- Iterative, inefficient if free paths cross many boundaries
- Simple volumes Piecewise-simple (e.g. homogeneous) volumes
- Unbiased Unbiased Biased

RAY MARCHING

- Iterative, inaccurate (or inefficient) for media with high frequencies
- Any volume

Common approach: sample optical thickness, find corresponding distance



Delta Tracking

(a.k.a. Woodcock tracking, pseudo scattering, hole tracking, null-collision method,...)

Delta tracking idea Add **FICTITIOUS MATTER** to homogenize medium

- albedo: $\alpha(\mathbf{x}) = 1$
- phase function: $f_p(\omega, \omega') = \delta(\omega \omega')$























































































Impact of Majorant





Impact of Majorant

Tight majorant = GOOD (few rejected collisions)





Impact of Majorant

Loose majorant = BAD (many expensive rejected collisions)





Delta Tracking

void preprocess()

majorant = findMaximumExtinction()

void <u>sampleFreePath</u>(\mathbf{x} , $\boldsymbol{\omega}$)

 $t = \Theta$

do:

// Sample distance to next tentative collision

t += -ln(1 - randf()) / majorant

// Compute probability of a real collision

 $Pr = getExtinction(\mathbf{x} + t \star \omega) / majorant$

while Pr < randf()</pre>

return t



Delta Tracking Summary

Unbiased, see [Coleman 68] for a proof

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Mathematical Verification of a Certain Monte Carlo Sampling Technique and Applications of the Technique to Radiation Transport Problems

W. A. Coleman

Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830 Received September 27, 1967 Revised November 10, 1967

The first section of this paper is a mathematical construction of a certain Monte Carlo procedure for sampling from the distribution

 $F(X) = \int_0^X \Sigma(x) \exp\left[-\int_0^x \Sigma[v] \, dv\right] \, dx, \quad 0 \le X$

The construction begins by defining a particular random variable λ . The distribution function of λ is developed and found to be identical to F(X). The definition of λ describes the sampling procedure. Depending on the behavior of $\Sigma(x)$, it may be more efficient to sample from F(X) by obtaining realizations of λ than by the more conventional procedure described in the paper.

Section II is a discussion of applications of the technique to problems in radiation transport where F(X) is frequently encountered as the distribution function for nuclear collisions. The first application is in charged particle transport where $\Sigma(x)$ is essentially a continuous function of x. An application in complex geometries where $\Sigma(x)$ is a step function, and changes values numerous times over a mean path, is also cited. Finally, it is pointed out that the technique has been used to improve the efficiency of estimating certain quantities, such as the number of absorptions in a material.

INTRODUCTION

In certain Monte Carlo problems it is necessary to obtain realizations (sample values) of a random variable having a distribution function^a given by

$$F(X) = \int_0^X \Sigma(x) \exp\left[-\int_0^x \Sigma(v) dv\right] dx , \quad 0 \le X,$$
(1)

where $\Sigma(x)$ is any real valued function having the For each value of η define properties:

(a)
$$0 \leq \Sigma(x)$$
 for $0 \leq x$.

(b)
$$\lim_{y\to\infty} \int_0^y \Sigma(x) dx = \infty$$
.

(c) $\Sigma(x)$ is bounded; there is an M > 0 with $0 \le 1$ $\Sigma(x) \leq M$ for all x.

^aIf F(X) is a distribution function it is nondecreasing, $F(-\infty) = 0$, and $F(\infty) = 1$. Many authors refer to such unctions as cumulative distribution functions.

where

Restriction (a) ensures that F(x) is a nondecreasing function of x, while (b) ensures that $F(\infty) = 1$.

One scheme for obtaining realizations of a random variable having the distribution F(X) is as follows. Consider the random variable η which has distribution

$$F_{\eta}(Y) = \int_0^T e^{-v} dv, \quad 0 \leq Y \quad .$$

$$\theta = \phi^{-1}(\eta)$$
 ,

$$\eta = \phi(\theta) = \int_0^{\theta} \Sigma(u) du$$
.

The random variable θ has the distribution F(X)given in Eq. (1). To obtain a realization of θ one might first sample from $F_{\eta}(Y)$, realizing η_1 . Then (2)

$$\theta_1 = \phi^{-1}(\eta_1) \quad .$$

MONTE CARLO SAMPLING TECHNIQUE

Sampling from $F_{\eta}(Y)$ is common practice in Monte Carlo calculations. However, the solution of Eq. (2) for θ_1 , given η_1 , may be rather laborious.

In practice it is often easier to obtain realizations from Eq. (1) by another procedure. This procedure is described in Sec. I in terms of the definition of a certain random variable λ . whose distribution is identical to that given in Eq. (1). In most applications it is fairly easy to argue that λ must be distributed according to Eq. (1) for physical reasons. The development in Sec. I is intended to provide a mathematical perspective for understanding existing applications and to encourage recognition of new applications. Section II is a summary of three current applications.

I. DEVELOPING THE DISTRIBUTION FUNCTION FOR λ

The purpose of this section is to construct the distribution function of a random variable λ whose through G. values are the termination points of a certain random walk to be described presently. The construction is based on the following hypotheses:

A. Let $\Sigma(x)$ be as described in conjunction with the distribution in Eq. (1).

B. Let $(\xi_1, \xi_2, \ldots, \xi_n, \ldots)$ denote an infinite sequence of totally independent random variables having a common distribution function,

$$P(\xi_i \leq X) = F_{\xi}(X) = \int_0^X M \ e^{-Mx} \ dx$$
,

 $0 \leq X; i = 1, 2, \ldots$

where M is a fixed upper bound of $\Sigma(x)$. C. Define $\sigma(x) \equiv \Sigma(x)/M$ and $\alpha(x) = 1 - \sigma(x)$, where $0 \leq x$, to simplify notation.

D. Let $(\rho_1, \rho_2, \ldots, \rho_n, \ldots)$ denote an infinite sequence of totally independent random variables having a common uniform distribution function, $P(\rho_i \leq R) = F_0(R) = R, \quad 0 \leq R \leq 1;$

$$i = 1, 2, \ldots$$

E. Let $(\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots)$ denote the infinite sequence of random variables which are the cumulative sums of the ξ_i :

$$\xi_i = \sum_{j=1}^{i} \xi_j = \zeta_{i-1} + \xi_i, \quad i = 1, 2, \ldots,$$

$$\zeta_0 \equiv \xi_0 \equiv 0 \quad .$$

F. Denote the minimum value of *n* for which

$$\rho_n \leq \sigma(\zeta_n)$$
, $n = 1, 2, ...,$

G. Let λ denote the random variable ζ_N . The values of λ are defined as those values of the ζ_n for which n takes on the value N.

The hypotheses A through G form a constructive definition of λ . They describe explicitly the procedure for obtaining realizations of λ . Let x_i , r_i, z_i , and L denote realizations of ξ_i, ρ_i, ζ_i , and λ , respectively. Using this notation, the procedure is as follows:

1) Assign i the value 1, z_0 the value 0. 2) Generate x_i and r_i .

3) Calculate $z_i = z_{i-1} + x_i$.

4) If $r_i \leq \sigma(z_i)$, stop and assign L the value z_i ; otherwise increment i by 1 and proceed to step 2.

For brevity in all of the discussion that follows, the procedure outlined above will be referred to as the λ procedure. The distribution function for λ will now be constructed using the hypotheses A

Denote the event for which N = 1 and $\lambda \leq Z$ by

$$E_1 = \{\rho_1 \leq \sigma(\zeta_1), \zeta_1 \leq Z\},$$

where Z is an arbitrary fixed value in the range of λ . Similarly denote the event for which N = 2 and $\lambda \leq Z$ by

$$E_2 = \{\rho_1 > \sigma(\zeta_1), \ \rho_2 \leq \sigma(\zeta_2), \ \zeta_2 \leq Z \}.$$

This notation is extended to describe the events for general N > 1 and $\lambda \leq Z$:

$$E_n = \{\rho_1 > \sigma(\zeta_1), \rho_2 > \sigma(\zeta_2), \ldots, \rho_{n-1} > \sigma(\zeta_{n-1}), \\ \rho_n \leq \sigma(\zeta_n), \zeta_n \leq Z \}$$

The event $\{\lambda \leq Z\}$ can occur in any of the mutually exclusive ways $E_1, E_2, \ldots, E_n, \ldots$. Hence, the distribution function for λ may be written as

$$P[\lambda \leq Z] = F_{\lambda}(Z) = \sum_{n=1}^{\infty} P(E_n)$$
.

Each of the joint probabilities $P(E_n)$, n = 1, 2, ...,may be expressed in terms of the random walk increments ξ_i , i = 1, 2, ...

$$P(E_1) = P[\rho_1 \leq \sigma(\xi_1), \xi_1 \leq Z]$$

$$P(E_n) = P\left[\rho_1 > \sigma(\xi_1), \ldots, \rho_{n-1} > \sigma\left\{\sum_{i=1}^{n} \xi_i\right\}, \\ \rho_n \leq \sigma\left\{\sum_{i=1}^{n} \xi_i\right\}, \\ \xi_n \leq Z - \sum_{i=1}^{n-1} \xi_i\right].$$

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The probability that $\rho_1 \leq \sigma(\xi_1)$ and tegral of the conditional probability that $\rho_1 \leq \sigma(\xi_1)$ given $\xi_1 = x_1$ with respect to the marginal distribution $F_{\xi}(x_1)$:

$$P(E_1) = \int_0^Z P[\rho_1 \le \sigma(\xi_1) | \xi_1 = x_1] dF_{\xi}(x_1) = \int_0^Z \sigma(x_1) M e^{-Mx_1} dx_1 \quad . \tag{4}$$

Similarly,

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$$P(E_{n}) = \int_{0}^{Z} \int_{0}^{Z-x_{1}} \dots \int_{0}^{Z-\frac{n-1}{2}x_{i}} P\left[\rho_{1} > \sigma(\xi_{1}), \dots, \rho_{n-1} > \sigma\left\{\sum_{i=1}^{n-1} \xi_{i}\right\}\right] ,$$

$$\rho_{n} \leq \sigma\left\{\sum_{i=1}^{n} \xi_{i}\right\} \mid \xi_{1} = x_{1}, \dots, \xi_{n} = x_{n} dF_{\xi_{1}}, \dots, \xi_{n}(x_{1}, \dots, x_{n}) .$$
(5)

 $F_{\xi_1}, \ldots, \xi_n(x_1, \ldots, x_n)$ denotes the joint distribution function of the variables ξ_1, \ldots, ξ_n . The integral limits in Eq. (5) are determined by first noting that $0 < \xi_i$, and, hence $\zeta_{i-1} < \zeta_i$, $i = 1, 2, \ldots$ For the event E_n to occur, it is necessary that $\zeta_n < Z$, which implies $\zeta_1 < \ldots < \zeta_n < Z$. In terms of ξ_i , it is necessary that $\xi_i < Z - \sum_{j=1}^{i-1} \xi_j$ for $i = 2, 3, \ldots, n$.

Since $\rho_1, \rho_2, \ldots, \rho_n$ are totally independent, the integrand in Eq. (5) is equal to $\binom{n-1}{n-1}$

$$P[\rho_1 > \sigma(\xi_1) | \xi_1 = x_1] \dots P\left[\rho_{n-1} > \sigma\left\{\sum_{i=1}^n \xi_i\right\} | \xi_1 = x_1, \dots, \xi_{n-1} = x_{n-1}\right]$$
$$\times P\left[\rho_n \leq \sigma\left\{\sum_{i=1}^n \xi_i\right\} | \xi_1 = x_1, \dots, \xi_n = x_n\right]$$

Also ξ_1, \ldots, ξ_n are totally independent and have a common distribution

$$F_{\xi_{1},\ldots,\xi_{n}}(x_{1},\ldots,x_{n}) = F_{\xi_{1}}(x_{1})\ldots F_{\xi_{n}}(x_{n}) = F_{\xi}(x_{1})\ldots F_{\xi}(x_{n}) .$$

$$P(E_n) = \int_0^Z \int_0^{Z-x_1} \dots \int_0^{Z-\sum_{i=1}^X i} P[\rho_1 > \sigma(\xi_1) | \xi_1 = x_1]$$

$$\dots P[\rho_{n-1} > \sigma \left\{ \sum_{i=1}^{n-1} \xi_i \right\} | \xi_1 = x_1, \dots, \xi_{n-1} = x_{n-1}]$$

$$\times P\left[\rho_n \le \sigma \left\{ \sum_{i=1}^n \xi_i \right\} | \xi_1 = x_1, \dots, \xi_n = x_n \right] dF_{\xi}(x_1) \dots dF_{\xi}(x_n)$$

$$= \int_0^Z \int_0^{Z-x_1} \dots \int_0^{Z-\sum_{i=1}^n x_i} [1 - \sigma(x_1)]$$

(3)

and

It is convenient to proceed with the probabilities expressed in terms of the variables
$$\zeta_1, \ldots, \zeta_n$$
.
The transformations from ξ_1, \ldots, ξ_n are direct. Introducing $\alpha(x)$ for brevity, the expressions for $P(E_1)$ and $P(E_n)$, $n \ge 2$, become

$$P(E_1) = \int_0^2 \sigma(z_1) M e^{mz_1} dz_1$$
,

$$P(E_n) = \int_0^Z dz_n M^n \sigma(z_n) e^{-Mz_n} \int_0^{z_n} dz_{n-1} \alpha(z_{n-1}) \int_0^{z_{n-1}} dz_{n-2} \dots \int_0^{z_2} dz_1 \alpha(z_1)$$

$$= \int_0^Z dz_1 \sigma(z_1) M^n e^{-Mz_1} \int_0^{z_1} dz_2 \alpha(z_2) \int_0^{z_2} \dots \int_0^{z_{n-1}} dz_n \alpha(z_n) .$$

 $\dots \left[1 - \sigma \left\{\sum_{i=1}^{n-1} x_i\right\}\right] \sigma \left\{\sum_{i=1}^n x_i\right\} Me^{-Mx_1} \dots Me^{-Mx_n}$

¹See for example, WILLIAM FELLER, An Introduction to Probability Theory and its Applications, Vol. II, p. 154 ff (1966).

MONTE CARLO SAMPLING TECHNIQUE

It will now be proved that

$$\int_0^{z_1} dz_2 \, \alpha(z_2) \, \int_0^{z_2} \ldots \, \int_0^{z_{n-1}} dz_n \, \alpha(z_n) \, = \, \frac{\left[\int_0^{z_1} \alpha(v) \, dv\right]^{n-1}}{(n-1)!}$$

Equation (7) is true for n = 2 by inspection. For n = 3,

$$\int_{0}^{z_{1}} dz_{2} \alpha(z_{2}) \int_{0}^{z_{2}} dz_{3} \alpha(z_{3}) = \int_{0}^{z_{1}} dz_{2} \frac{d}{dz_{2}} \frac{\left[\int_{0}^{z_{2}} \alpha(z) dz\right]^{2}}{2} = \frac{\left[\int_{0}^{z_{1}} \alpha(z) dz\right]^{2}}{2}$$

Assuming Eq. (7) to be true for arbitrary n, it can be shown to hold for n + 1 by multiplying Eq. (7) by $\alpha(z_1)$ and integrating from 0 to z.

$$\int_{0}^{z} dz_{1} \alpha(z_{1}) \int_{0}^{z_{1}} dz_{2} \dots \int_{0}^{z_{n-1}} dz_{n} \alpha(z_{n}) = \int_{0}^{z} dz_{1} \alpha(z_{1}) \left[\frac{\int_{0}^{z_{1}} dz_{n}}{(n-1)} \right]_{0}^{z_{n-1}} dz_{n-1} \alpha(z_{n}) = \int_{0}^{z} dz_{1} \alpha(z_{1}) \left[\frac{\int_{0}^{z_{1}} dz_{n}}{(n-1)} \right]_{0}^{z_{n-1}} dz_{n-1} \alpha(z_{n}) dz_{n-1} = \frac{\left[\int_{0}^{z} \alpha(v) dv \right]^{n}}{n!} = \frac{\left[\int_{0}^{z} \alpha(v) dv \right]^{n}}{n!}$$

It follows that Eq. (7) holds for arbitrary $n \ge 2$. Substituting the identity [Eq. (7)] into Eq. (6) gives

$$P(E_n) = \int_0^Z dz_1 \,\sigma(z_1) \, M^n \, e^{-Mz_1} \, \frac{\left[\int_0^{z_1} \alpha(v) \, dv\right]^{n-1}}{(n-1)!} \quad , \quad 2$$

Equation (3) becomes

$$P(\lambda \leq Z) = \sum_{n=1}^{\infty} P(E_n) = P(E_1) + \sum_{n=2}^{\infty} \int_0^Z dz_1 \sigma(z_1) M^n e^{-Mz_1} \frac{\left[\int_0^{z_1} \alpha(v) dv\right]^{n-1}}{(n-1)!}$$
$$= \sum_{n=0}^{\infty} \int_0^Z dz_1 \sigma(z_1) M e^{-Mz_1} \frac{\left[M \int_0^{z_1} \alpha(v) dv\right]^n}{n!}$$
$$= \int_0^Z dz_1 \sigma(z_1) M e^{-Mz_1} \exp\left[M \int_0^{z_1} \alpha(v) dv\right] = \int_0^Z dz_1 M \sigma(z_1) \exp\left[\int_0^Z \Sigma(v) dv\right] dz .$$

Hence λ has the distribution

II. APPLICATIONS OF THE TECHNIQUE TO RADIATION TRANSPORT PROBLEMS

The λ procedure described in Sec. I is useful as a Monte Carlo technique in solving certain transport problems. This section is a summary of three situations in which the λ procedure has been utilized. In each case $\Sigma(z)$ is a nuclear cross section and z is a relative position variable to be determined. The value of $\Sigma(z)$ determines the relative frequency of nuclear collisions per unit of particle track length

High-Energy Charged Particle Transport

of type p undergoing a nonelastic collision with a Denote the macroscopic nonelastic cross section, stationary nucleus of type N depends upon p, N, under these conditions, by $\Sigma_1(z)$. The distance

and the kinetic energy E of the incident particle. The kinetic energy of a charged particle varies between nuclear events due to interactions with electrons. For the more massive charged particles, such as protons and alphas, the kinetic energy is usually assumed to be a continuous. decreasing function of position. Consider a material composed uniformly of one nuclear species N. Assume the kinetic energy E_0 of a particular type of particle p at a position $z_0 = 0$ is known. The kinetic energy E of p at an arbitrary point $z > z_0$ is a function of z.

$E = f_{p,N,E_0}(z) \quad .$

The macroscopic cross section for a particle Each of the variables p, N, and E_0 has been fixed.

(6)

$$(x - x) = E(x) - E(x) - E(x) - E(x)$$

$$F_{\xi_1,\ldots,\xi_n}(x_1,\ldots,x_n) = F_{\xi_1}(x_1)$$

$$\int_{0}^{Z} dz_{1} \sigma(z_{1}) M e^{-Mz_{1}} \underbrace{\left[M \int_{0}^{z} \alpha(v) dv \right]}_{n!}$$

$$z_{1} \sigma(z_{1}) M e^{-Mz_{1}} \exp \left[M \int_{0}^{z_{1}} \alpha(v) dv \right]$$

$$(z) \exp \left[- \int_{0}^{z} \Sigma(v) dv \right] dz \quad .$$
ution function given in Eq. (1).

COLEMAN
$$\xi_1 \leq Z$$
 may be expressed as the interval





 $\leq n$.



Delta Tracking Summary

Unbiased, see [Coleman 68] for a proof

Majorant extinction

- defines the combined homogeneous volume
- must bound the real extinction
- loose majorants lead to many fictitious collisions

