Participating media
Course announcements

• Take-home quiz 6 posted, due Tuesday 3/14 at 23:59 (after spring break).

• Programming assignment 3 posted, due Friday 3/17 at 23:59 (after spring break).
  - How many of you have looked at/started/finished it?
  - Any questions?

• Suggest topics for Friday’s reading group on Piazza.
Overview of today’s lecture

• Participating media.

• Scattering material characterization.

• Volume rendering equation.

• Ray marching.

• Volumetric path tracing.

• Delta tracking.
Slide credits

Most of these slides were directly adapted from:

• Wojciech Jarosz (Dartmouth).
Fog
Clouds & Crepuscular rays
Fire
Underwater
Surface or Volume?

source: Flickr
Antelope Canyon, Az.
Aerial (Atmospheric) Perspective

Wojciech Jarosz

Henrik Wann Jensen
Thus, if one is to be five times as distant, make it five times bluer.

—Treatise on Painting, Leonardo Da Vinci, pp 295, circa 1480.
Nebula
Emission
Absorption
Scattering

http://coclouds.com
Defining Participating Media

Typically, we do not model particles of a medium explicitly (wouldn’t fit in memory, completely impractical to ray trace)

The properties are described statistically using various coefficients and densities

- Conceptually similar idea as microfacet models
Defining Participating Media

Homogeneous:

- Infinite or bounded by a surface or simple shape
Defining Participating Media

Heterogeneous (spatially varying coefficients):
- Procedurally, e.g., using a noise function
- Simulation + volume discretization, e.g., a voxel grid
Radiance

The main quantity we are interested in for rendering is radiance.

Previously: radiance *remains constant* along rays between surfaces.

\[
L_i(x, \bar{\omega}) = L_0(y, -\bar{\omega}) \\
y = r(x, \bar{\omega})
\]
Radiance

The main quantity we are interested in for rendering is radiance

Now: radiance may *change* along rays between surfaces

\[
L_i(x, \bar{\omega}) \neq L_o(y, -\bar{\omega})
\]

\[
y = r(x, \bar{\omega})
\]
Participating Media
How much light is *lost/gained* along the differential beam due to interactions of light with the medium?
Differential Beam Segment

\[ \bar{\omega} \]

Outgoing light

\[ dz \]

\[ dA \]

Incoming light
Absorption

\[ dL(x, \omega) = -\sigma_a(x) L(x, \omega) dz \]

\(\sigma_a(x)\) : absorption coefficient \( [m^{-1}]\)
Out-scattering

\[ dL(x, \bar{\omega}) = -\sigma_s(x)L(x, \bar{\omega})dz \]

\( \sigma_s(x) \): scattering coefficient \( [m^{-1}] \)
In-scattering

\[ dL(x, \omega) = \sigma_s(x) L_s(x, \omega) dz \]

- \( \sigma_s(x) \) : scattering coefficient \([m^{-1}]\)
- \( L_s(x, \omega) \) : in-scattered radiance
Emission

\[ dL(x, \vec{\omega}) = \sigma_a(x) L_e(x, \vec{\omega}) \, dz \]

*Sometimes modeled without the absorption coefficient just by specifying a “source” term

* \( \sigma_a(x) \): absorption coefficient \([m^{-1}]\)

* \( L_e(x, \vec{\omega}) \): emitted radiance
Radiative Transfer Equation (RTE)

\[
dL(x, \omega) = \frac{-\sigma_a(x) L(x, \omega) dz - \sigma_s(x) L(x, \omega) dz}{\sigma_a(x) L_e(x, \omega) dz + \sigma_s(x) L_s(x, \omega) dz}
\]
Losses (Extinction)

Absorption
\[ dL(x, \bar{\omega}) = -\sigma_a(x) L(x, \bar{\omega}) dz - \sigma_s(x) L(x, \bar{\omega}) dz \]
\[ = -\sigma_t(x) L(x, \bar{\omega}) dz \]

\( \sigma_t(x) \) : extinction coefficient \([m^{-1}]\)

: total loss of light per unit distance

What about a beam with a finite length?
Extinction Along a Finite Beam

\[ dL(x, \bar{\omega}) = -\sigma_t(x)L(x, \bar{\omega})dz \quad // \text{Assume constant } \sigma_t(x), \text{ reorganize} \]

\[ \frac{dL(x, \bar{\omega})}{L(x, \bar{\omega})} = -\sigma_t dz \quad // \text{Integrate along beam from 0 to } z \]

\[ \ln(L_z) - \ln(L_0) = -\sigma_t z \]

\[ \ln \left( \frac{L_z}{L_0} \right) = -\sigma_t z \quad // \text{Exponentiate} \]

\[ \frac{L_z}{L_0} = e^{-\sigma_t z} \]
Beer-Lambert Law

Expresses the remaining radiance after traveling a finite distance through a medium with constant extinction coefficient

The fraction is referred to as the *transmittance*

Think of this as fractional visibility between points

\[
\frac{L_z}{L_0} = e^{-\sigma_t z}
\]

- Radiance at distance \(z\)
- Radiance at the beginning of the beam
Transmittance

Homogeneous volume:

\[ T_r(x, y) = e^{-\sigma_t \|x - y\|} \]

Heterogeneous volume (spatially varying \( \sigma_t \)):

\[ T_r(x, y) = e^{-\int_0^\|x - y\| \sigma_t(t) \, dt} \]

Optical thickness
Transmittance

Homogeneous volume:

\[ T_r(x, y) = e^{-\sigma_t ||x-y||} \]

Heterogeneous volume (spatially varying \( \sigma_t \)):

\[ T_r(x, y) = e^{-\int_0^{||x-y||} \sigma_t(t) dt} \]

Transmittance is multiplicative:

\[ T_r(x, z) = T_r(x, y)T_r(y, z) \]
Radiative Transfer Equation (RTE)

\[ dL(x, \bar{\omega}) = -\sigma_a(x)L(x, \bar{\omega})dz - \sigma_s(x)L(x, \bar{\omega})dz + \sigma_a(x)L_e(x, \bar{\omega})dz + \sigma_s(x)L_s(x, \bar{\omega})dz \]
Volume Rendering Equation

\[ L(x, \omega) = T_{r}(x, x_z) L(x_z, \omega) \]

Reduced (background) surface radiance
Volume Rendering Equation

\[ L(x, \omega) = T_r(x, x_z)L(x_z, \omega) \]

\[ + \int_0^z T_r(x, x_t)\sigma_a(x_t)L_e(x_t, \omega)dt \]

Accumulated emitted radiance
Volume Rendering Equation

\[ L(x, \bar{\omega}) = T_r(x, x_z) L(x_z, \bar{\omega}) \]

\[ + \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \bar{\omega}) dt \]

\[ + \int_0^z T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \bar{\omega}) dt \]

Accumulated in-scattered radiance
Volume Rendering Equation

\[ L(x, \vec{\omega}) = T_r(x, x_z) L(x_z, \vec{\omega}) + \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \vec{\omega}) dt + \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \vec{\omega}', \vec{\omega}) L_i(x_t, \vec{\omega}') d\vec{\omega}' dt \]

Accumulated in-scattered radiance
Volume Rendering Equation

\[ L(x, \bar{\omega}) = T_r(x, x_z) L(x_z, \bar{\omega}) \]

\[ + \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \bar{\omega}) dt \]

\[ + \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega}) L_i(x_t, \bar{\omega}') d\bar{\omega}' dt \]
Scattering in Media
Phase Function $f_p$

Describes distribution of scattered light

Analog of BRDF but for scattering in media

Integrates to unity (unlike BRDF)

$$\int_{S^2} f_p(x, \omega', \omega') d\omega' = 1$$

Why do we have this property?

*We will use the same convention that phase function direction vectors always point away from the shading point $x$. Many publications, however, use a different convention for phase functions, in which direction vectors “follow” the light, i.e. one direction points towards $x$ and the other away from $x$. When reading papers, be sure to clarify the meaning of the vectors to avoid misinterpretation.*
Isotropic Scattering

Uniform scattering, analogous to Lambertian BRDF

\[ f_p(\vec{\omega'}, \vec{\omega}) = \frac{1}{4\pi} \]

Where does this value come from?
Anisotropic Scattering

Quantifying anisotropy \((g\), “average cosine”\):

\[
g = \int_{S^2} f_p(x, \tilde{\omega}', \tilde{\omega}) \cos \theta \, d\tilde{\omega}'
\]

where:

\[
\cos \theta = -\tilde{\omega} \cdot \tilde{\omega}'
\]

- \(g = 0\) : isotropic scattering (on average)
- \(g > 0\) : forward scattering
- \(g < 0\) : backward scattering
Henyey-Greenstein Phase Function

Anisotropic scattering

\[ f_{\text{PHG}}(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g \cos \theta)^{3/2}} \]

\begin{align*}
g < 0 & \quad \text{(Left diagram)} \\
g = 0 & \quad \text{(Middle diagram)} \\
g > 0 & \quad \text{(Right diagram)}
\end{align*}
Henyey-Greenstein Phase Function

Anisotropic scattering

\[ f_{\text{HG}}(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g \cos \theta)^{3/2}} \]

\[ g = -0.5 \quad \text{g} = 0 \quad \text{g} = 0.8 \]
Henyey-Greenstein Phase Function

Linear plot

Log plot
Henyey-Greenstein Phase Function

Empirical phase function

Introduced for intergalactic dust

Very popular in graphics and other fields
Schlick’s Phase Function

Empirical phase function

Faster approximation of HG

\[ f_{p, \text{Schlick}}(\theta) = \frac{1}{4\pi} \frac{1 - k^2}{(1 - k \cos \theta)^2} \]

\[ k = 1.55g - 0.55g^3 \]
Schlick’s Phase Function

Empirical phase function

Faster approximation of HG
Lorenz-Mie Scattering

If the diameter of scatterers is on the order of light wavelength, we cannot neglect the wave nature of light.

Solution to Maxwell’s equations for scattering from any spherical dielectric particle.

Explains many phenomena.

Complicated:

- Solution is an infinite analytic series.
Lorenz-Mie Phase Function

Sphere diameter = 1μm

Sphere diameter = 10μm

Sphere diameter = 100μm

Data obtained from http://www.philiplaven.com/mieplot.htm
Rainbows
Lorenz-Mie Approximations

Hazy atmosphere

\[ f_{p\text{ hazy}}(\theta) = \frac{1}{4\pi} \left( 5 + \left( \frac{1 + \cos \theta}{2} \right)^8 \right) \]

Murky atmosphere

\[ f_{p\text{ murky}}(\theta) = \frac{1}{4\pi} \left( 17 + \left( \frac{1 + \cos \theta}{2} \right)^{32} \right) \]
Lorenz-Mie Approximations

Hazy atmosphere

\[ f_{p \text{ hazy}}(\theta) = \frac{1}{4\pi} \left( 5 + \left( \frac{1 + \cos \theta}{2} \right)^8 \right) \]

Murky atmosphere

\[ f_{p \text{ murky}}(\theta) = \frac{1}{4\pi} \left( 17 + \left( \frac{1 + \cos \theta}{2} \right)^{32} \right) \]
Rayleigh Scattering

Approximation of Lorenz-Mie for tiny scatterers that are typically smaller than 1/10th the wavelength of visible light

Used for atmospheric scattering, gasses, transparent solids

Highly wavelength dependent
Rayleigh Phase Function

\[ f_{p\text{Rayleigh}}(\theta) = \frac{3}{16\pi} (1 + \cos^2 \theta) \]

Scattering at right angles is half as likely as scattering forward or backward.
Rayleigh Scattering

\[ \sigma_{\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2 \]

- Wavelength
- Index of refraction
- Density of scatterers
- Diameter of scatterers
Rayleigh Scattering

\[ \sigma_{sRayleigh}(\lambda, d, \eta, \rho) = \rho \frac{2\pi^5 d^6}{3\lambda^4} \left( \frac{\eta^2 - 1}{\eta^2 + 2} \right)^2 \]
Examples
Examples

Steam

Forward scattering

Smoke

Backward scattering
Examples

Isotropic scattering
Examples

Forward scattering
Why is the Sky Blue?
Why is the Sunset Red?

Atmosphere

Earth
Rayleigh Scattering
Media Properties (Recap)

Given:
- Absorption coefficient $\sigma_a(x)$ $[m^{-1}]$
- Scattering coefficient $\sigma_s(x)$ $[m^{-1}]$
- Phase function $f_p(x, \bar{\omega}', \bar{\omega})$ $[sr^{-1}]$

Derived:
- Extinction coefficient $\sigma_t(x) = \sigma_a(x) + \sigma_s(x)$ $[m^{-1}]$
- Albedo $\alpha(x) = \sigma_s(x)/\sigma_t(x)$ [none]
- Mean-free path $1/\sigma_t(x)$ [m]
- Transmittance $T_r(x,y) = e^{-\int_{0}^{||x-y||} \sigma_t(t)dt}$ [none]
Homogeneous Isotropic Medium

Given:
- Absorption coefficient \( \sigma_a \) \( [m^{-1}] \)
- Scattering coefficient \( \sigma_s \) \( [m^{-1}] \)
- Phase function \( \frac{1}{4\pi} \) \( [sr^{-1}] \)

Derived:
- Extinction coefficient \( \sigma_t = \sigma_a + \sigma_s \) \( [m^{-1}] \)
- Albedo \( \alpha = \sigma_s / \sigma_t \) \( [\text{none}] \)
- Mean-free path \( 1 / \sigma_t \) \( [m] \)
- Transmittance \( T_r(x, y) = e^{-\sigma_t ||x-y||} \) \( [\text{none}] \)
What is this?
Crepuscular Rays
Anti-Crepuscular Rays
Crepuscular rays from space
Solving the Volume Rendering Equation
Complexity Progression

homogeneous vs. heterogeneous

scattering

- none
- fake ambient
- single
- multiple
Volume Rendering Equation

\[ L(x, \bar{\omega}) = T_r(x, x_z) L(x_z, \bar{\omega}) \]

\[ + \int_0^Z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \bar{\omega}) dt \]

\[ + \int_0^Z T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \bar{\omega}) dt \]

Accumulated in-scattered radiance

Attenuated background radiance

Accumulated emitted radiance
Purely absorbing media

\[ L(x, \omega) = T_r(x, x_z) L(x_z, \omega) \]

Attenuated background radiance
Fog
Participating Media

\[
L(x, \bar{\omega}) = \int_0^s T_r(x \leftrightarrow x_t) \sigma_s(x_t) L_i(x_t, \bar{\omega}) \, dt + T_r(x \leftrightarrow x_s) L(x_s, \bar{\omega})
\]

\[
L(x, \bar{\omega}) = \sigma_s \int_0^s T_r(x \leftrightarrow x_t) L_i(x_t, \bar{\omega}) \, dt + T_r(x \leftrightarrow x_s) L(x_s, \bar{\omega})
\]

\[
L(x, \bar{\omega}) = \sigma_s \int_0^s e^{-t \sigma_t} L_i(x_t, \bar{\omega}) \, dt + e^{-s \sigma_t} L(x_s, \bar{\omega})
\]
\[ L(x, \omega) = \sigma_s \int_0^s e^{-t\sigma_t} L_t(x_t, \omega) \, dt + e^{-s\sigma_t} L(x_s, \omega) \]
Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant:

\[ L(x, \omega) = \sigma_s \int_0^s e^{-t \sigma_t} L_i(x_t, \omega) \, dt + e^{-s \sigma_t} L(x_s, \omega) \]
Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant:

\[
L(x, \tilde{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} L_i(x_t, \tilde{\omega}) \, dt + e^{-s\sigma_t} L(x_s, \tilde{\omega})
\]

\[
L(x, \tilde{\omega}) = \sigma_s L_i \left( \int_0^s e^{-t\sigma_t} \, dt \right) + e^{-s\sigma_t} L(x_s, \tilde{\omega})
\]

\[
L(x, \tilde{\omega}) = \sigma_s L_i \left( \frac{1 - e^{-s\sigma_t}}{\sigma_t} \right) + e^{-s\sigma_t} L(x_s, \tilde{\omega})
\]

\[
L(x, \tilde{\omega}) = \text{lerp} \left( \frac{\sigma_s L_i}{\sigma_t}, L(x_s, \tilde{\omega}), e^{-s\sigma_t} \right)
\]
OpenGL Fog
Fog
Volume Rendering Equation

\[ L(x, \bar{\omega}) = T_r(x, x_z) L(x_z, \bar{\omega}) \]
\[ + \int_0^z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \bar{\omega}) dt \]
\[ + \int_0^z T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \bar{\omega}) dt \]

Accumulated in-scattered radiance
In-scattered Radiance

\[ L(x, \bar{\omega}) = \int_0^z T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \bar{\omega}) \, dt \]

\[ L_s(x_t, \bar{\omega}) = \int_{S^2} f_p(x_t, \bar{\omega}, \bar{\omega}') L_i(x_t, \bar{\omega}') \, d\bar{\omega}' \]

Single scattering

- \( L_i \) arrives directly from a light source (direct illum.)
  i.e.:
  \[ L_i(x, \bar{\omega}) = T_r(x, r(x, \bar{\omega})) L_e(r(x, \bar{\omega}), -\bar{\omega}) \]

Multiple scattering

- \( L_i \) arrives through multiple bounces (indirect illum.)
Single Scattering

\[ L(x, \tilde{\omega}) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \tilde{\omega}', \tilde{\omega}) T_r(x_t, x_e) L_e(x_e, -\tilde{\omega}') d\tilde{\omega}' dt \]
Single Scattering

\[
L(x, \bar{\omega}) = \int_{0}^{z} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega}) T_r(x_t, x_e) L_e(x_e, -\bar{\omega}') d\bar{\omega}' dt
\]
Single Scattering

\[ L(x, \bar{\omega}) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega}) T_r(x_t, x_e) L_e(x_e, -\bar{\omega}') d\bar{\omega}' dt \]
Single Scattering

\[ L(x, \bar{\omega}) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega}) T_r(x_t, x_e) L_e(x_e, -\bar{\omega}') d\bar{\omega}' dt \]
Single Scattering

\[ L(x, \bar{\omega}) = \int_{0}^{z} T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega}) T_r(x_t, x_e) L_e(x_e, -\bar{\omega}') d\bar{\omega}' dt \]

(Semi-)analytic solutions:
- Sun et al. [2005]
- Pegoraro et al. [2009, 2010]

Numerical solutions:
- Ray-marching
- Equiangular sampling
Analytic Single Scattering

\[
L(x, \bar{\omega}) = \int_0^z T_r(x, x_t) \sigma_s(x_t) \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega}) T_r(x_t, x_e) L_e(x_e, -\bar{\omega}') d\bar{\omega}' dt
\]

Assumptions:

- Homogeneous medium
- Point or spot light
- Relatively simple phase function
- No occlusion

\[
L(x, \bar{\omega}) = \frac{\Phi}{4\pi} \frac{1}{4\pi} \sigma_s \int_0^z e^{-\sigma_t \|x, x_t\|} \frac{e^{-\sigma_t \|x_t, x_p\|}}{\|x_t, x_p\|^2} dt
\]
Analytic Single Scattering
Analytic Single Scattering
Analytic Single Scattering

\[ L_m(x_a, x_b, \hat{\omega}) = \frac{\kappa_s}{\hbar} e^{\kappa_s (x_a - x_h)} 2^N \sum_{n=0}^{N-1} c(n) \sum_{k=0}^{2n} d(n, k) \int_{v_a}^{v_b} \frac{e^{-H_v}}{(v^2 + 1)^{n+1}} v^k dv \]

\[ \int \frac{e^{a\nu}}{(v^2 + 1)^m} v^n dv = \frac{1}{2^{m-1}} \sum_{l=0}^{m-1} \frac{1}{2^l} \binom{m-1+l}{m-1} \left( \sum_{k=0}^{\min\{m-1-l, n\}} \binom{n}{k} \frac{a^{m-1-l-k}}{(m-1-l-k)!} E(a, v, m-n-l+k) \right) \]

\[ -e^{a\nu} \sum_{j=1}^{m-1-l-k} \frac{(j-1)!}{(m-1-l-k)!} \frac{a^{m-1-l-k-j}}{(v^2 + 1)^j} \sum_{i=(m-n-l+k-j) \mod 2}^{\leq j} \binom{-1}{n-m-l+k-i}^2 \binom{j}{i} v^i \]

\[ + e^{a\nu} \sum_{k=0}^{n-m+l} \binom{n}{k} \sum_{j=0}^{n-m+l-k} \frac{1}{j!} (-a)^{n-m+l-k-j} \sum_{i=(m+n+l-k-j) \mod 2}^{\leq j} \binom{-1}{m+n+l+k-i}^2 \binom{j}{i} v^i \]

No shadows, implementation nightmare, computationally intensive...

Let’s try brute force!
Ray-Marching

\[ L(x, \bar{\omega}) = \int_0^z T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \bar{\omega}) dt \]

Approximate with Riemann sum
Ray-Marching

\[ L(x, \bar{\omega}) \approx \sum_{i=1}^{N} T_r(x, x_{t,i}) \sigma_s(x_{t,i}) L_s(x_{t,i}, \bar{\omega}) \Delta t \]
Ray-Marching

\[ L(x, \bar{\omega}) \approx \sum_{i=1}^{N} T_r(x, x_{t,i}) \sigma_s(x_{t,i}) L_s(x_{t,i}, \bar{\omega}) \Delta t \]
Ray-Marching

\[ L(x, \omega) \approx \sum_{i=1}^{N} T_r(x, x_{t,i}) \sigma_s(x_{t,i}) L_s(x_{t,i}, \omega) \Delta t \]

Homogeneous volume: \[ T_r(x, x_{t,i}) = e^{-\sigma_t \|x, x_{t,i}\|} \]
Ray-Marching

\[ L(x, \bar{\omega}) \approx \sum_{i=1}^{N} T_r(x, x_{t,i}) \sigma_s(x_{t,i}) L_s(x_{t,i}, \bar{\omega}) \Delta t \]

Heterogeneous volume: \[ T_r(x, x_{t,i}) = T_r(x, x_{t,i-1}) e^{-\sigma_t(x_{t,i}) \Delta t} \]

Assume constant extinction along each segment.
Ray-Marching

\[ L(x, \omega) \approx \sum_{i=1}^{N} T_r(x, x_{t,i}) \sigma_s(x_{t,i}) L_s(x_{t,i}, \omega) \Delta t \]
Ray-Marching

\[ L_s(x_t, \bar{\omega}) = \int_{S^2} f_p(x_t, \bar{\omega}', \bar{\omega}) L_i(x_t, \bar{\omega}') d\bar{\omega}' \]
Ray-Marching

\[ L_s(x_t, \bar{\omega}) \approx \frac{1}{M} \sum_{j=0}^{M} \frac{f_p(x_t, \bar{\omega}', \bar{\omega})}{p(\bar{\omega}_j')} L_i(x_t, \bar{\omega}_j') \]
Ray-Marching

Single scattering:

\[ L_i(x_t, \vec{\omega}) = T_r(x_t, x_e) L_e(x_e, -\vec{\omega}) \]

Another ray-marching needed to estimate the transmittance along the connection ray (in heterogeneous media)
Ray-Marching in Heterogeneous Media

Marching towards the light source
- Connections are expensive, many, and uniformly distributed along the primary ray
Decoupled Transmittance and In-scattering

1. Ray-march and cache transmittance

- Choose step-size w.r.t. frequency content to accurately capture variations

Piece-wise approximation of $T_r(x, x_t)$
Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

- Distribute samples $\propto$ (part of) the integrand
Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

- Distribute samples $\propto$ (part of) the integrand

\[ p(x_t) \propto T_r(x, x_t) \]
Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

- Distribute samples $\propto$ (part of) the integrand

$$p(x_t) \propto \frac{1}{d^2}$$

$d$ : distance to light
Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

- Distribute samples $\propto$ (part of) the integrand

\[ p(x_t) \propto \frac{1}{d^2} \]

\(d\) : distance to light
Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

- Distribute samples $\propto$ (part of) the integrand

$$p(x_t) \propto \frac{1}{d^2}$$

$d$ : distance to light
Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

- Distribute samples \( \propto (\text{part of}) \) the integrand

Equi-angular sampling [Kulla and Fajardo 2012]

\[ p(x_t) \propto \frac{1}{d^2} \]

\( d \): distance to light
Decoupled Transmittance and In-scattering

Ray-marching

Equiangular sampling

Images courtesy of Kulla and Fajardo
Multiple Bounces

Same concept as in recursive Monte Carlo ray tracing, but taking into account volumetric scattering

Exponential growth:
Visual Break

Single scattering  Multiple scattering
Volumetric Path Tracing
Volumetric Path Tracing

Motivation:
- Same as with standard path tracing: avoid the exponential growth

Paths can:
- Reflect/refract off surfaces
- Scatter inside a volume
Volume Rendering Equation

\[ L(x, \omega) = \int_0^Z T_r(x, x_t) \sigma_a(x_t) L_e(x_t, \omega) \, dt + \int_0^Z T_r(x, x_t) \sigma_s(x_t) L_s(x_t, \omega) \, dt + T_r(x, x_z) L(x_z, \omega) \]

- Accumulated emitted radiance
- Accumulated in-scattered radiance
- Attenuated background radiance
Volume Rendering Equation

\[ L(x, \omega) = \int_0^Z T_r(x, x_t) \left[ \sigma_a(x_t)L_e(x_t, \omega) + \sigma_s(x_t)L_s(x_t, \omega) \right] \, dt \]

\[ + T_r(x, x_z)L(x_z, \omega) \]

Accumulated emitted + in-scattered radiance

Attenuated background radiance
Volume Rendering Equation

\[ L(x, \bar{\omega}) = \int_0^z T_r(x, x_t) \left[ \sigma_a(x_t)L_e(x_t, \bar{\omega}) + \sigma_s(x_t)L_s(x_t, \bar{\omega}) \right] \, dt + T_r(x, x_z)L(x_z, \bar{\omega}) \]
1-Sample Monte Carlo Estimator

\[
\langle L(x, \omega) \rangle = \frac{T_r(x, x_t)}{p(t)} \left[ \sigma_a(x_t) L_e(x_t, \omega) + \sigma_s(x_t) L_s(x_t, \omega) \right] \\
+ \frac{T_r(x, x_z)}{P(z)} L(x_z, \omega)
\]

- \( p(t) \) - probability density of distance \( t \)
- \( P(z) \) - probability of exceeding distance \( z \)
1-Sample Monte Carlo Estimator

\[ \langle L(x, \vec{\omega}) \rangle = \frac{T_r(x, x_t)}{p(t)} \left[ \sigma_a(x_t) L_e(x_t, \vec{\omega}) + \sigma_s(x_t) \frac{f_p(\vec{\omega}, \vec{\omega}_i) L(x_t, \vec{\omega}_i)}{p(\vec{\omega}_i)} \right] \]

\[ + \frac{T_r(x, x_z)}{P(z)} L(x_z, \vec{\omega}) \]

\[ p(t) \text{ - probability density of distance } t \]

\[ P(z) \text{ - probability of exceeding distance } z \]

\[ p(\vec{\omega}_i) \text{ - probability density of direction } \vec{\omega}_i \]
Volumetric Path Tracing

1. Sample distance to next interaction

2. Scatter in the volume or bounce off a surface
Volumetric Path Tracing with NEE
Sampling the Phase Function

Isotropic:
- Uniform sphere sampling

Henyey-Greenstein:
- Using the inversion method we can derive

\[ \cos \theta = \frac{1}{2g} \left( 1 + g^2 - \left( \frac{1 - g^2}{1 - g + 2g \xi_1} \right)^2 \right) \]

\[ \phi = 2\pi \xi_2 \]
- PDF is the value of the HG phase function
Free-path Sampling

Free-path (or free-flight distance):
- Distance to the next interaction within the medium
- Dense media (e.g. milk): short mean-free path
- Thin media (e.g. atmosphere): long mean-free path

Ideally, we want to sample proportional to (part of) integrand, e.g. transmittance:

\[
p(x_t|(x, \bar{\omega})) \propto T_r(x, x_t) \]

\[
p(t) \propto T_r(t) \text{ simplified notation for brevity} \]
Free-path Sampling

Homogeneous media: \( T_r(t) = e^{-\sigma_t t} \)

- **PDF:** \( p(t) \propto e^{-\sigma_t t} \)
  \[
  p(t) = \frac{e^{-\sigma_t t}}{\int_0^\infty e^{-\sigma_t s} \, ds} = \sigma_t e^{-\sigma_t t}
  \]

- **CDF:** \( P(t) = \int_0^t \sigma_t e^{-\sigma_t s} \, ds = 1 - e^{-\sigma_t t} \)

- **Inverted CDF:** \( P^{-1}(\xi) = -\frac{\ln(1 - \xi)}{\sigma_t} \)
Free-path Sampling

Homogeneous media:

\[ T_r(t) = e^{-\sigma t t} \]

Recipe:

- Generate random number \( \xi \)
- Sample distance \( t = -\frac{\ln(1 - \xi)}{\sigma_t} \)
- Compute PDF \( p(t) = \sigma_t e^{-\sigma_t t} \)
Free-path Sampling

Homogeneous media: \[ T_r(t) = e^{-\sigma_i t} \]

Recipe:
- Generate random number \( \xi \)
- Sample distance \( t = -\frac{\ln(1 - \xi)}{\sigma_t} = s \)
- Compute PDF \( p(t) = \sigma_t e^{-\sigma_i t} = e^{-\sigma_i s} \) Note: This is now a probability, not a probability density!

Surface hit before reaching \( t \)
Volumetric PT for Homogeneous Volumes

Color $vPT(x, \omega)$

$t_{max} = \text{nearestSurface}(x, \omega)$

$t = -\log(1 - \text{randf}()) / \sigma_t$  // Sample free path

if $t < t_{max}$:  // Volume interaction

$x += t \ast \omega$

$p_{df_t} = \sigma_t \ast \exp(-\sigma_t \ast t)$

$(\omega', p_{df_{\omega'}}) = \text{samplePF}(\omega)$

return $Tr(t) / p_{df_t} \ast (\sigma_a \ast L_e(x, \omega) + \sigma_s \ast PF(\omega, \omega') \ast vPT(x, \omega') / p_{df_{\omega'}})$

else:  // Surface interaction

$x += t_{max} \ast \omega$

$Pr_{t_{max}} = \exp(-\sigma_t \ast t_{max})$

$(\omega', p_{df_{\omega'}}) = \text{sampleBRDF}(n, \omega)$

return $Tr(t_{max}) / Pr_{t_{max}} \ast (L_e(x, \omega) + BRDF(\omega, \omega') \ast vPT(x, \omega') / p_{df_{\omega'}})$

\[
\langle L(x, \bar{\omega}) \rangle = \frac{T_r(x, x_t)}{p(t)} \left[ \sigma_a(x_t)L_e(x_t, \bar{\omega}) + \sigma_s(x_t)\frac{f_p(\bar{\omega}, \bar{\omega}_i)L(x_t, \bar{\omega}_i)}{p(\bar{\omega}_i)} \right] + \frac{T_r(x, x_z)}{P(z)}L(x_z, \bar{\omega})
\]
Volumetric PT for Homogeneous Volumes

\[
\text{Color } v\text{PT}(\mathbf{x}, \omega)
\]

\[t_{\max} = \text{nearestSurface}(\mathbf{x}, \omega)\]

\[t = -\log(1 - \text{randf}()) / \sigma_t \quad // \text{Sample free path}\]

\[
\begin{align*}
&\text{if } t < t_{\max}: // \text{Volume interaction} \\
&\quad \mathbf{x} += t \times \omega \\
&\quad pdf_t = \sigma_t \times \exp(-\sigma_t \times t) \\
&\quad (\omega', pdf_{\omega'}) = \text{samplePF}(\omega)
\end{align*}
\]

\[
// \text{Note: transmittance and PF cancel out with PDFs except for a constant factor } 1/\sigma_t
\]

\[
\text{return } T_r(t) / pdf_t \ast (\sigma_a \ast L_e(\mathbf{x}, \omega) + \sigma_s \ast \text{PF}(\omega, \omega') \ast v\text{PT}(\mathbf{x}, \omega') / pdf_{\omega'})
\]

\[
\begin{align*}
&\text{else: } // \text{Surface interaction} \\
&\quad \mathbf{x} += t_{\max} \times \omega \\
&\quad Pr_{t_{\max}} = \exp(-\sigma_t \times t_{\max}) \\
&\quad (\omega', pdf_{\omega'}) = \text{sampleBRDF}(\mathbf{n}, \omega)
\end{align*}
\]

\[
// \text{Note: transmittance and prob of sampling the distance cancel out}
\]

\[
\text{return } T_r(t_{\max}) / Pr_{t_{\max}} \ast (L_e(\mathbf{x}, \omega) + \text{BRDF}(\omega, \omega') \ast v\text{PT}(\mathbf{x}, \omega') / pdf_{\omega'})
\]

\[
\langle L(\mathbf{x}, \bar{\omega}) \rangle = \frac{T_r(\mathbf{x}, \mathbf{x}_t)}{p(t)} \left[ \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \bar{\omega}) + \sigma_s(\mathbf{x}_t) \frac{f_p(\bar{\omega}, \bar{\omega}_i) L(\mathbf{x}_t, \bar{\omega}_i)}{p(\bar{\omega}_i)} \right] + \frac{T_r(\mathbf{x}, \mathbf{x}_z)}{P(z)} L(\mathbf{x}_z, \bar{\omega})
\]

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Volumetric PT for Homogeneous Volumes

Color \( vPT(x, \omega) \)

\( t_{\text{max}} = \text{nearestSurface}(x, \omega) \)
\( t = -\log(1 - \text{randf}()) / \sigma_t \) // Sample free path

\textbf{if} \( t < t_{\text{max}} \): // Volume interaction
\( x += t \times \omega \)
\( pdf_t = \sigma_t \times \exp(-\sigma_t \times t) \)
\((\omega', pdf_{\omega'}) = \text{samplePF}(\omega) \)
\textbf{// Note: transmittance and PF cancel out with PDFs except for a constant factor 1/\( \sigma_t \)}
\textbf{return} \( \sigma_a/\sigma_t \times L_e(x, \omega) + \sigma_s/\sigma_t \times vPT(x, \omega') \)

\textbf{else:} // Surface interaction
\( x += t_{\text{max}} \times \omega \)
\( Pr_{t_{\text{max}}} = \exp(-\sigma_t \times t_{\text{max}}) \)
\((\omega', pdf_{\omega'}) = \text{sampleBRDF}(n, \omega) \)
\textbf{// Note: transmittance and prob of sampling the distance cancel out}
\textbf{return} \( L_e(x, \omega) + \text{BRDF}(\omega, \omega') \times vPT(x, \omega') / pdf_{\omega'} \)

\[ \langle L(x, \bar{\omega}) \rangle = \frac{T_r(x,x_t)}{p(t)} \left[ \sigma_a(x_t)L_e(x_t,\bar{\omega}) + \sigma_s(x_t)\frac{f_p(\bar{\omega},\bar{\omega}_i)L(x_t,\bar{\omega}_i)}{p(\bar{\omega}_i)} \right] + \frac{T_r(x,x_z)}{P(z)}L(x_z,\bar{\omega}) \]
What about heterogeneous media?
Free-path Sampling

Heterogeneous media: \[ T_r(t) = e^{\int_0^t -\sigma_t(s)ds} \]

- Closed-form solutions exist only for simple media
  - e.g. linearly or exponentially varying extinction
- Other solutions:
  - Regular tracking (3D DDA)
  - Ray marching
  - Delta tracking
Free-path Sampling

How to sample the flight distance to the next interaction?

\[ T(t) = e^{-\int_0^t \sigma_t(s) \, ds} = \begin{cases} P(X > t) \\ P(X \leq t) = F(t) \end{cases} \]

Random variable representing flight distance

CDF

Partition of unity

\[ F(t) = 1 - T(t) \]

Recipe for generating samples
Free-path Sampling

Cumulative distribution function (CDF)

\[ F(t) = 1 - T(t) = 1 - e^{-\tau(t)} \]

Probability density function (PDF)

\[ p(t) = \frac{dF(t)}{dt} = \frac{d}{dt} \left( 1 - e^{-\tau(t)} \right) = \sigma_t(t)e^{-\tau(t)} \]

Inverted cumulative distr. function (CDF⁻¹)

\[ \zeta = 1 - e^{-\tau(t)} \]

\[ \int_0^t \sigma_t(s) ds = -\ln(1 - \zeta) \]

Approaches for finding t:
1) ANALYTIC (closed-form CDF⁻¹)
2) SEMI-ANALYTIC (regular tracking)
3) APPROXIMATE (ray marching)
Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

\[ \int_{0}^{t} \sigma_t(s) \, ds = - \ln(1 - \xi) \]

\[ \sum_{i=1}^{k} \sigma_{t,i} \Delta_i = - \ln(1 - \xi) \]

Regular tracking:
1) Draw a random number \( \xi \)
2) While LHS < RHS
   move to the next intersection
3) Find the exact location
   in the last segment analytically
Ray Marching

Find the collision distance approximately

\[ \int_0^t \sigma_t(s) \, ds = -\ln(1 - \xi) \]

\[ \sum_{i=1}^k \sigma_{t,i} \Delta = -\ln(1 - \xi) \]

Ray marching:
1) Draw a random number \( \xi \)
2) While \( \text{LHS} < \text{RHS} \)
   make a (fixed-size) step
3) Find the exact location in the last segment analytically
Ray Marching

Find the collision distance approximately

$$\int_0^t \sigma_t(s) \, ds = -\ln(1 - \xi)$$

$$\sum_{i=1}^k \sigma_{t,i} \Delta = -\ln(1 - \xi)$$

Ray marching:

1) Draw a random number $\xi$
2) While LHS < RHS
   make a (fixed-size) step
3) Find the exact location
   in the last segment analytically

LHS > RHS

General volume

LHS = RHS

Sampled collision

Start
Ray Marching

Find the collision distance approximately

$$\int_0^t \sigma_t(s) \, ds = -\ln(1 - \xi)$$

$$\sum_{i=1}^k \sigma_{t,i} \Delta = -\ln(1 - \xi)$$

Ray marching:
1) Draw a random number $\xi$
2) While $LHS < RHS$
   make a (fixed-size) step
3) Find the exact location in the last segment analytically

General volume

LHS > RHS

Sampled collision $LHS= RHS$

Start
## Free-path Sampling

<table>
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<tr>
<th>ANALYTIC CDF⁻¹</th>
<th>REGULAR TRACKING</th>
<th>RAY MARCHING</th>
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<tr>
<td>Efficient &amp; simple, limited to few volumes</td>
<td>Iterative, inefficient if free paths cross many boundaries</td>
<td>Iterative, inaccurate (or inefficient) for media with high frequencies</td>
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<tr>
<td>Simple volumes (e.g. homogeneous)</td>
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<td>Any volume</td>
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<tr>
<td>Unbiased</td>
<td>Unbiased</td>
<td>Biased</td>
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</tbody>
</table>

**Common approach:** sample optical thickness, find corresponding distance
Delta Tracking

(a.k.a. Woodcock tracking, pseudo scattering, hole tracking, null-collision method,...)
Delta tracking idea

Add **FICTITIOUS MATTER** to homogenize medium

- albedo: $\alpha(\mathbf{x}) = 1$

- phase function: $f_p(\omega, \omega') = \delta(\omega - \omega')$

![Diagram of incident light, fictitious particle, and outgoing light with text: Presence of fictitious matter does not impact light transport.](image-url)
Homogenization

Volume bounds

Real particle
Homogenization
Homogenization

Volume bounds

Fictitious particle

Real particle
Homogenization
Homogenization

Volume bounds

Fictitious particle

Real particle
Homogenization

- Volume bounds
- Fictitious particle
- Real particle
Homogenization
Homogenization
Homogenization
Homogenization

Volume bounds

Fictitious particle

Real particle
Homogenization

Volume bounds
Fictitious particle
Real particle
Homogenization

Volume bounds

Real particle
Stochastic Sampling

Volume bounds

Real medium
Stochastic Sampling

\[ \overline{\sigma} = \sigma_t(x) + \sigma_n(x) \]
Stochastic Sampling

\[ P_r(x) = \frac{\sigma_t(x)}{\bar{\sigma}} \quad P_n(x) = \frac{\sigma_n(x)}{\bar{\sigma}} \]

Majorant \( \bar{\sigma} = \sigma_t(x) + \sigma_n(x) \)

\[ \ln(1 - \zeta) \]

Distance

Extinction
Stochastic Sampling

\[ P_r(x) = \frac{\sigma_t(x)}{\bar{\sigma}} \quad P_n(x) = \frac{\sigma_n(x)}{\bar{\sigma}} \]

Majorant \[ \bar{\sigma} = \sigma_t(x) + \sigma_n(x) \]
Stochastic Sampling

\[ P_r(x) = \frac{\sigma_t(x)}{\bar{\sigma}} \quad P_n(x) = \frac{\sigma_n(x)}{\bar{\sigma}} \]

Majorant \( \bar{\sigma} = \sigma_t(x) + \sigma_n(x) \)
Stochastic Sampling

\[ P_r(x) = \frac{\sigma_t(x)}{\bar{\sigma}} \quad P_n(x) = \frac{\sigma_n(x)}{\bar{\sigma}} \]

Majorant \[ \bar{\sigma} = \sigma_t(x) + \sigma_n(x) \]
Impact of Majorant

\[
\bar{\sigma} = \sigma_t(x) + \sigma_n(x)
\]
Impact of Majorant

Tight majorant = GOOD
(few rejected collisions)

$$\tilde{\sigma} = \sigma_t(x) + \sigma_n(x)$$
Impact of Majorant

Loose majorant = BAD
(many expensive rejected collisions)
void preprocess()

majorant = findMaximumExtinction()

void sampleFreePath(x, ω)

t = 0

do:

// Sample distance to next tentative collision

t += -ln(1 - randf()) / majorant

// Compute probability of a real collision

Pr = getExtinction(x + t*ω) / majorant

while Pr < randf()

return t
Delta Tracking Summary

Unbiased, see [Coleman 68] for a proof
Delta Tracking Summary

Unbiased, see [Coleman 68] for a proof

Majorant extinction

- defines the combined homogeneous volume
- must bound the real extinction
- loose majorants lead to many fictitious collisions