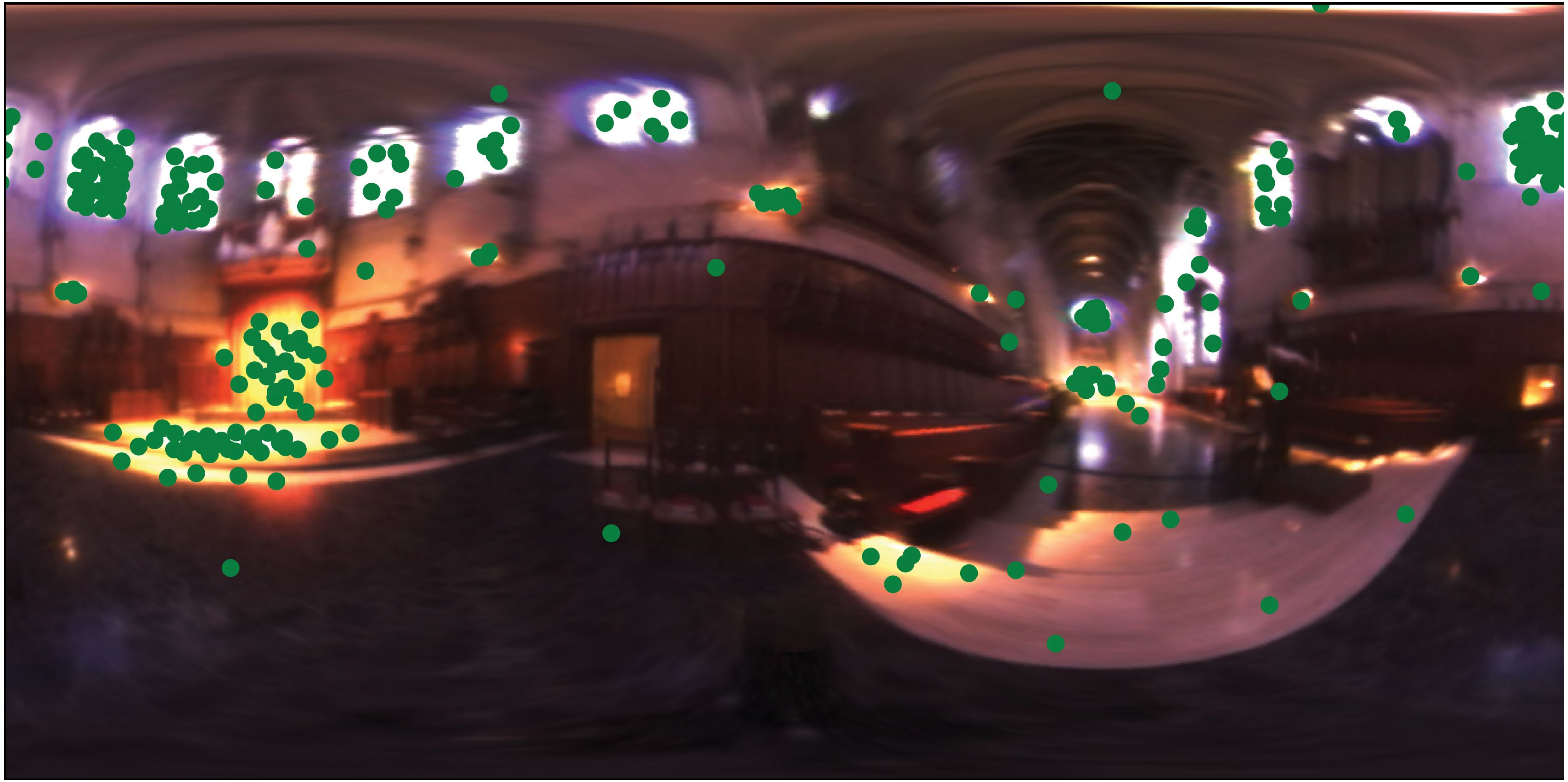
Direct illumination



15-468, 15-668, 15-868
Physics-based Rendering
Spring 2025, Lecture 10

http://graphics.cs.cmu.edu/courses/15-468

Course announcements

- Programming assignment 2 posted, due Friday 2/28 at 23:59.
  - How many of you have looked at/started/finished it?
  - Any questions?

Overview of today’s lecture

- Importance sampling the reflectance equation.
- BRDF importance sampling.
- Direct versus indirect illumination.
- Different forms of the reflectance equation.
- Environment lighting.
- Light sources.
- Mixture sampling.
- Multiple importance sampling.

Slide credits

Most of these slides were directly adapted from:

- Wojciech Jarosz (Dartmouth).

Reflection equation

$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

What terms can we importance sample?

- BRDF
- incident radiance
- cosine term

Reflection equation

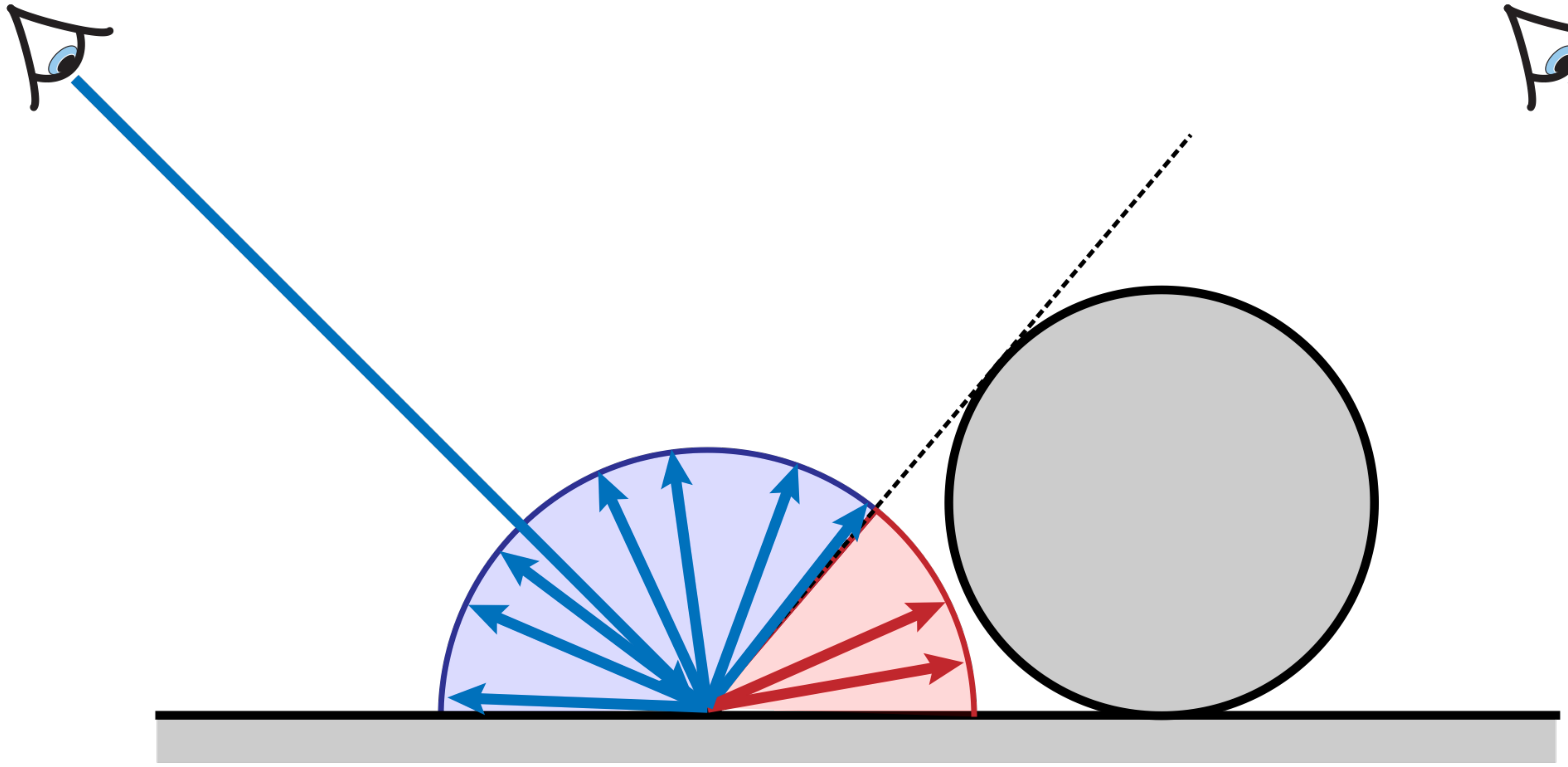
$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) L_i(\mathbf{x}, \vec{\omega}_i) \boxed{\cos\theta_i} d\vec{\omega}_i$$

What terms can we importance sample?

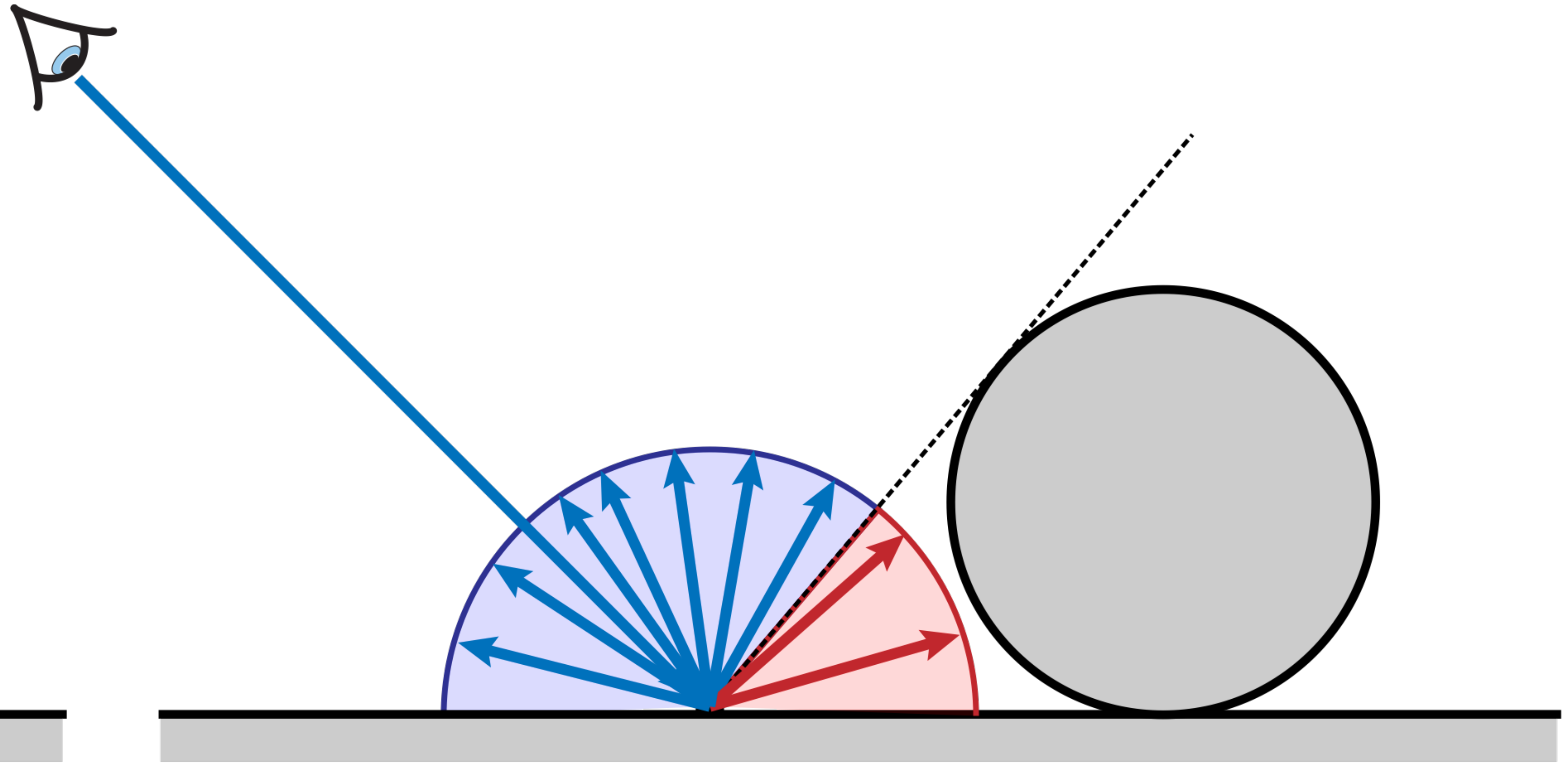
- BRDF
- incident radiance
- cosine term

This is what we did for ambient occlusion

Uniform hemispherical sampling



Cosine-weighted importance sampling



Reflection equation

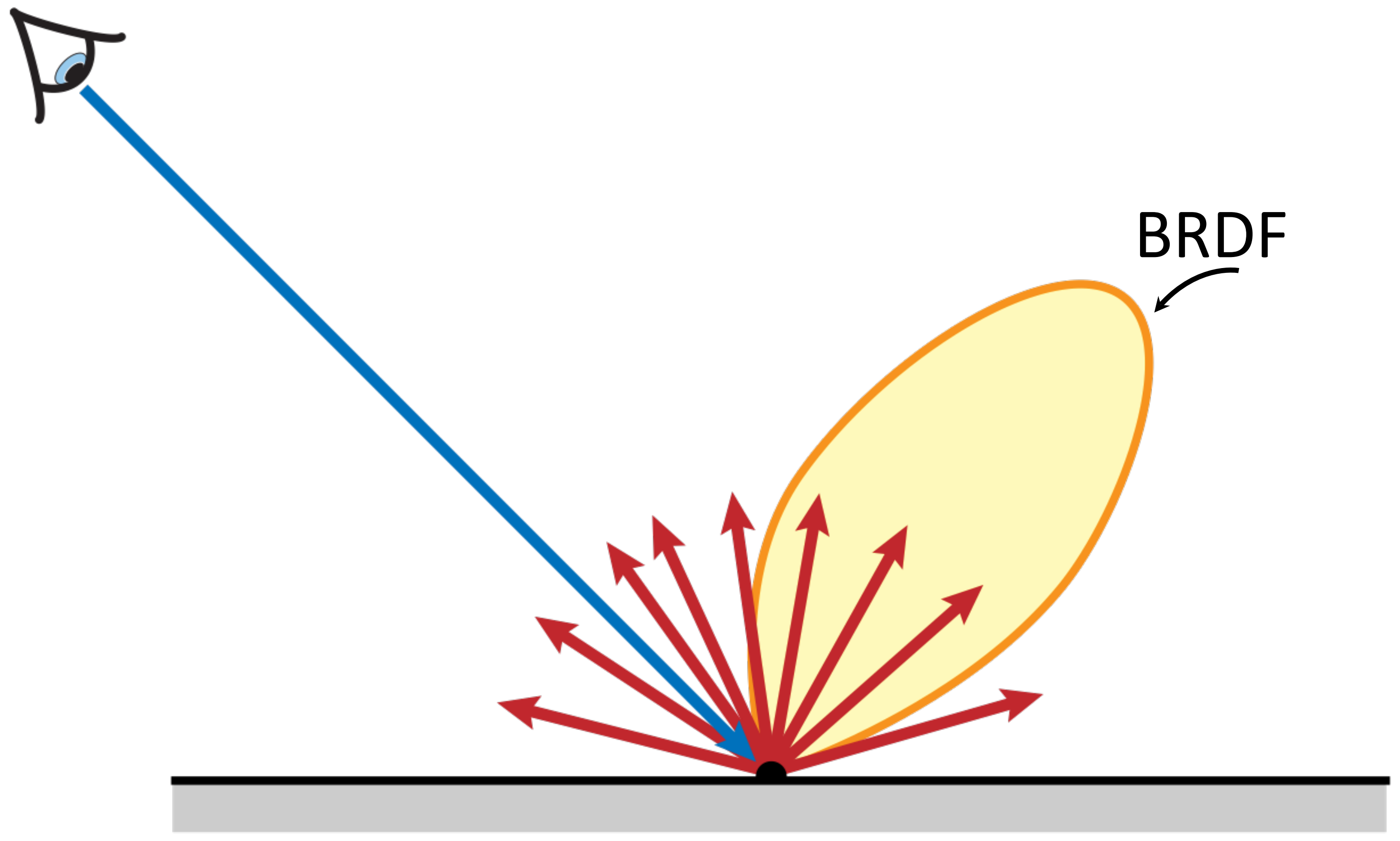
$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{H^2} \boxed{f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r)} L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

What terms can we importance sample?

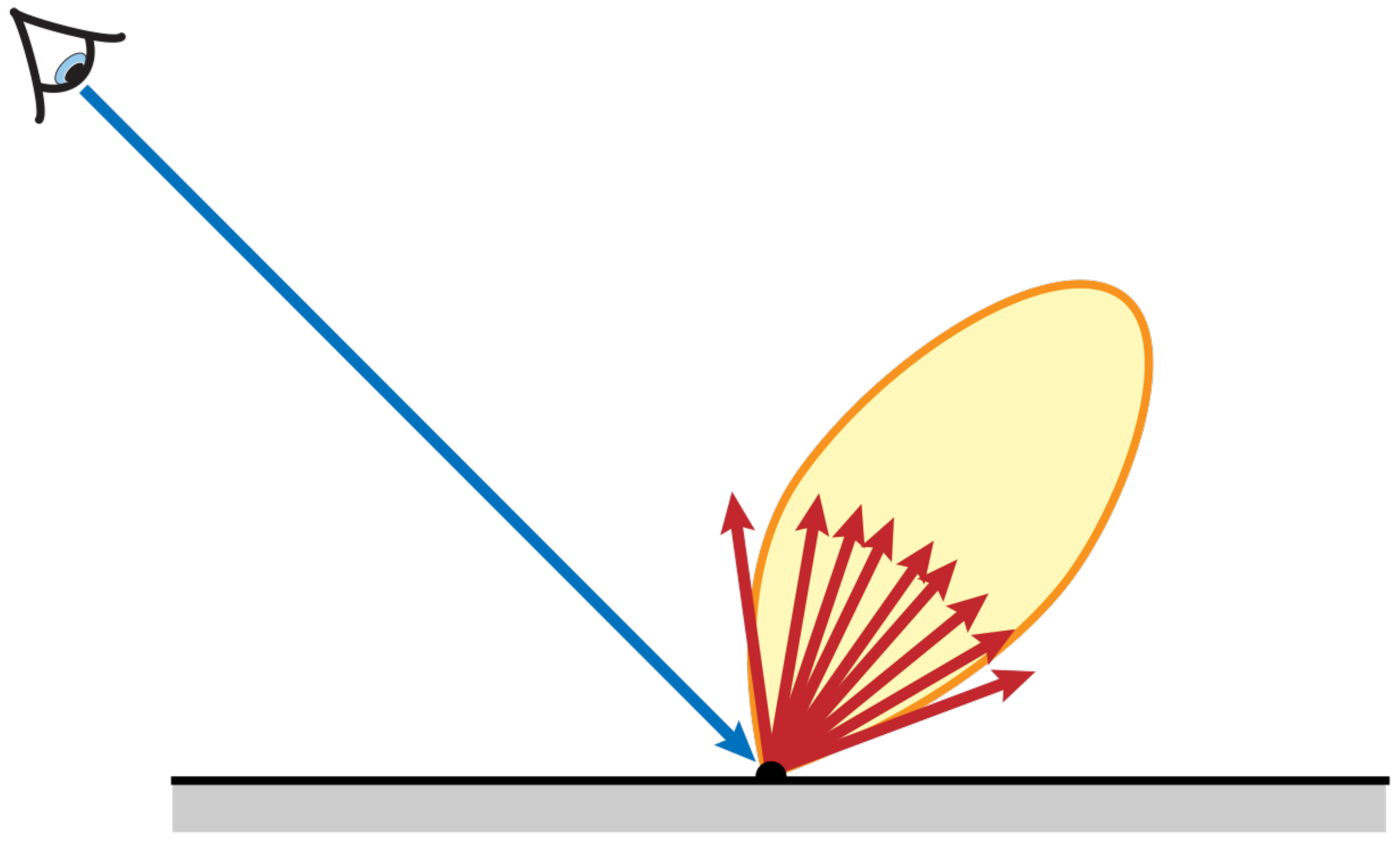
- BRDF
- incident radiance
- cosine term

Importance Sampling the BRDF

Cosine-weighted
importance sampling

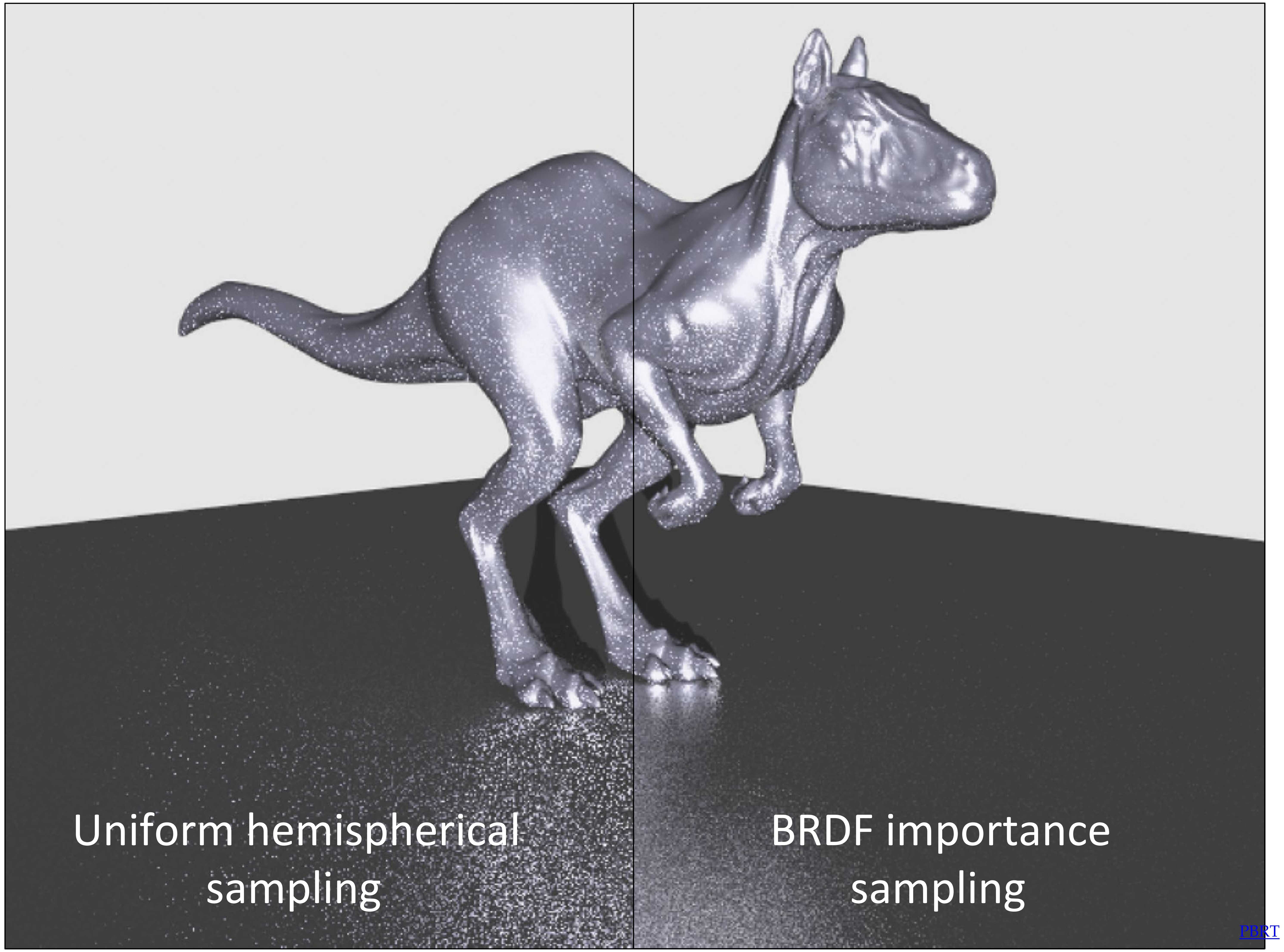


BRDF importance
sampling



$$p(\vec{\omega}_i) \underset{\sim}{\propto} f(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r)$$

Importance Sampling the BRDF



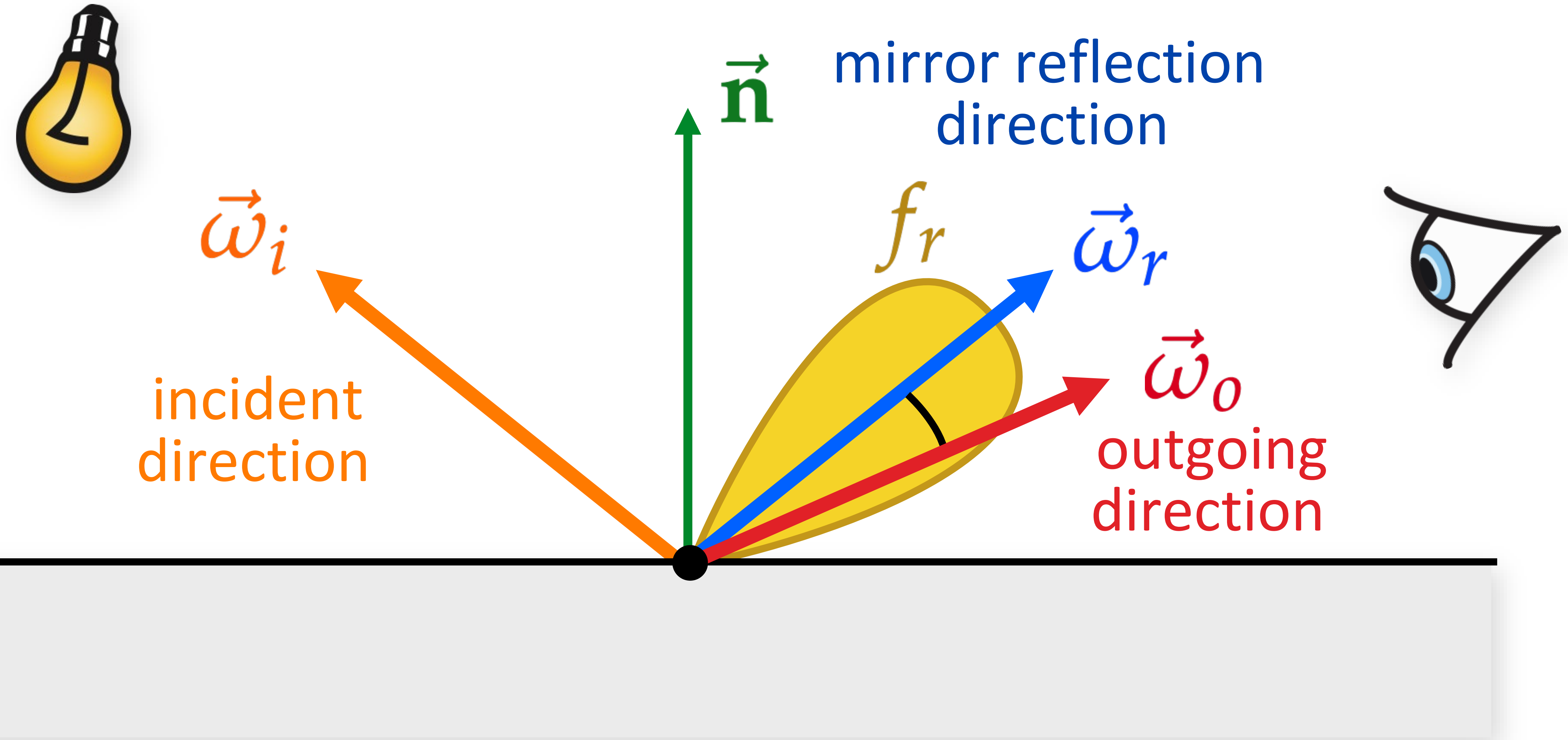
PBRT

Phong BRDF

Normalized exponentiated cosine lobe:

$$f_r(\vec{\omega}_o, \vec{\omega}_i) = \frac{e+2}{2\pi}(\vec{\omega}_r \cdot \vec{\omega}_o)^e$$

$$\vec{\omega}_r = (2\vec{\mathbf{n}}(\vec{\mathbf{n}} \cdot \vec{\omega}_i) - \vec{\omega}_i)$$



Phong BRDF

Normalized exponentiated cosine lobe:

$$f_r(\vec{\omega}_o, \vec{\omega}_i) = \frac{e+2}{2\pi}(\vec{\omega}_r \cdot \vec{\omega}_o)^e$$

$$\vec{\omega}_r = (2\vec{\mathbf{n}}(\vec{\mathbf{n}} \cdot \vec{\omega}_i) - \vec{\omega}_i)$$

Interpretation

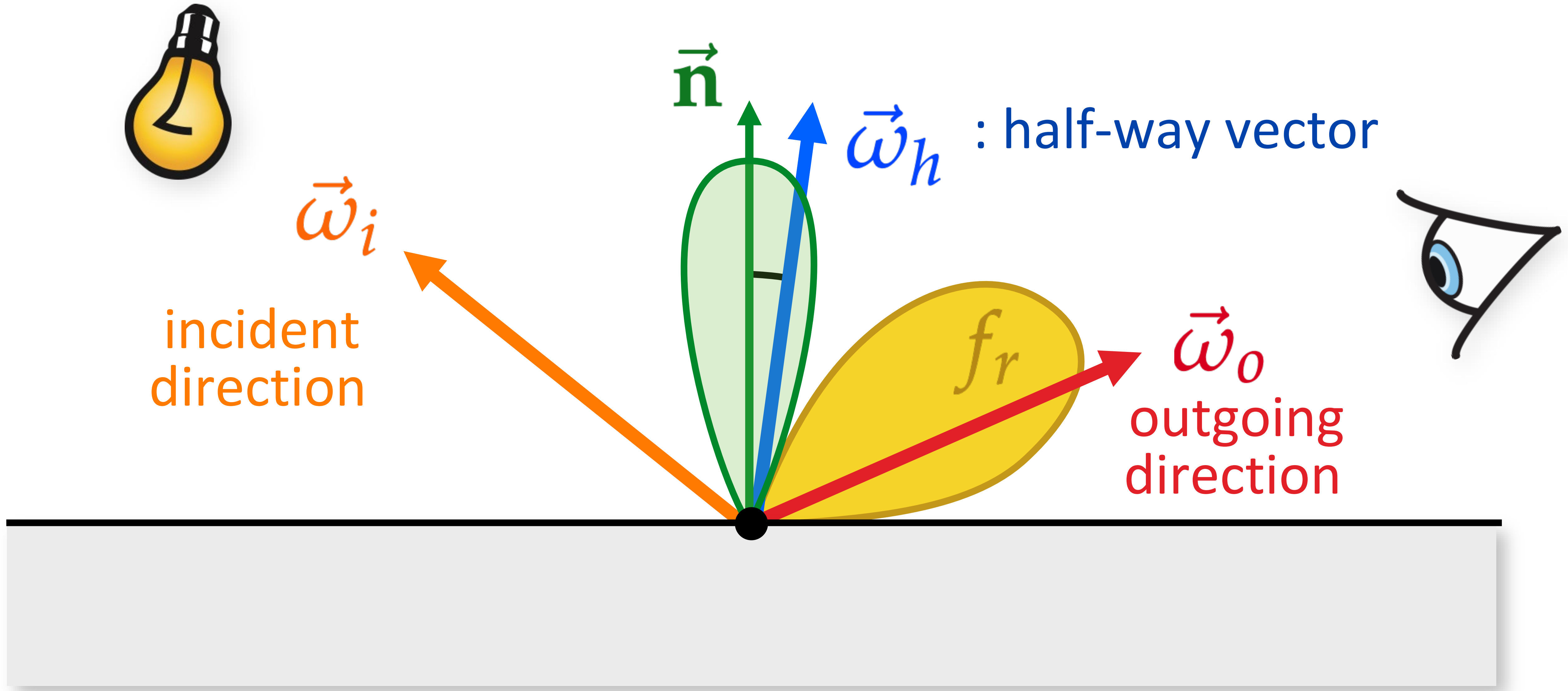
- randomize reflection rays in a lobe about mirror direction
- perfect mirror reflection of a blurred light

Blinn-Phong BRDF

Randomize normals instead of reflection directions

$$f_r(\vec{\omega}_o, \vec{\omega}_i) = \frac{e+2}{2\pi}(\vec{\omega}_h \cdot \vec{\mathbf{n}})^e$$

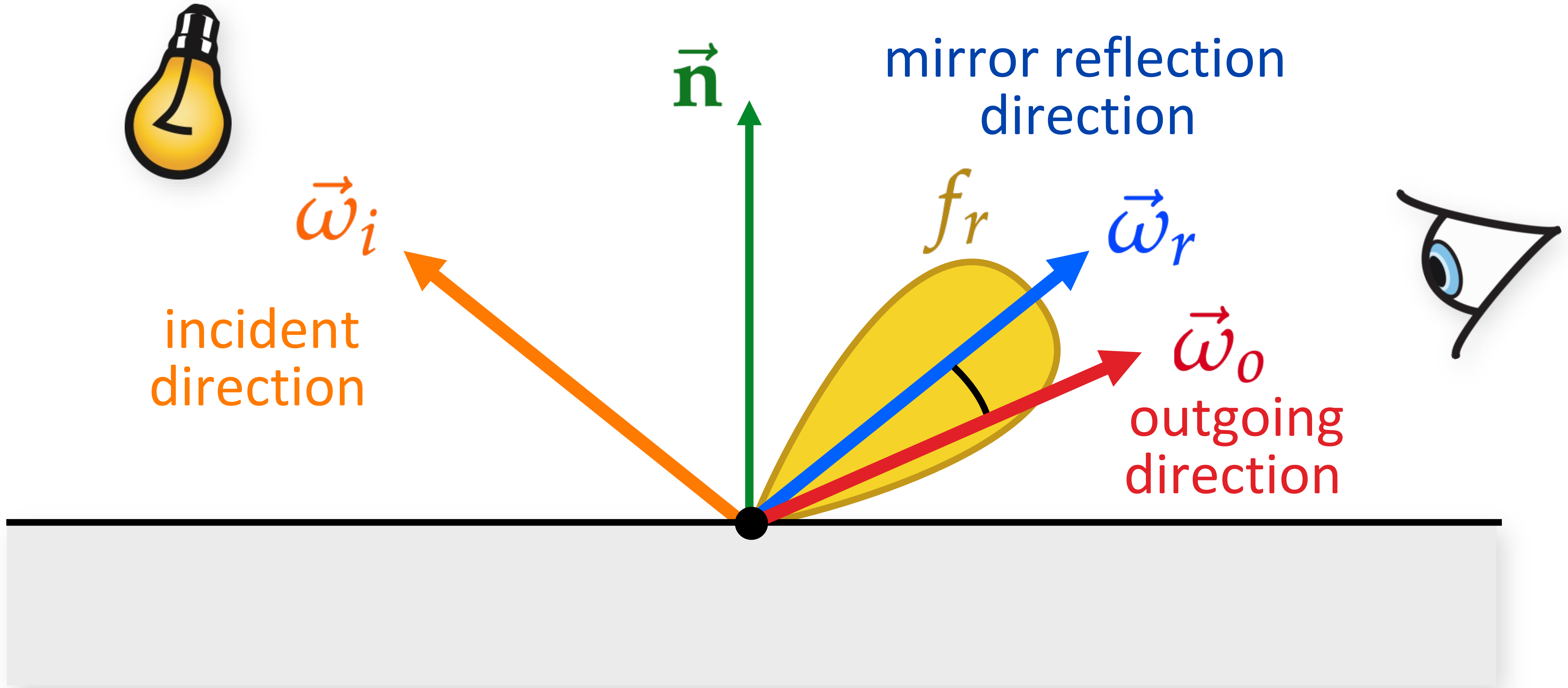
$$\vec{\omega}_h = \frac{\vec{\omega}_i + \vec{\omega}_o}{\|\vec{\omega}_i + \vec{\omega}_o\|}$$



Phong BRDF

$$f_r(\vec{\omega}_o, \vec{\omega}_i) = \frac{e+2}{2\pi}(\vec{\omega}_r \cdot \vec{\omega}_o)^e$$

$$\vec{\omega}_r = (2\vec{\mathbf{n}}(\vec{\mathbf{n}} \cdot \vec{\omega}_i) - \vec{\omega}_i)$$



Importance Sampling the BRDF

Recipe:

1. Express the desired distribution in a convenient coordinate system
    - requires computing the Jacobian
2. Compute marginal and conditional 1D PDFs
3. Sample 1D PDFs using the inversion method

Sampling the Blinn-Phong BRDF

$$f_r(\vec{\omega}_o, \vec{\omega}_i) = \frac{e+2}{2\pi}(\vec{\omega}_h \cdot \vec{\mathbf{n}})^e$$

Mirror reflection from random micro-normal

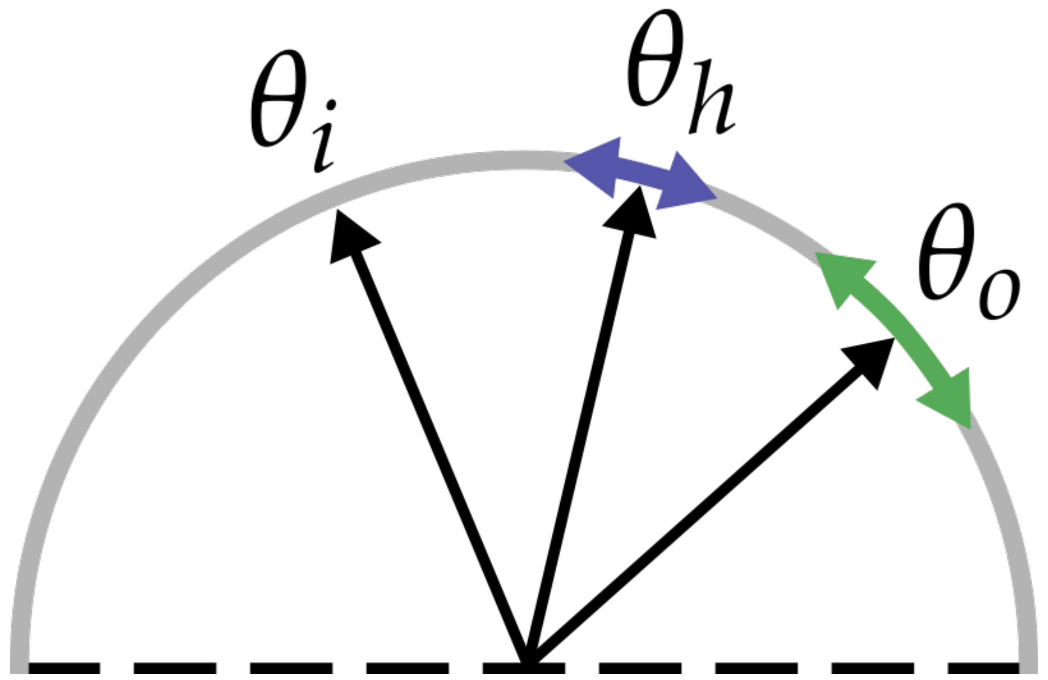
General recipe:

- randomly generate a ω_h, with PDF proportional to $\cos^e$
- reflect incident direction ω_i about ω_h to obtain ω_o
- convert PDF(ω_h) to PDF(ω_o) (change-of-variable)

Read PBRTv3 14.1

Half-direction transform

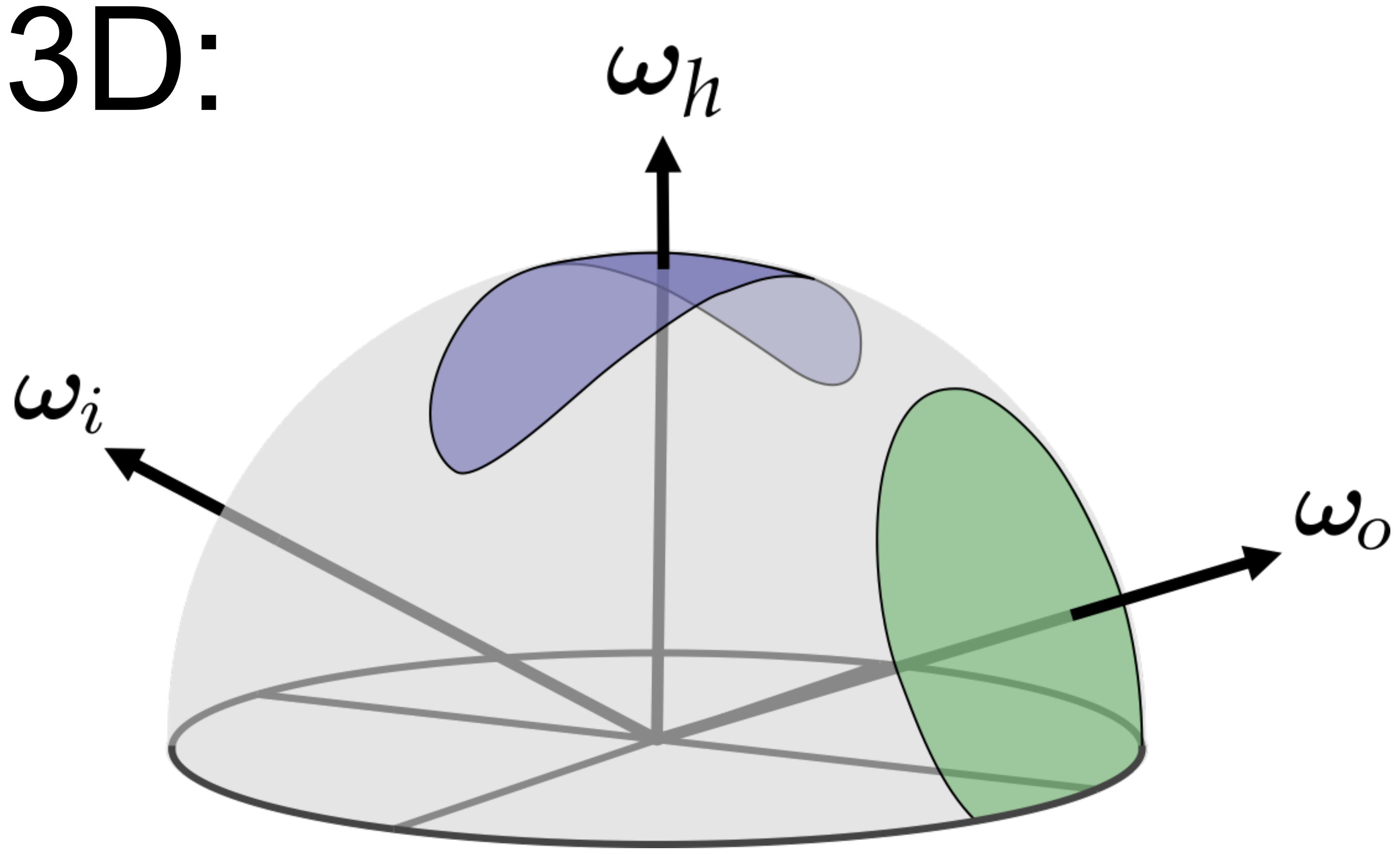
2D:



$$\theta_h := \frac{\theta_i + \theta_o}{2}$$

$$\frac{\mathrm{d}\theta_h}{\mathrm{d}\theta_o} = \text{ ?}$$

3D:



$$\boldsymbol{\omega}_h := \frac{\boldsymbol{\omega}_i + \boldsymbol{\omega}_o}{\|\boldsymbol{\omega}_i + \boldsymbol{\omega}_o\|}$$

$$\frac{\mathrm{d}\boldsymbol{\omega}_h}{\mathrm{d}\boldsymbol{\omega}_o} =$$

Reflection equation

$$L_r(\mathbf{x},\vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x},\vec{\omega}_i,\vec{\omega}_r)\boxed{L_i(\mathbf{x},\vec{\omega}_i)}\cos\theta_i \, d\vec{\omega}_i$$

What terms can we importance sample?

- BRDF
- incident radiance
- cosine term

Direct vs. Indirect illumination

Direct vs. Indirect Illumination

Where does L_i "come from"?

$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, \mathrm{d}\vec{\omega}_i$$

Direct vs. Indirect Illumination

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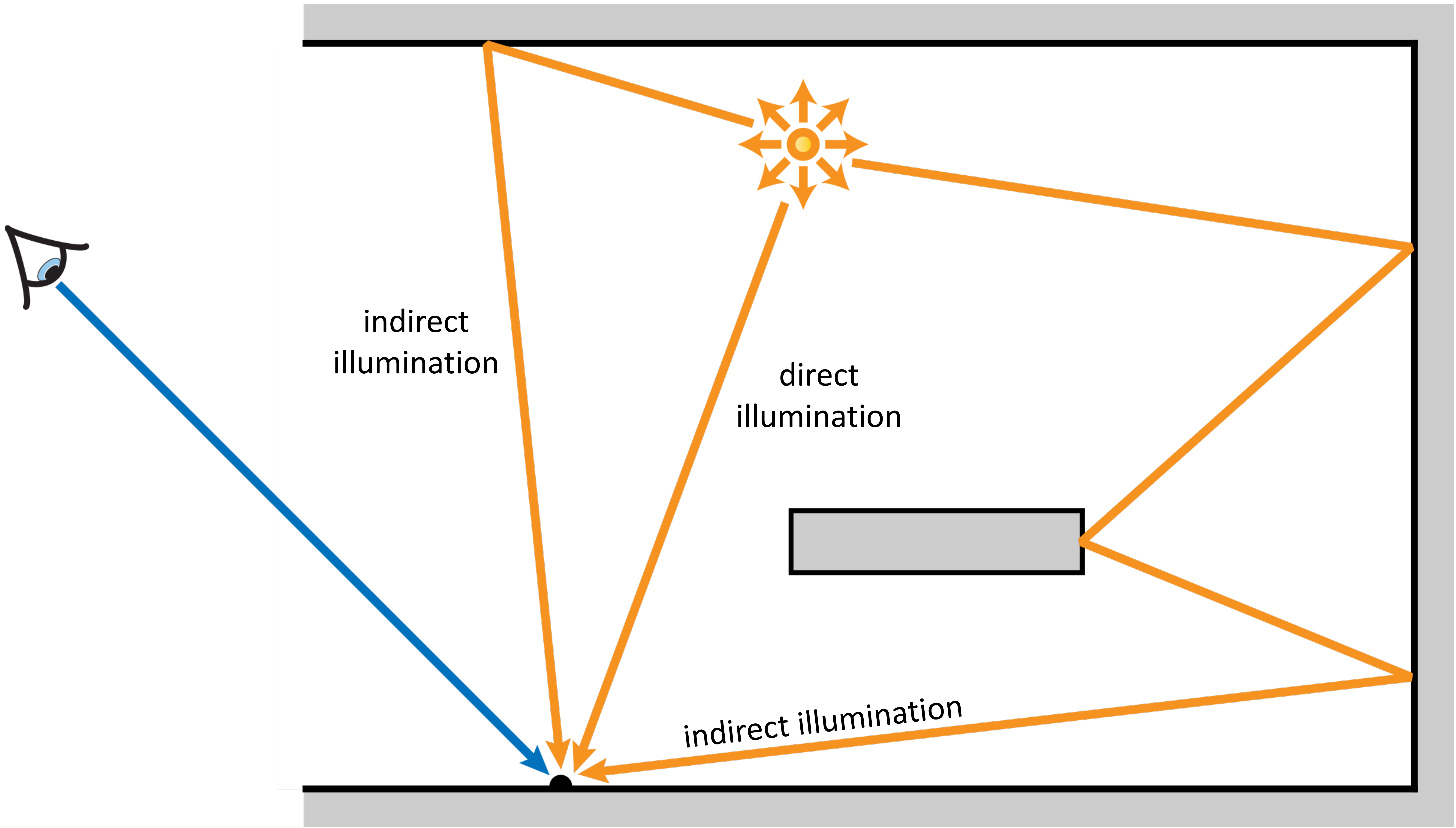
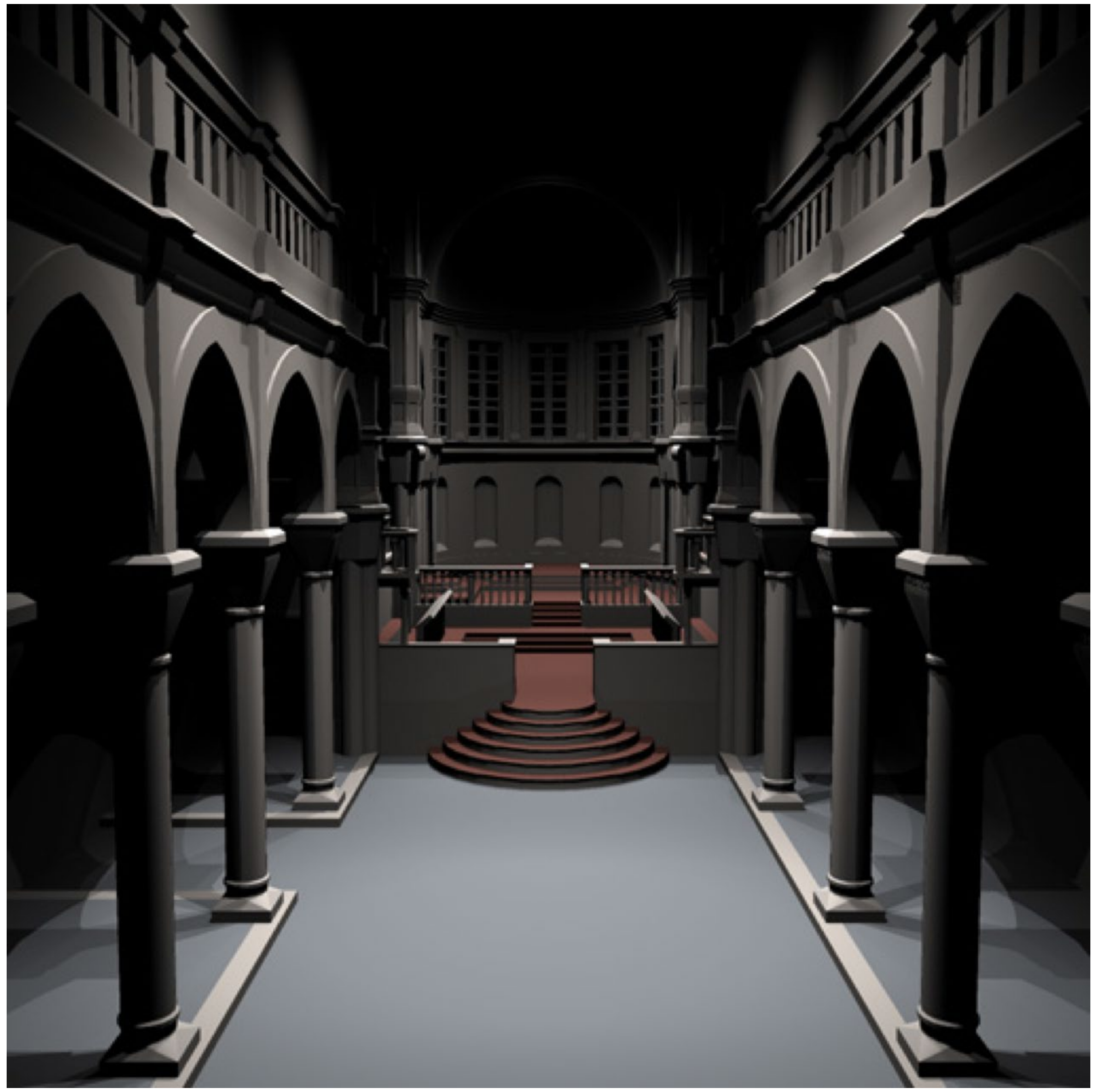


Direct vs. Indirect Illumination



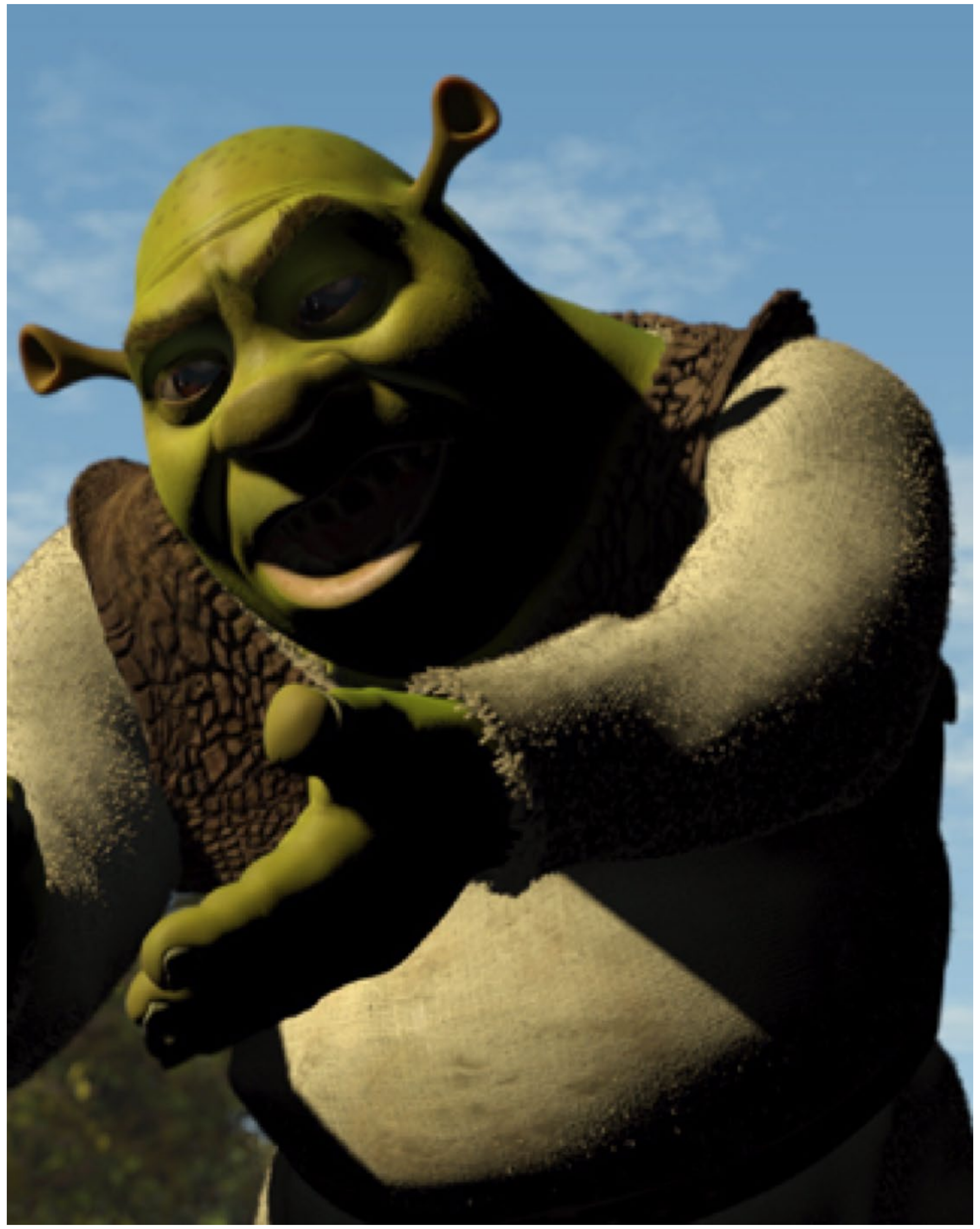
Direct illumination

Indirect illumination

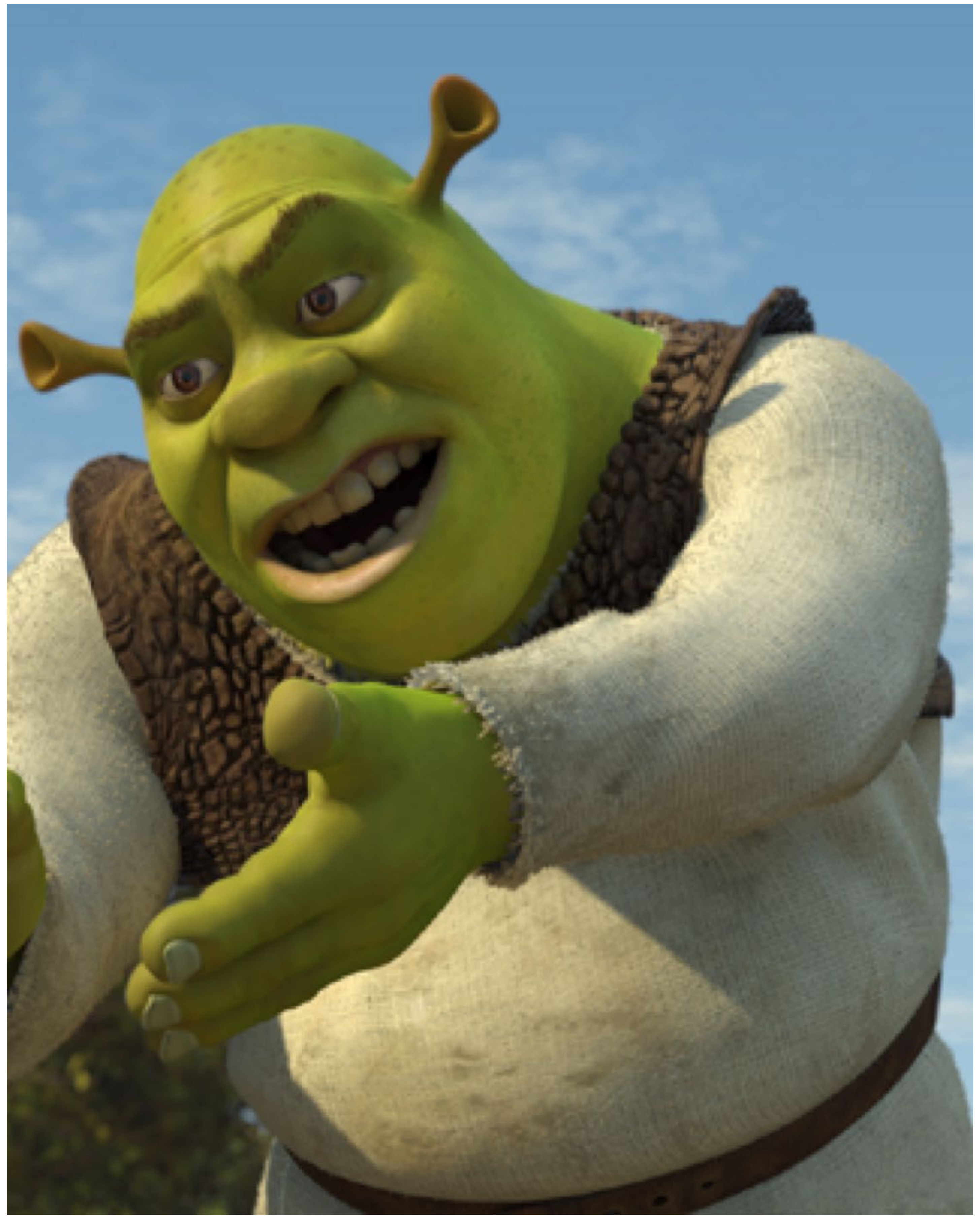
Direct + indirect illumination

Direct vs. Indirect Illumination

Direct illumination only



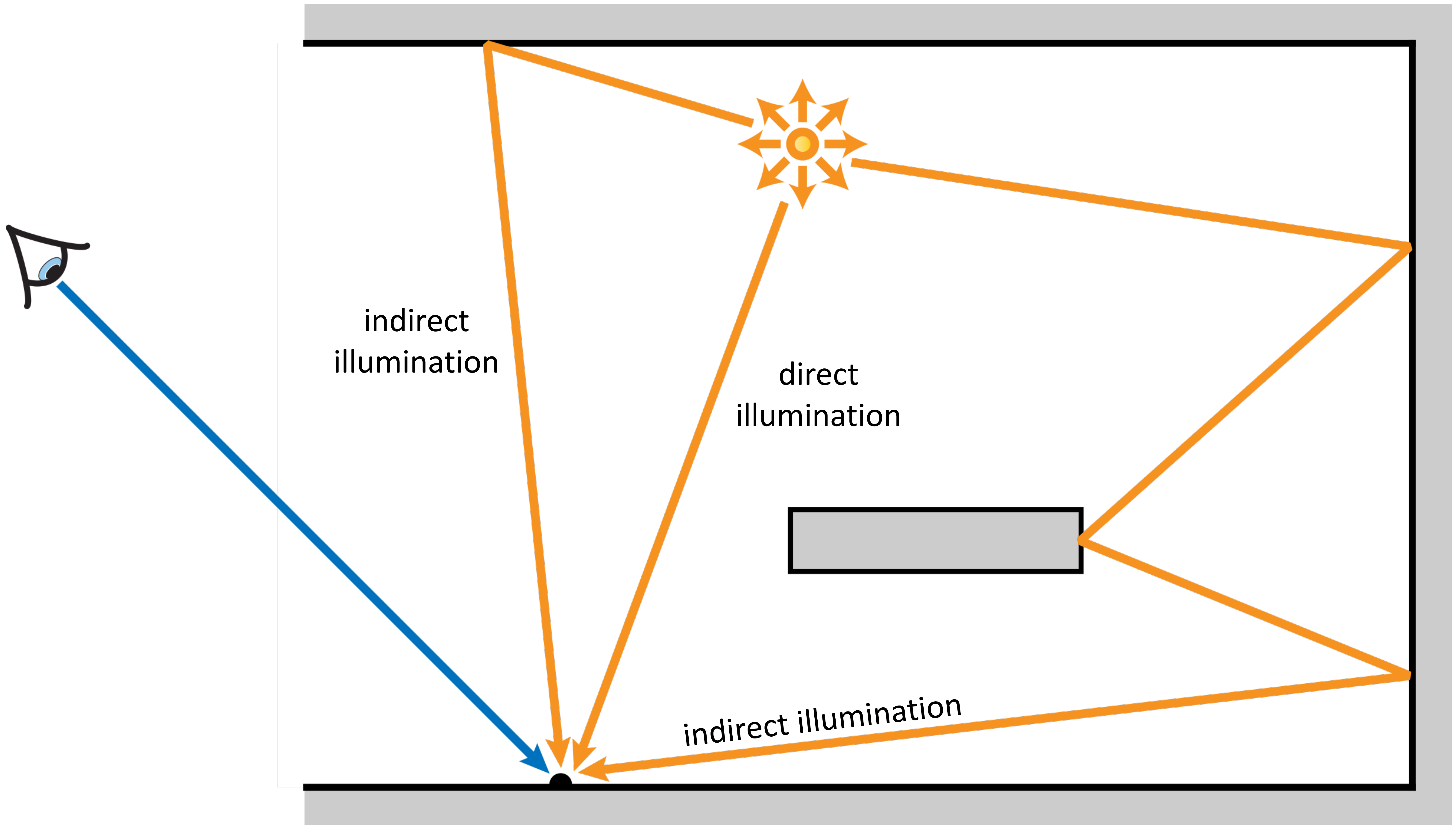
Direct + Indirect illumination



Images courtesy of PDI/DreamWorks

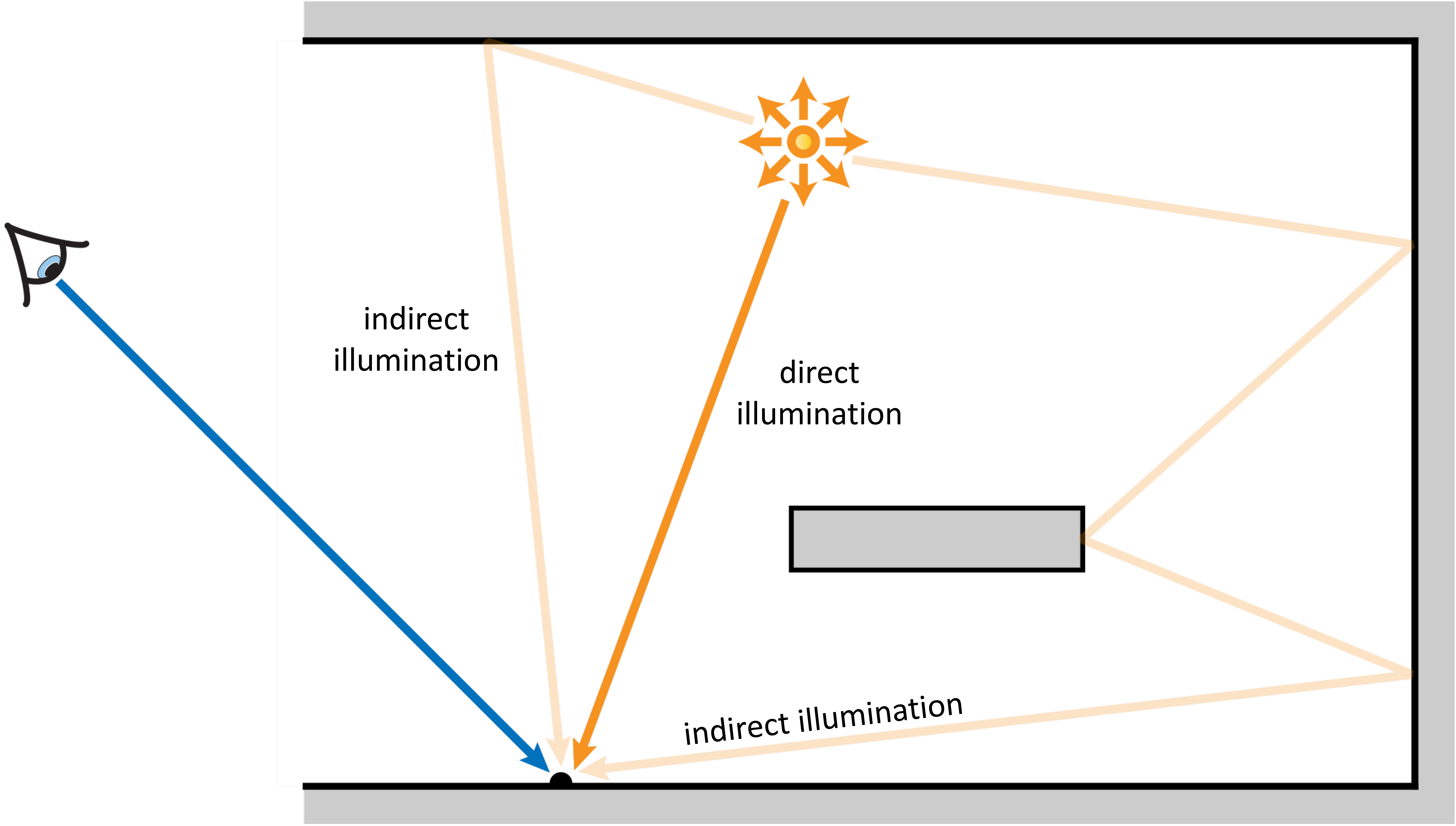
Importance Sampling Incident Radiance

Generally impossible, but…



Importance Sampling Incident Radiance

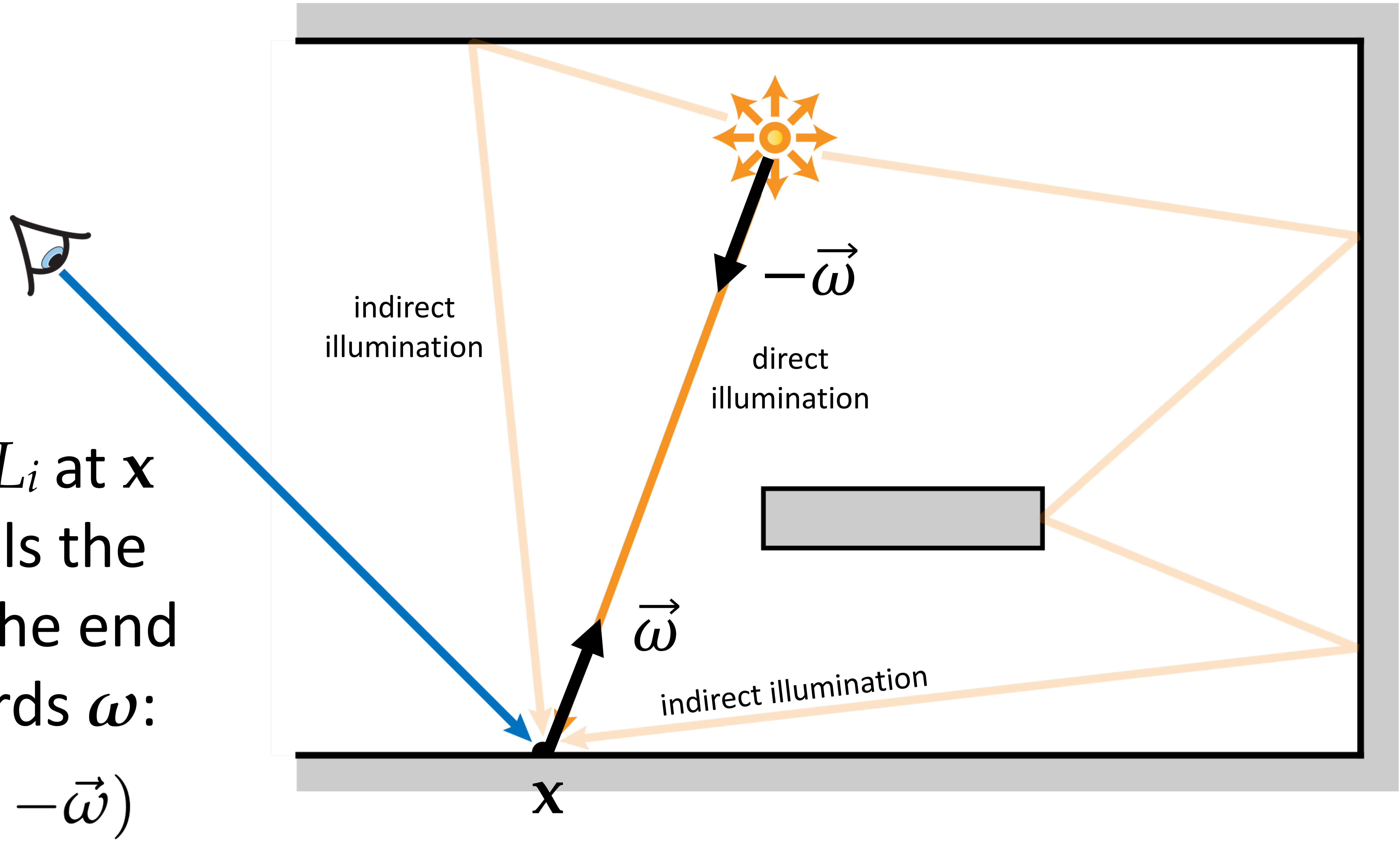
Generally impossible, but possible if we assume only direct illumination



Direct Illumination

$$L_r(\mathbf{x},\vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x},\vec{\omega}_i,\vec{\omega}_r)\, L_i(\mathbf{x},\vec{\omega}_i) \cos\theta_i \,\mathrm{d}\vec{\omega}_i$$

$$L_r(\mathbf{x},\vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x},\vec{\omega}_i,\vec{\omega}_r)\, L_e(r(\mathbf{x},\vec{\omega}_i),-\vec{\omega}_i) \cos\theta_i \,\mathrm{d}\vec{\omega}_i$$



The incident radiance L_i at $\mathbf{x}$ from direction $\boldsymbol{\omega}$ equals the *emitted* radiance L_e at the end of the ray from $\mathbf{x}$ towards $\boldsymbol{\omega}$:

$$L_i(\mathbf{x},\vec{\omega}) = L_e(r(\mathbf{x},\vec{\omega}),-\vec{\omega})$$

Direct Illumination

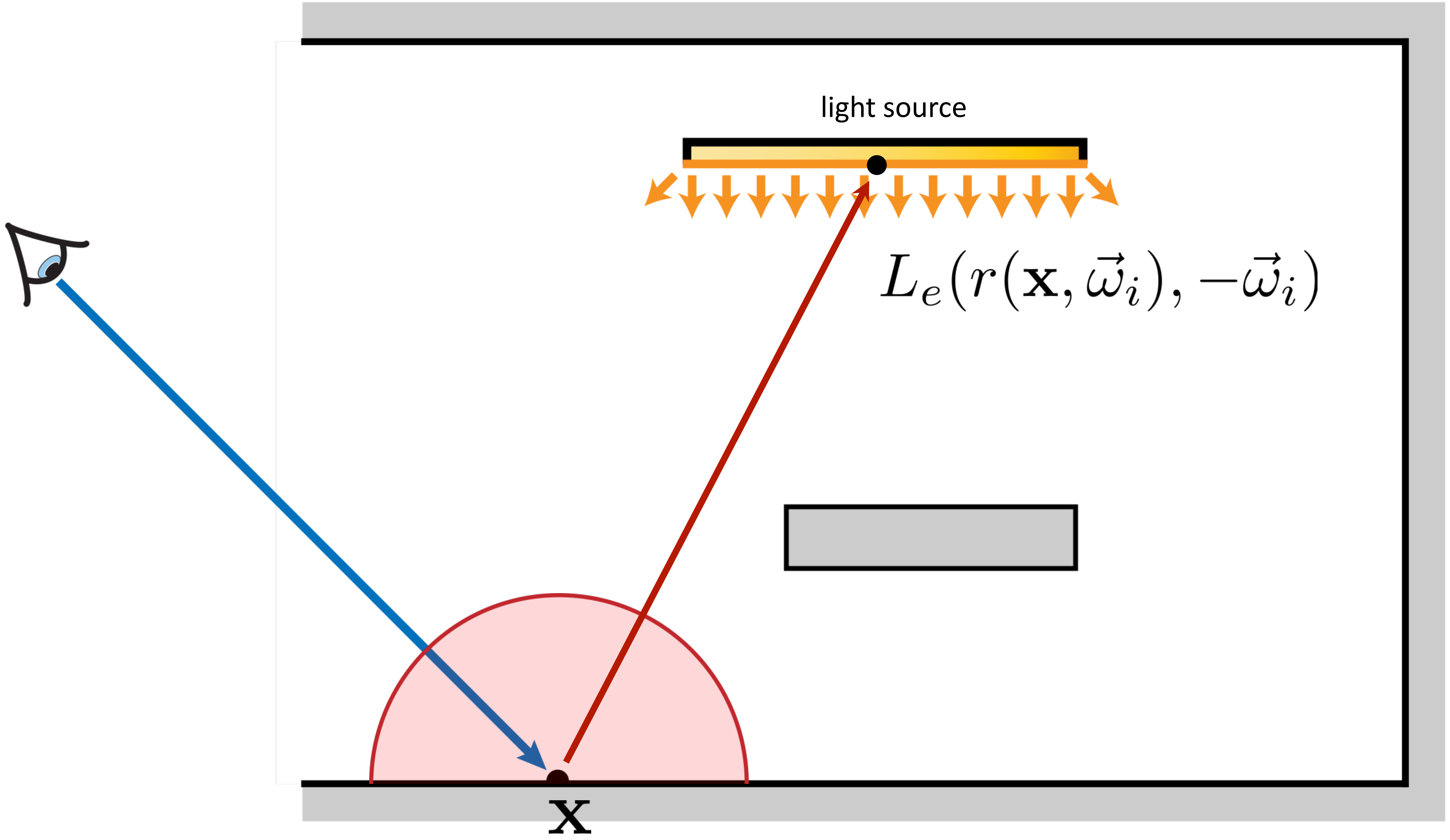
$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r)\, L_e(r(\mathbf{x}, \vec{\omega}_i), -\vec{\omega}_i) \cos\theta_i \, \mathrm{d}\vec{\omega}_i$$

How can we estimate the integral?

$$\langle L_r(\mathbf{x}, \vec{\omega}_r)^N \rangle = \frac{1}{N} \sum_{k=1}^{N} \frac{f_r(\mathbf{x}, \vec{\omega}_{i,k}, \vec{\omega}_r) L_e(r(\mathbf{x}, \vec{\omega}_{i,k}), -\vec{\omega}_{i,k}) \cos\theta_{i,k}}{p_\Omega(\vec{\omega}_{i,k})}$$

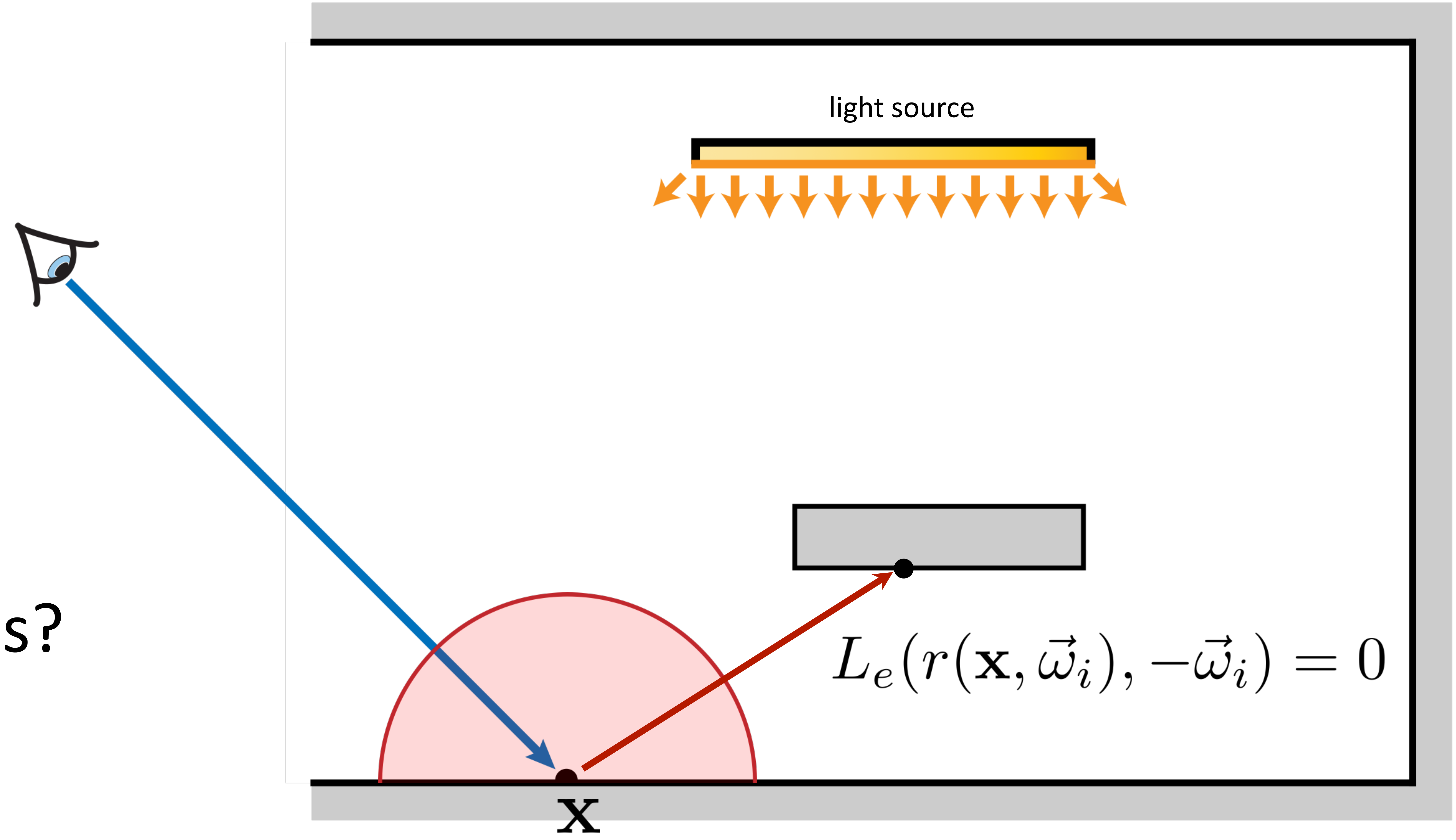
Direct Illumination

$$L_r(\mathbf{x},\vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x},\vec{\omega}_i,\vec{\omega}_r) \boxed{L_e(r(\mathbf{x},\vec{\omega}_i),-\vec{\omega}_i)} \cos\theta_i \, \mathrm{d}\vec{\omega}_i$$



Direct Illumination

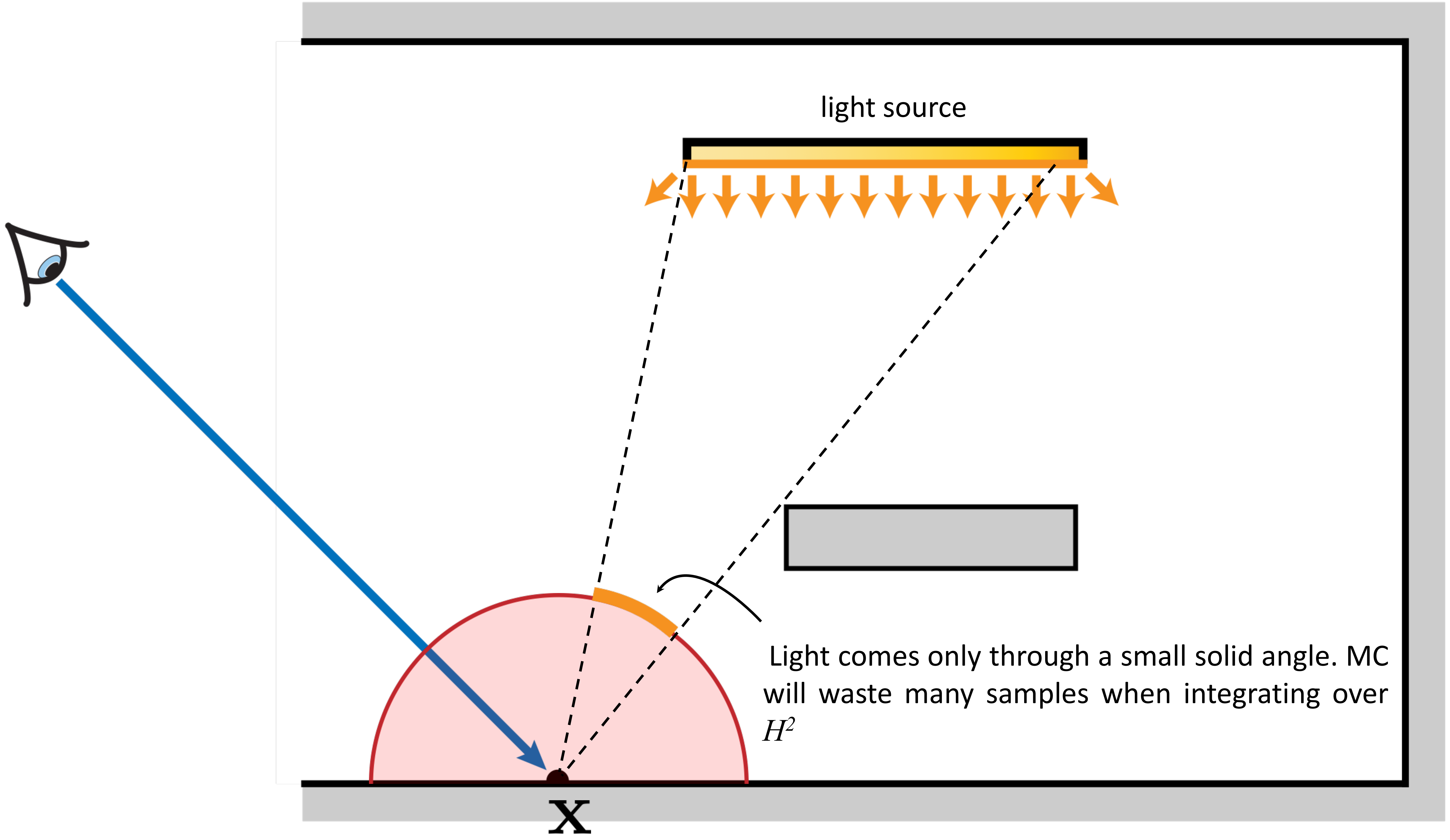
$$L_r(\mathbf{x},\vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x},\vec{\omega}_i,\vec{\omega}_r)\boxed{L_e(r(\mathbf{x},\vec{\omega}_i),-\vec{\omega}_i)}\cos\theta_i \,\mathrm{d}\vec{\omega}_i$$



Any problems?

Direct Illumination

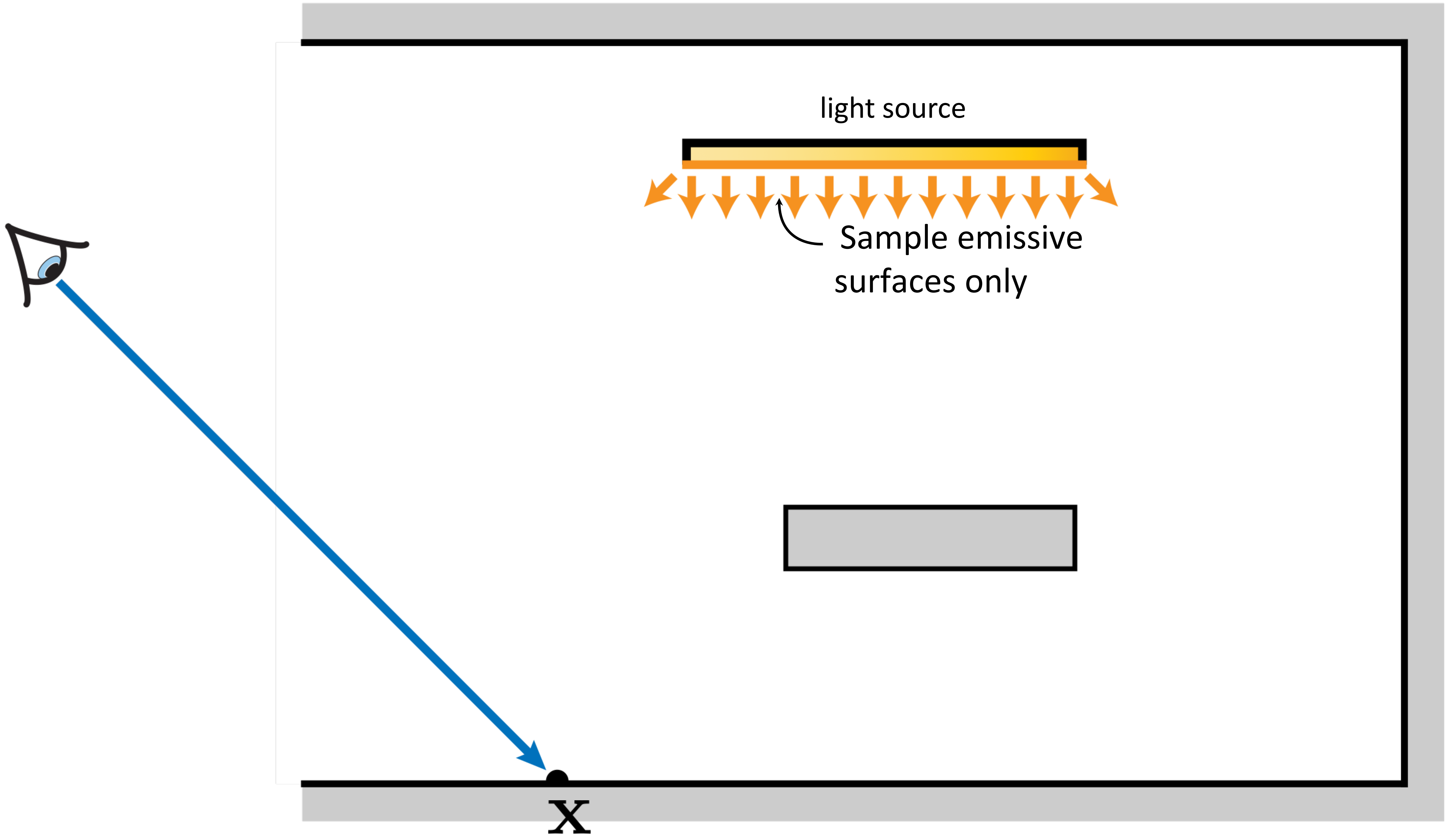
$$L_r(\mathbf{x},\vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x},\vec{\omega}_i,\vec{\omega}_r)\, L_e(r(\mathbf{x},\vec{\omega}_i),-\vec{\omega}_i) \cos\theta_i \,\mathrm{d}\vec{\omega}_i$$



Light comes only through a small solid angle. MC will waste many samples when integrating over H^2

Direct Illumination

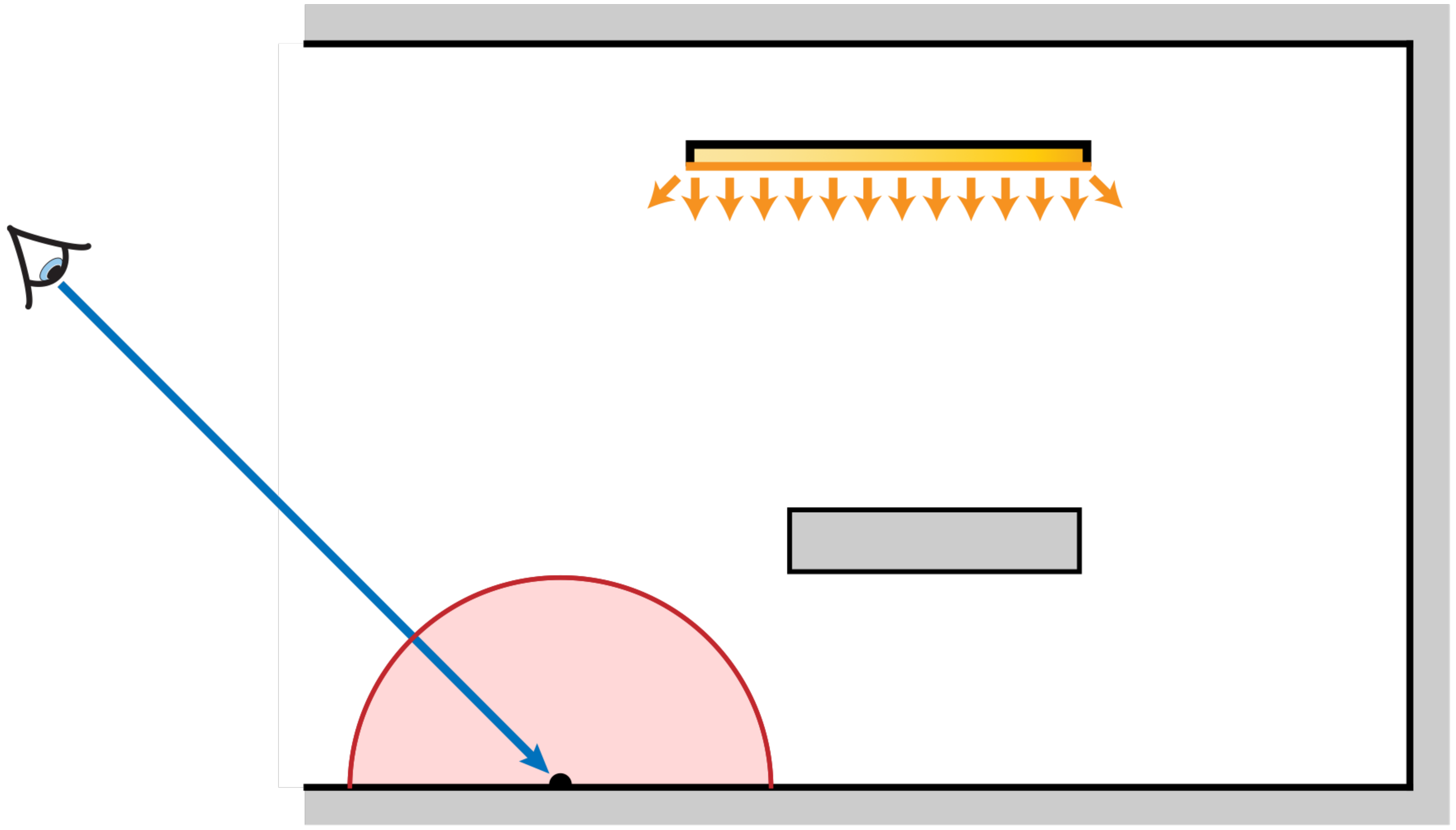
$$L_r(\mathbf{x},\vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x},\vec{\omega}_i,\vec{\omega}_r)\, L_e(r(\mathbf{x},\vec{\omega}_i),-\vec{\omega}_i)\cos\theta_i \,\mathrm{d}\vec{\omega}_i$$



For direct illumination, it would be better to explicitly sample emissive surfaces

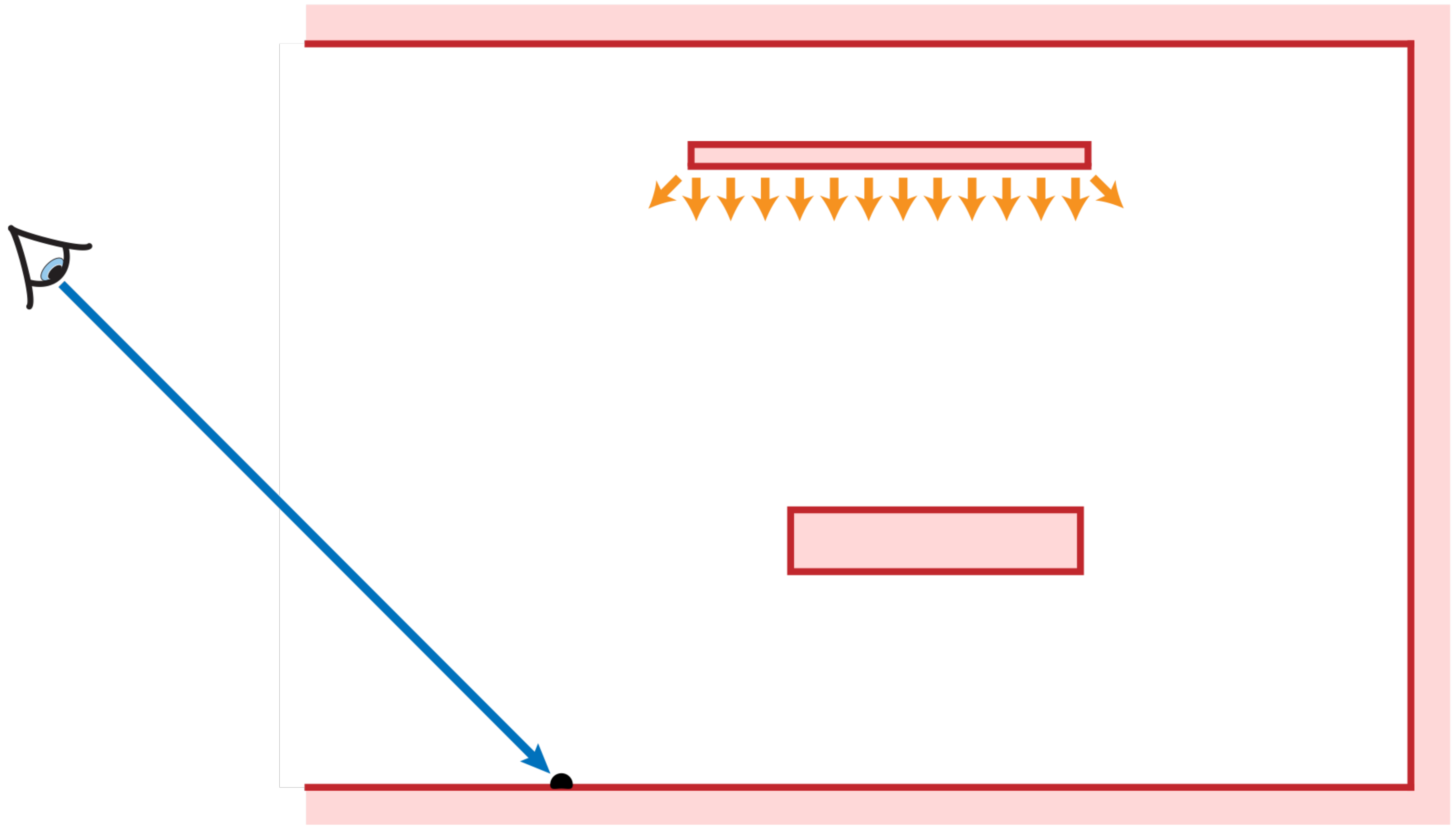
Forms of Reflection Equation

Hemispherical integration



$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

Surface Area integration

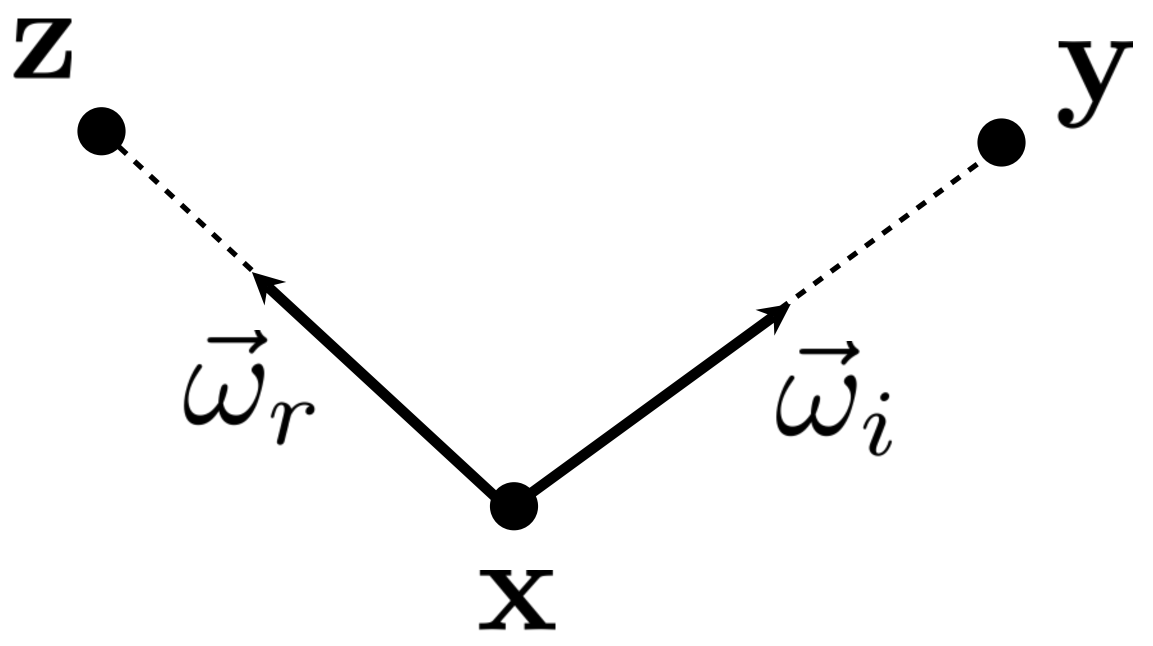


$$L_r(\mathbf{x}, \mathbf{z}) = \int_{A} f_r(\mathbf{x}, \mathbf{y}, \mathbf{z}) L_i(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) \, dA(\mathbf{y})$$

Forms of Reflection Equation

Change in notation:

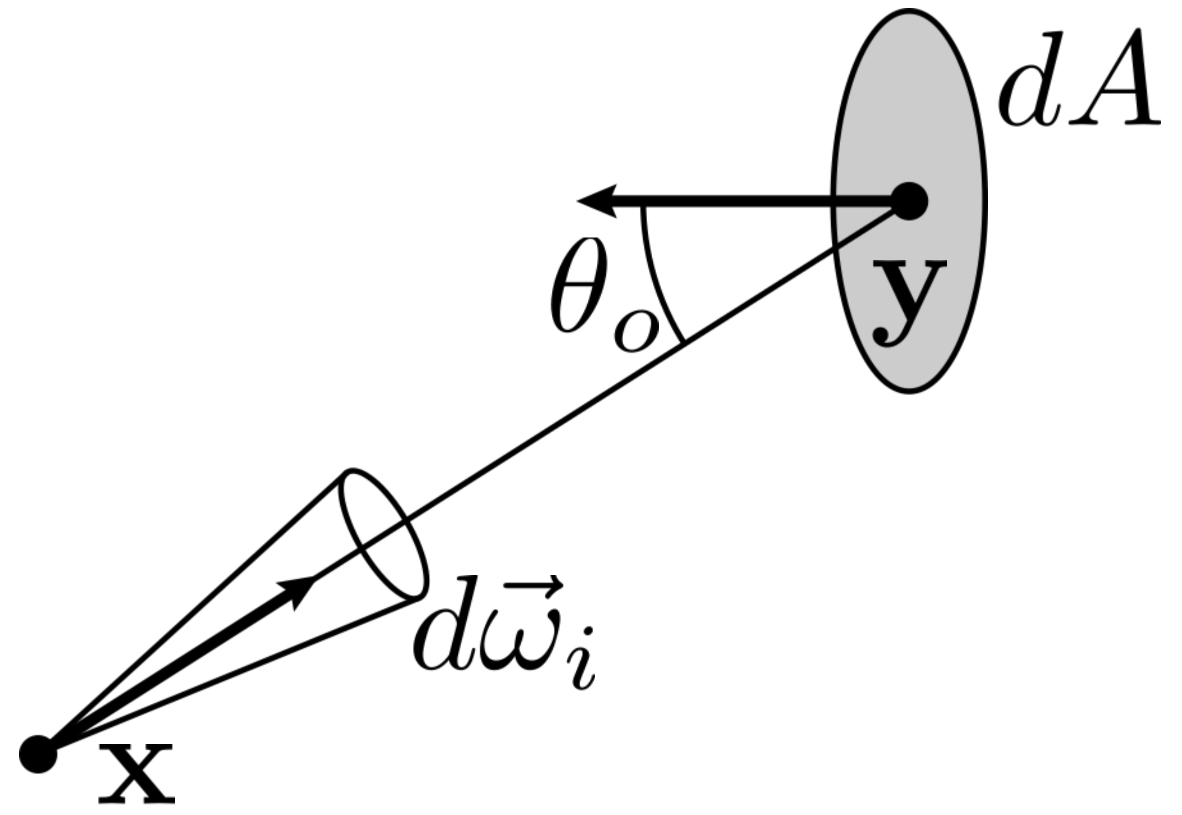
$$L_i(\mathbf{x}, \vec{\omega}_i) = L_i(\mathbf{x}, \mathbf{y})$$
$$L_r(\mathbf{x}, \vec{\omega}_r) = L_r(\mathbf{x}, \mathbf{z})$$
$$f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) = f_r(\mathbf{x}, \mathbf{y}, \mathbf{z})$$



Transform integral over directions into integral over surface area.

Jacobian determinant of the trans.:

$$d\vec{\omega}_i = \frac{|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2} dA$$



Forms of Reflection Equation

$$\begin{aligned} L_i(\mathbf{x}, \vec{\omega}_i) &= L_i(\mathbf{x}, \mathbf{y}) \\ L_r(\mathbf{x}, \vec{\omega}_r) &= L_r(\mathbf{x}, \mathbf{z}) \\ f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) &= f_r(\mathbf{x}, \mathbf{y}, \mathbf{z}) \\ d\vec{\omega}_i &= \frac{|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2} dA \end{aligned}$$

Hemispherical form:

$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

Surface area form:

$$L_r(\mathbf{x}, \mathbf{z}) = \int_A f_r(\mathbf{x}, \mathbf{y}, \mathbf{z}) L_i(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) \, dA(\mathbf{y})$$

Area Form of the Reflection Eq.

$$L_r(\mathbf{x},\mathbf{z}) = \int_A f_r(\mathbf{x},\mathbf{y},\mathbf{z}) L_i(\mathbf{x},\mathbf{y}) \boxed{G(\mathbf{x},\mathbf{y})}\, dA(\mathbf{y})$$

Geometry term:

$$G(\mathbf{x},\mathbf{y}) = \boxed{V(\mathbf{x},\mathbf{y})} \; \frac{|\cos\theta_i|\,|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2}$$

Visibility term:

$$V(\mathbf{x},\mathbf{y}) = \begin{cases} 1: & \text{visible} \\ 0: & \text{not visible} \end{cases}$$

Area Form of the Reflection Eq.

$$L_r(\mathbf{x},\mathbf{z}) = \int_A f_r(\mathbf{x},\mathbf{y},\mathbf{z}) L_i(\mathbf{x},\mathbf{y}) \boxed{G(\mathbf{x},\mathbf{y})}\, dA(\mathbf{y})$$

Original foreshortening term

Geometry term:

$$G(\mathbf{x},\mathbf{y}) = V(\mathbf{x},\mathbf{y}) \frac{|\boxed{\cos\theta_i}|\ |\boxed{\cos\theta_o}|}{\boxed{\|\mathbf{x}-\mathbf{y}\|^2}}$$

Jacobian determinant
of the transform

Visibility term:

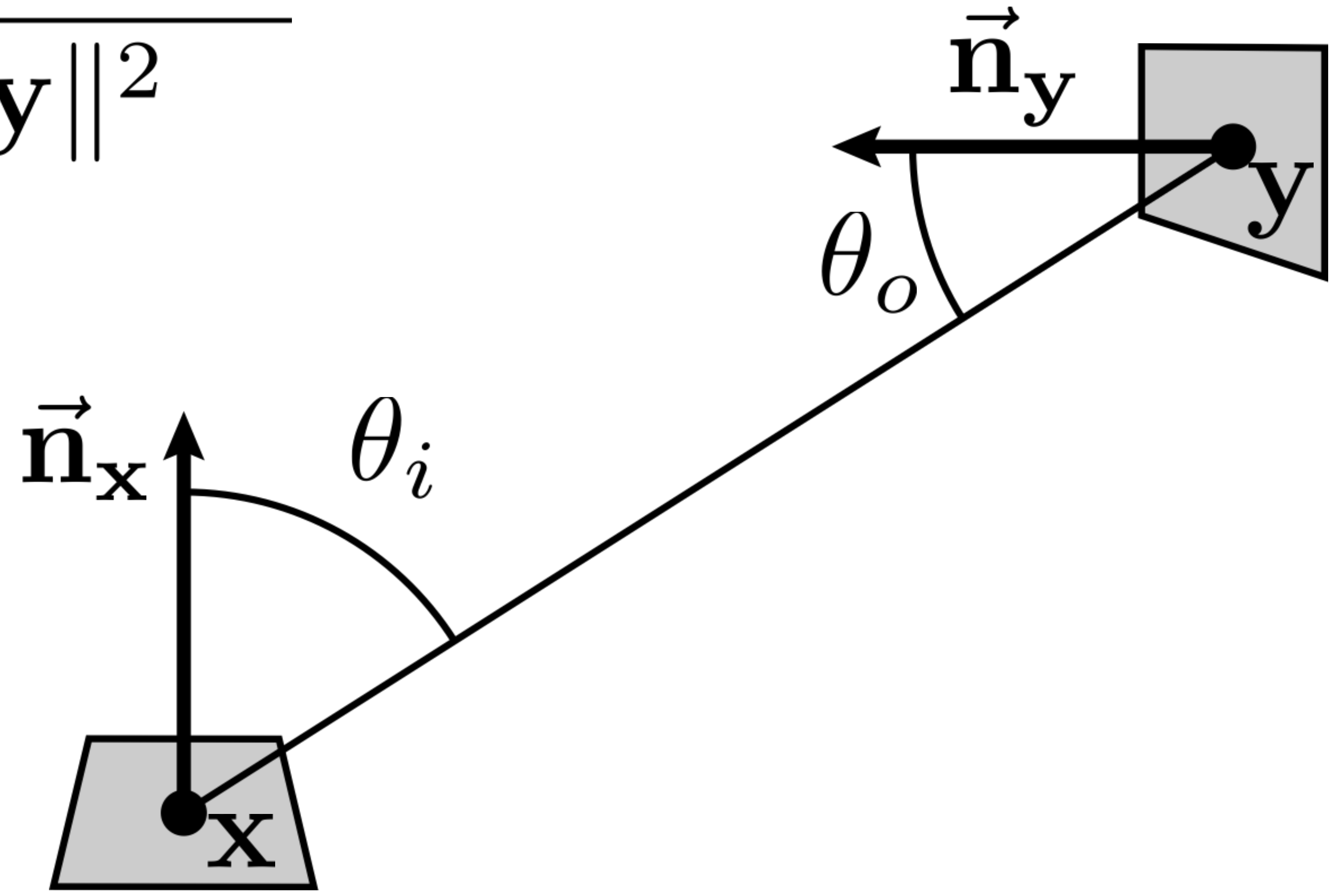
$$V(\mathbf{x},\mathbf{y}) = \begin{cases} 1: & \text{visible} \\ 0: & \text{not visible} \end{cases}$$

$$d\vec{\omega}_i = \frac{|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2} dA$$

Area Form of the Reflection Eq.

Interpreting

$$\frac{|\cos\theta_i|\,|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2}$$



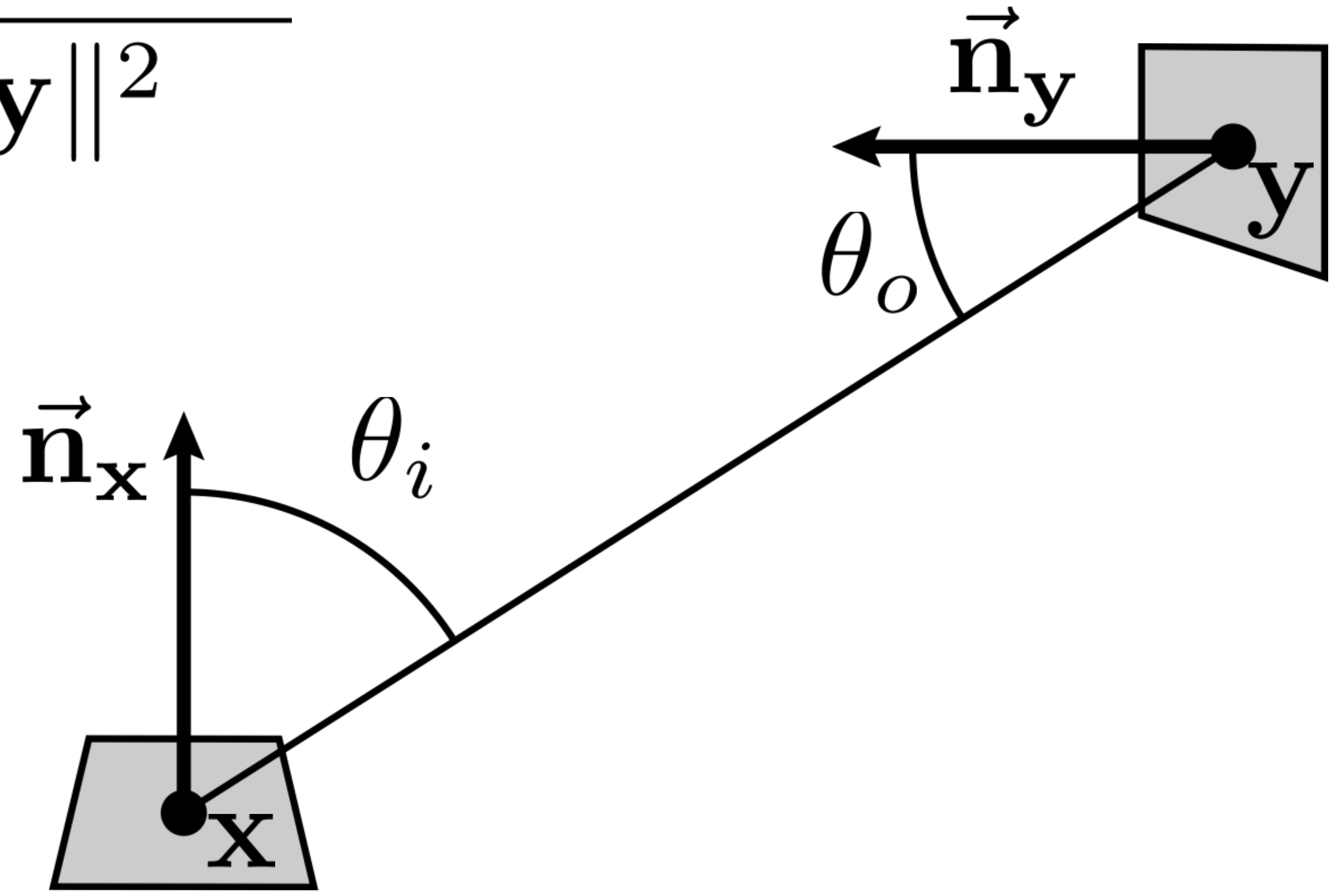
The chance that a photon emitted from a differential patch will hit another diff. patch decreases as:

- the patches face away from each other (numerator)
- the patches move away from each other (denominator)

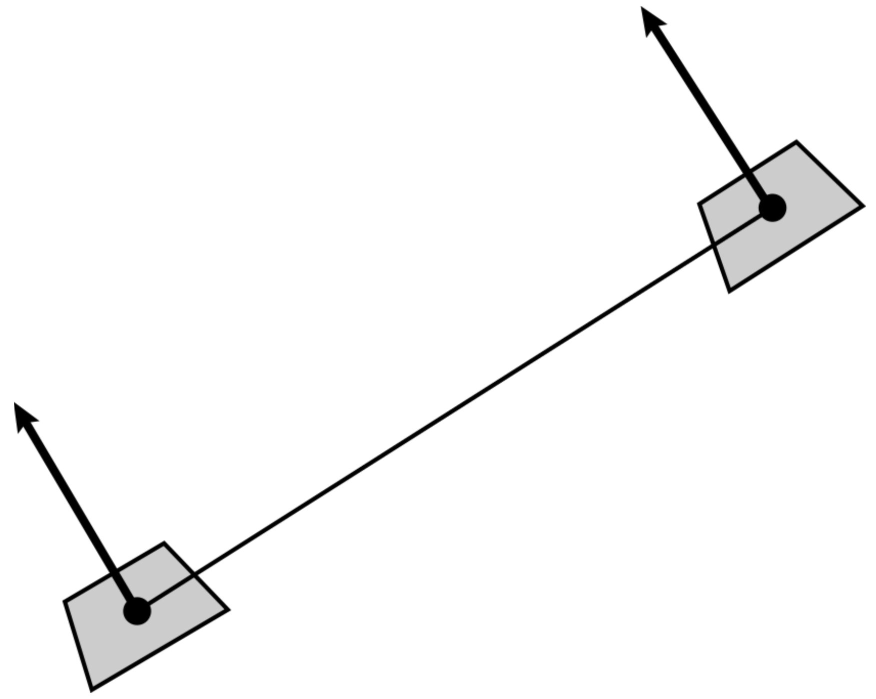
Area Form of the Reflection Eq.

Interpreting

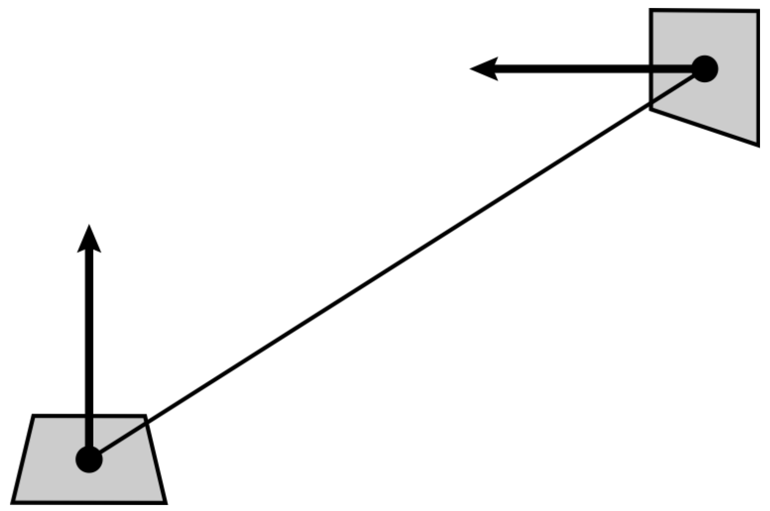
$$\frac{|\cos\theta_i|\,|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2}$$



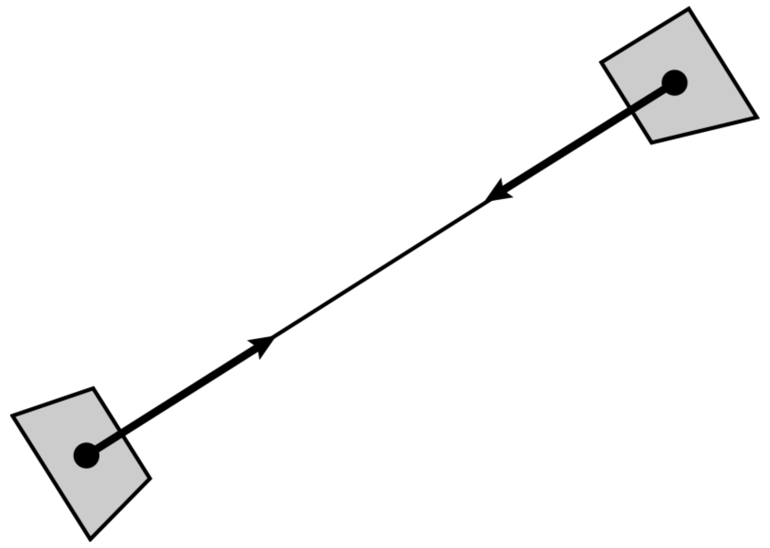
numerator = 0



0 < numerator < 1

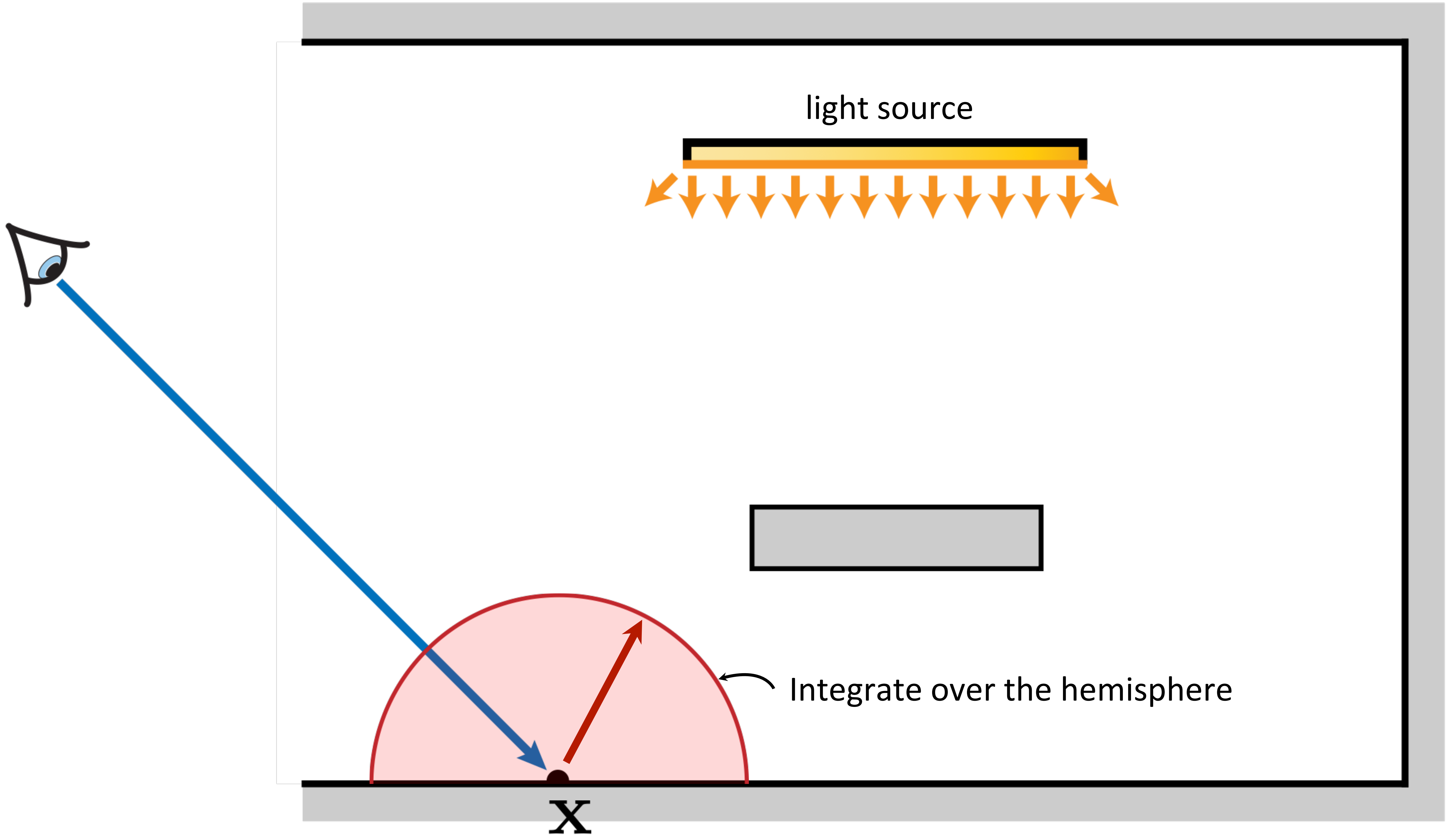


numerator = 1



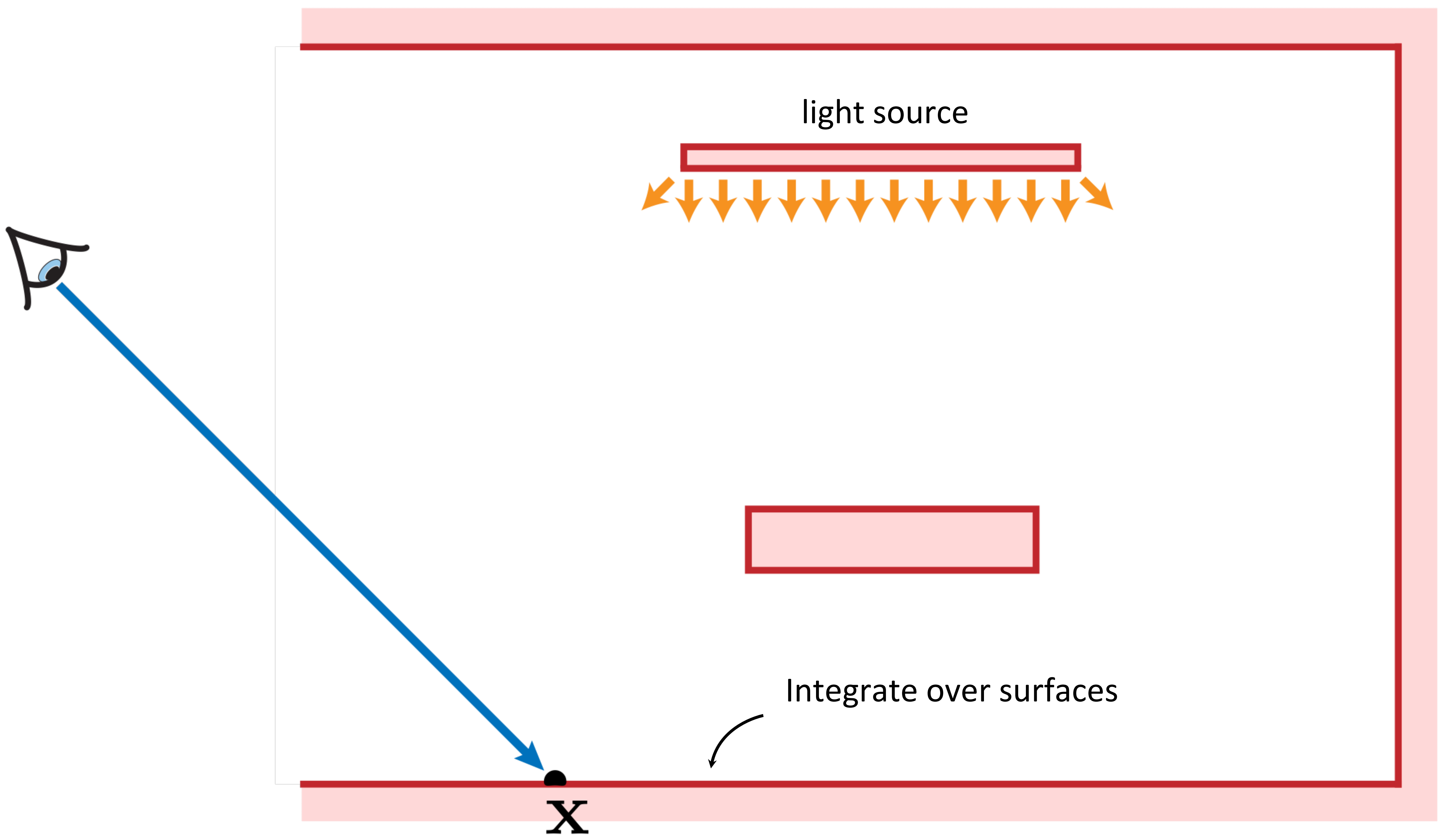
Direct Illumination

$$L_r(\mathbf{x}, \vec{\omega}_r) = \boxed{\int_{H^2}} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) L_e(r(\mathbf{x}, \vec{\omega}_i), -\vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$



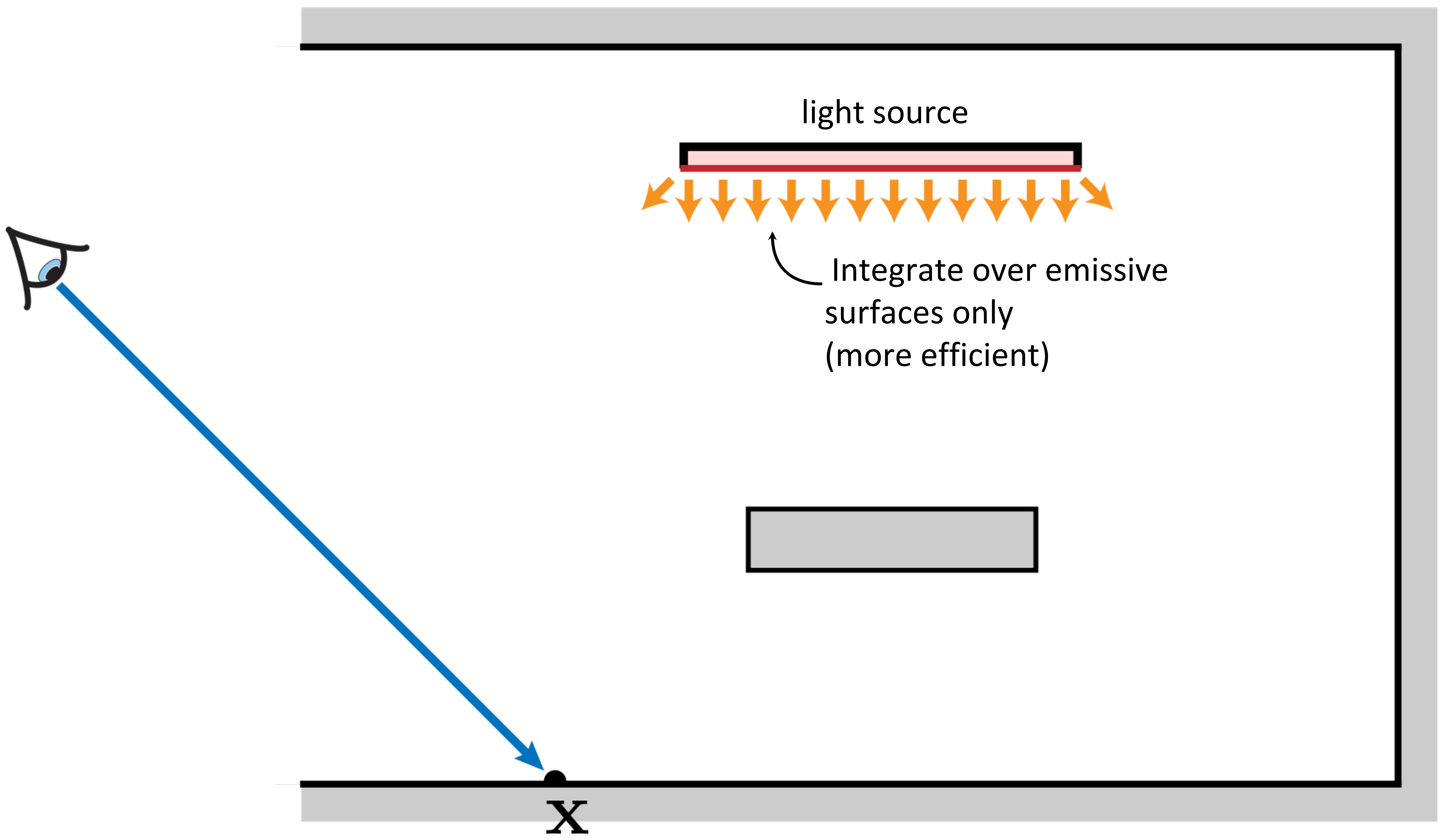
Direct Illumination

$$L_r(\mathbf{x},\mathbf{z}) = \boxed{\int_A} f_r(\mathbf{x},\mathbf{y},\mathbf{z}) L_e(\mathbf{y},\mathbf{x}) V(\mathbf{x},\mathbf{y}) \frac{|\cos\theta_i|\,|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2} dA(\mathbf{y})$$



Direct Illumination

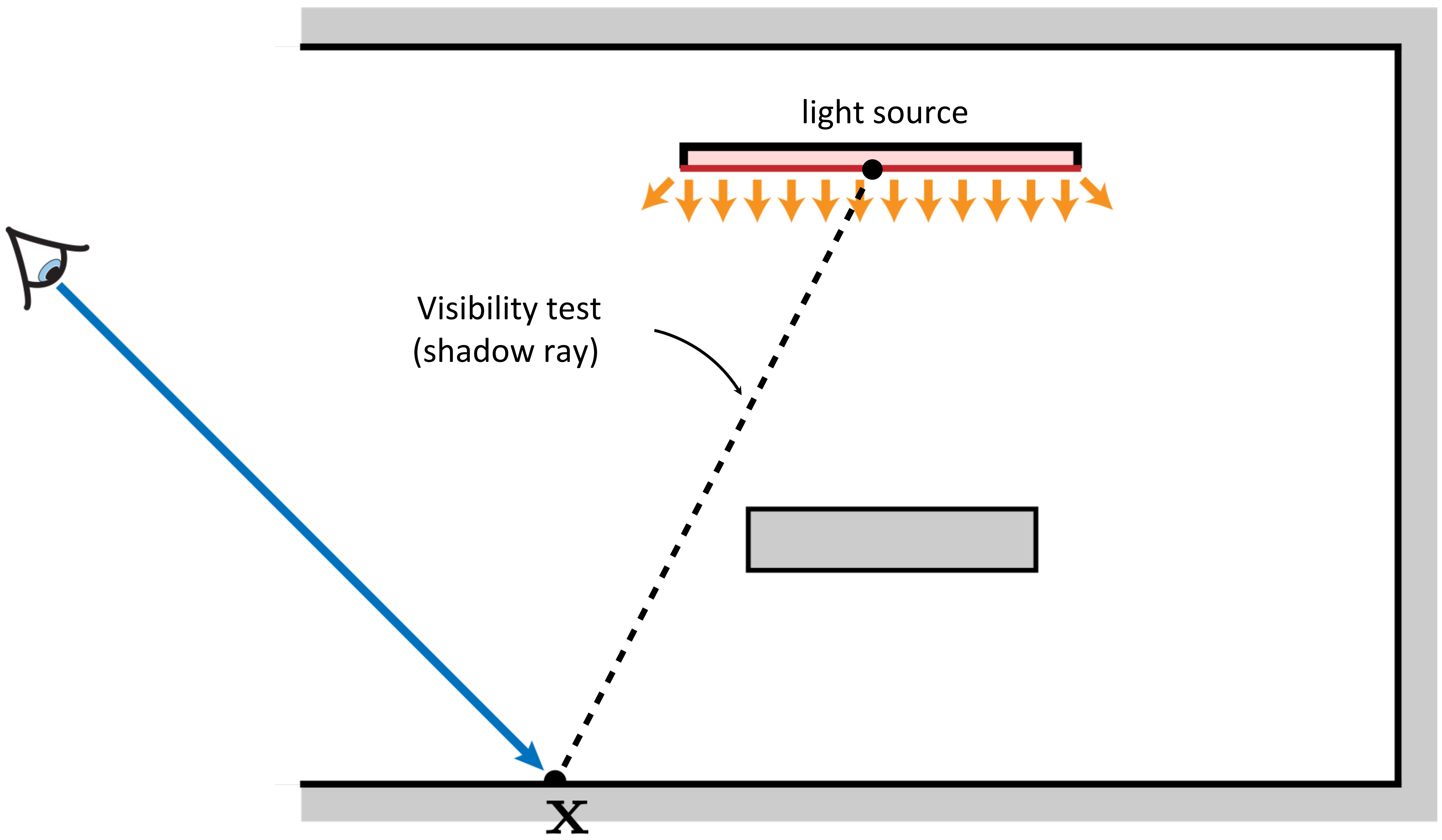
$$L_r(\mathbf{x},\mathbf{z}) = \boxed{\int_{A_e}} f_r(\mathbf{x},\mathbf{y},\mathbf{z}) L_e(\mathbf{y},\mathbf{x}) V(\mathbf{x},\mathbf{y}) \frac{|\cos\theta_i|\,|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2} dA(\mathbf{y})$$



Integrate over emissive surfaces only (more efficient)

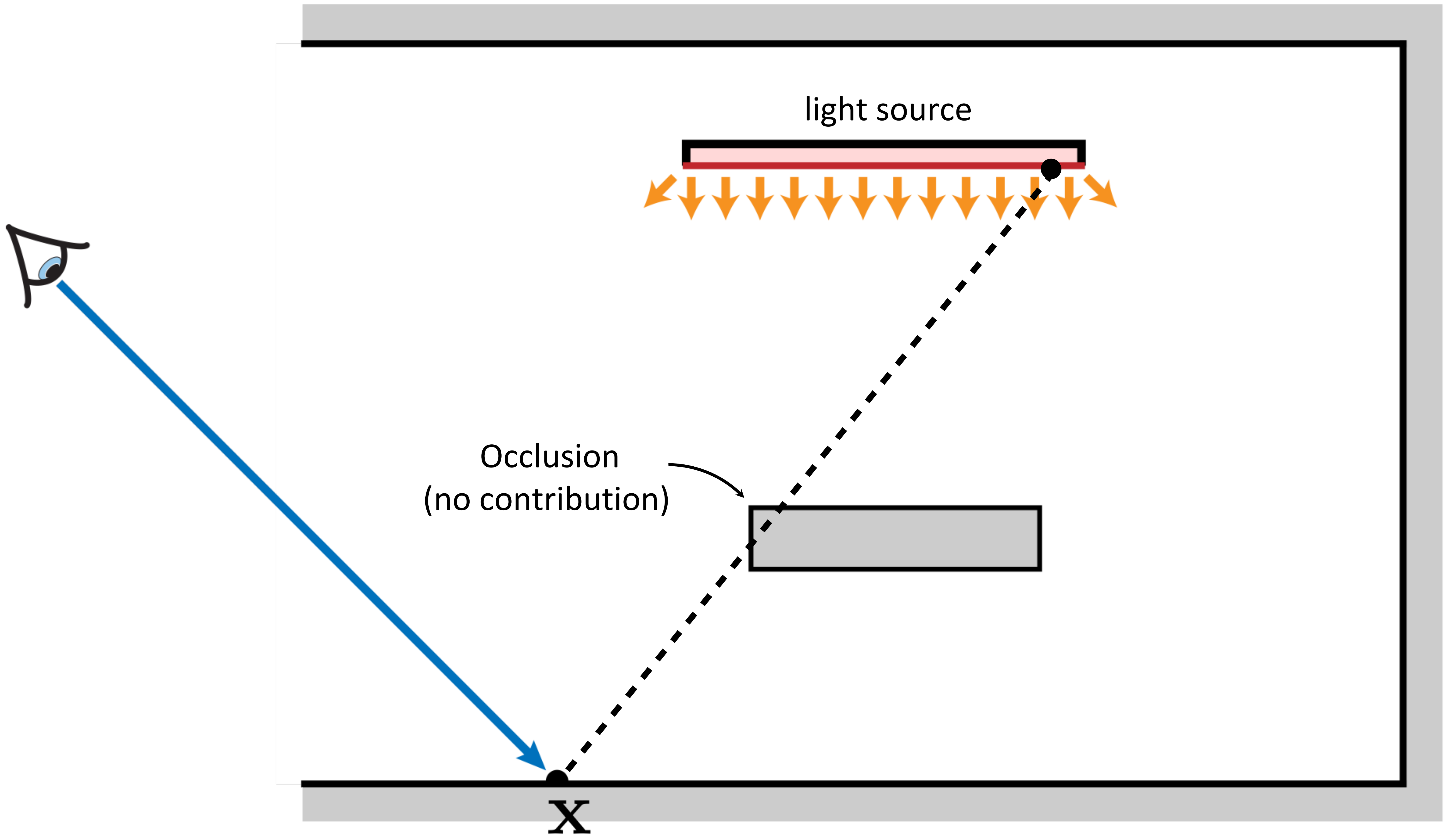
Direct Illumination

$$L_r(\mathbf{x},\mathbf{z}) = \int_{A_e} f_r(\mathbf{x},\mathbf{y},\mathbf{z}) L_e(\mathbf{y},\mathbf{x}) \boxed{V(\mathbf{x},\mathbf{y})} \frac{|\cos\theta_i|\,|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2} dA(\mathbf{y})$$

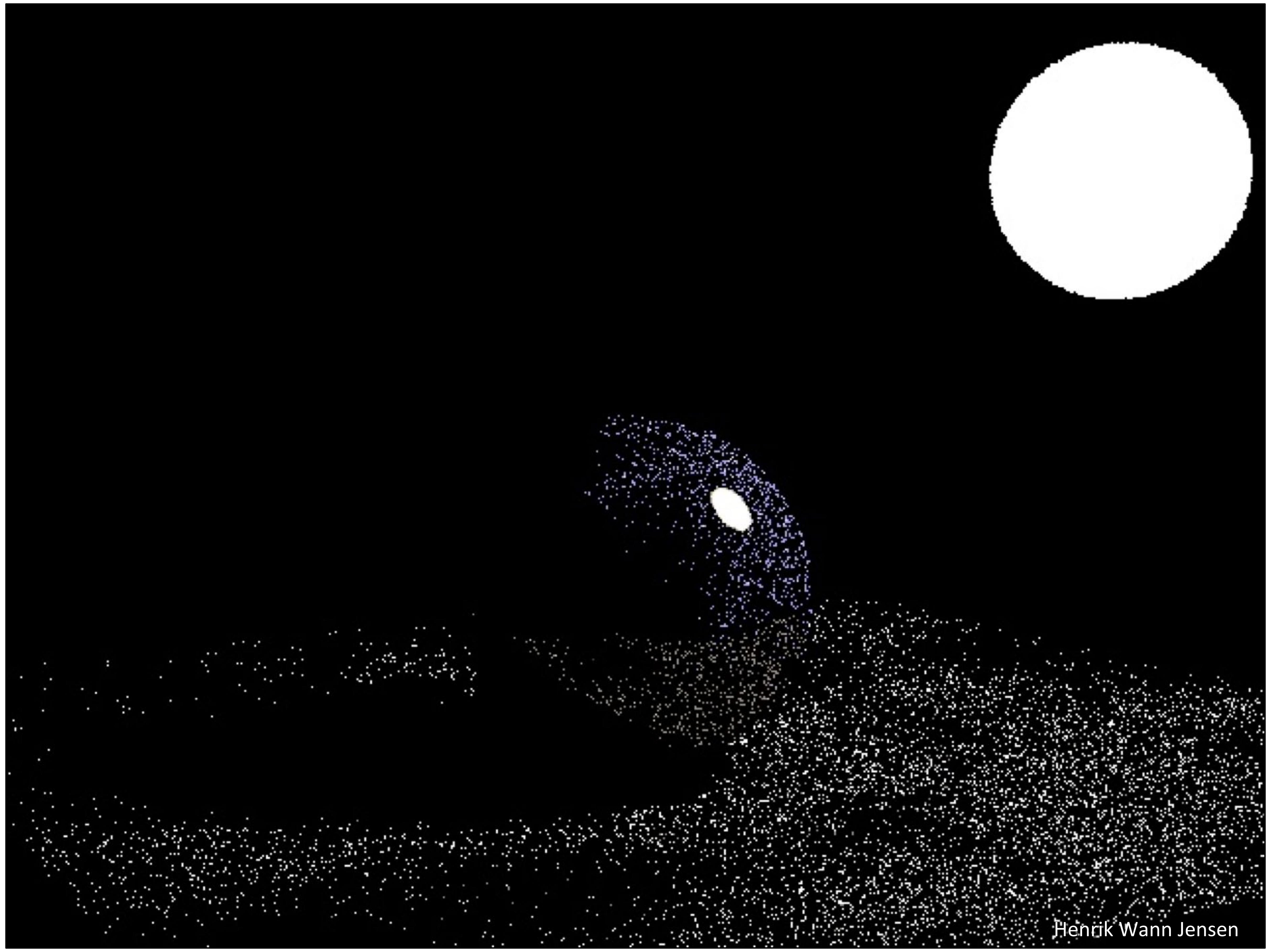


Direct Illumination

$$L_r(\mathbf{x},\mathbf{z}) = \int_{A_e} f_r(\mathbf{x},\mathbf{y},\mathbf{z}) L_e(\mathbf{y},\mathbf{x}) \boxed{V(\mathbf{x},\mathbf{y})} \frac{|\cos\theta_i|\,|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2} dA(\mathbf{y})$$

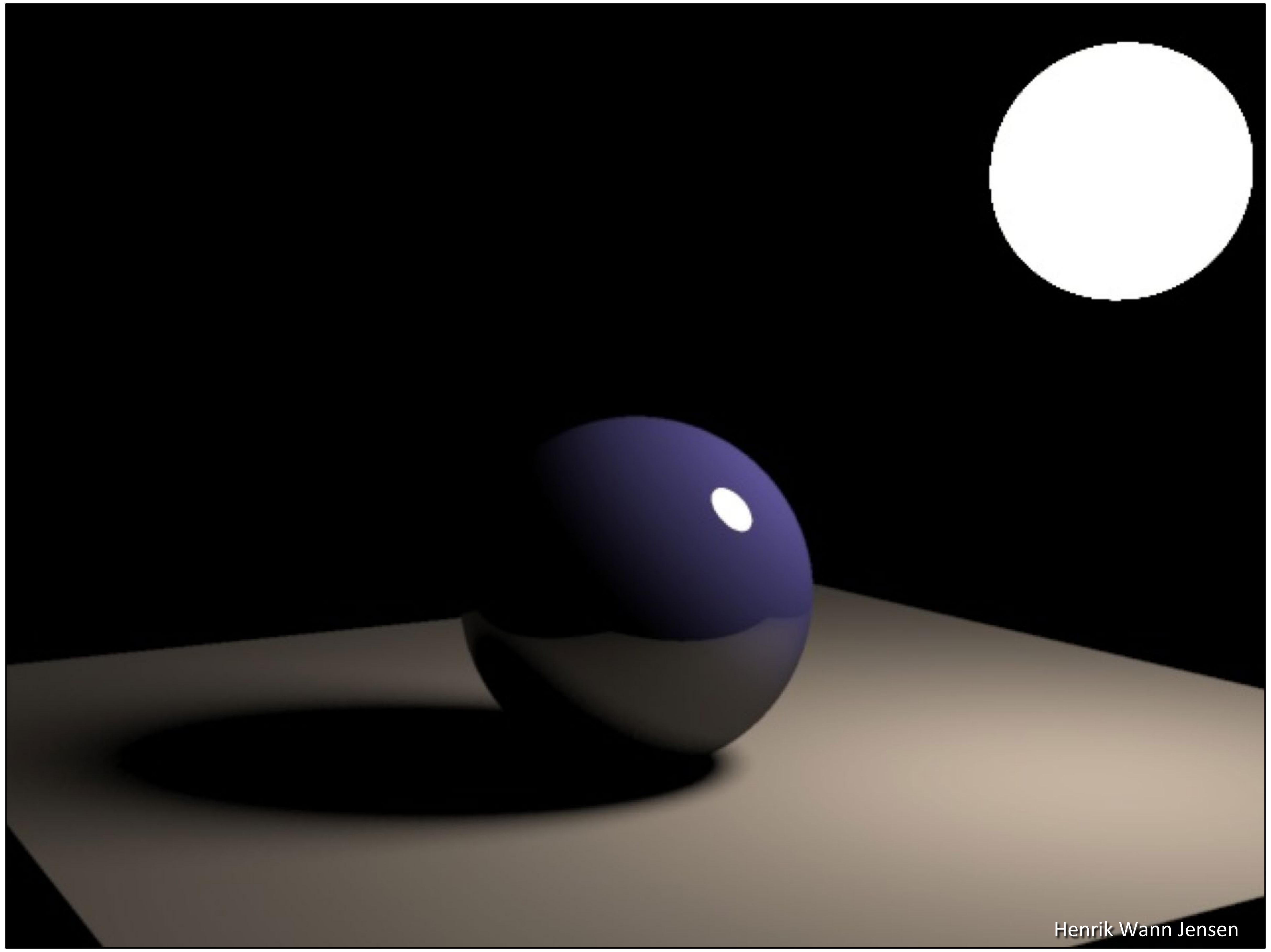


Direct Illumination



Sampling the hemisphere

Direct Illumination

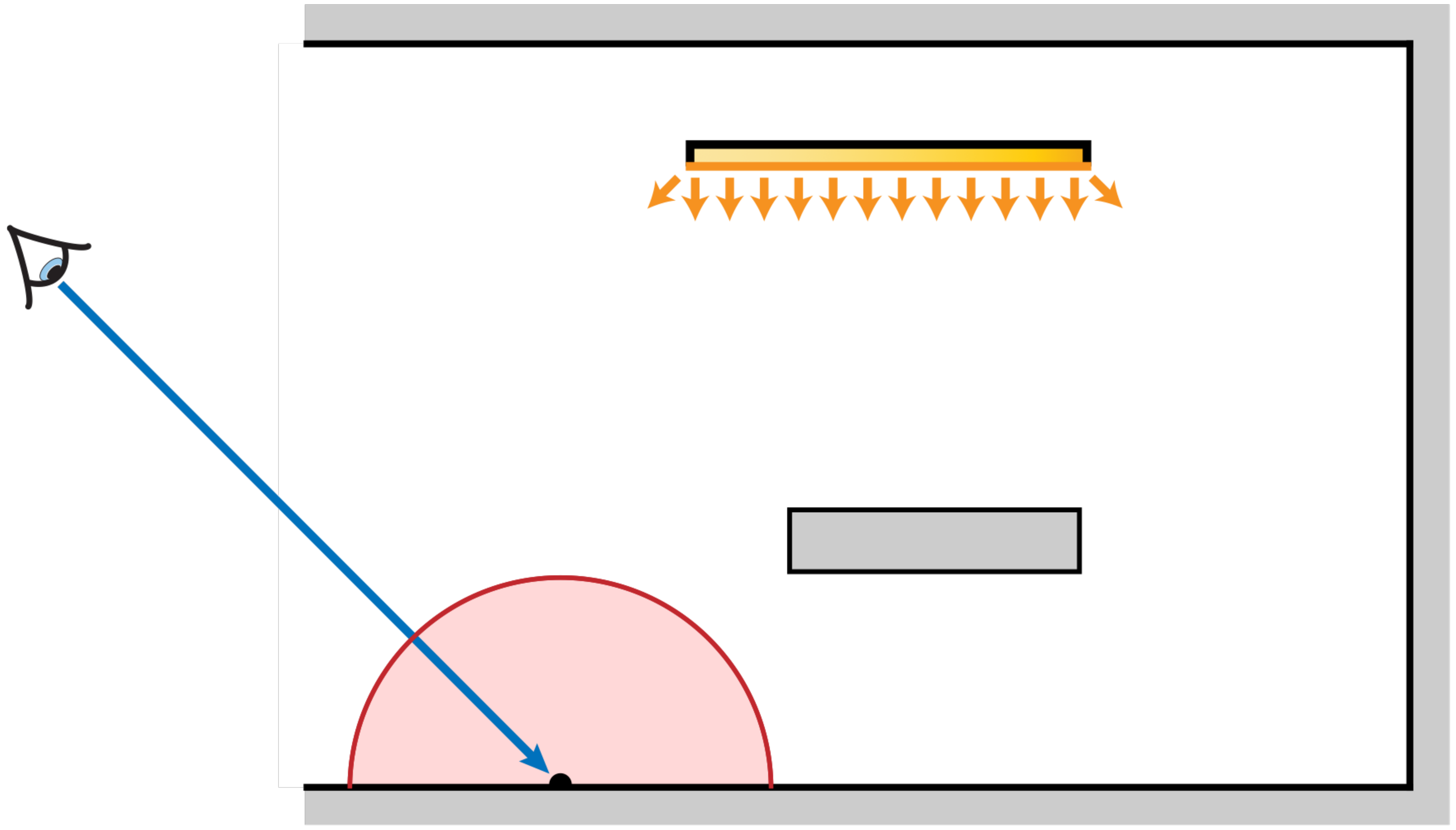


Henrik Wann Jensen

Sampling the area of the light

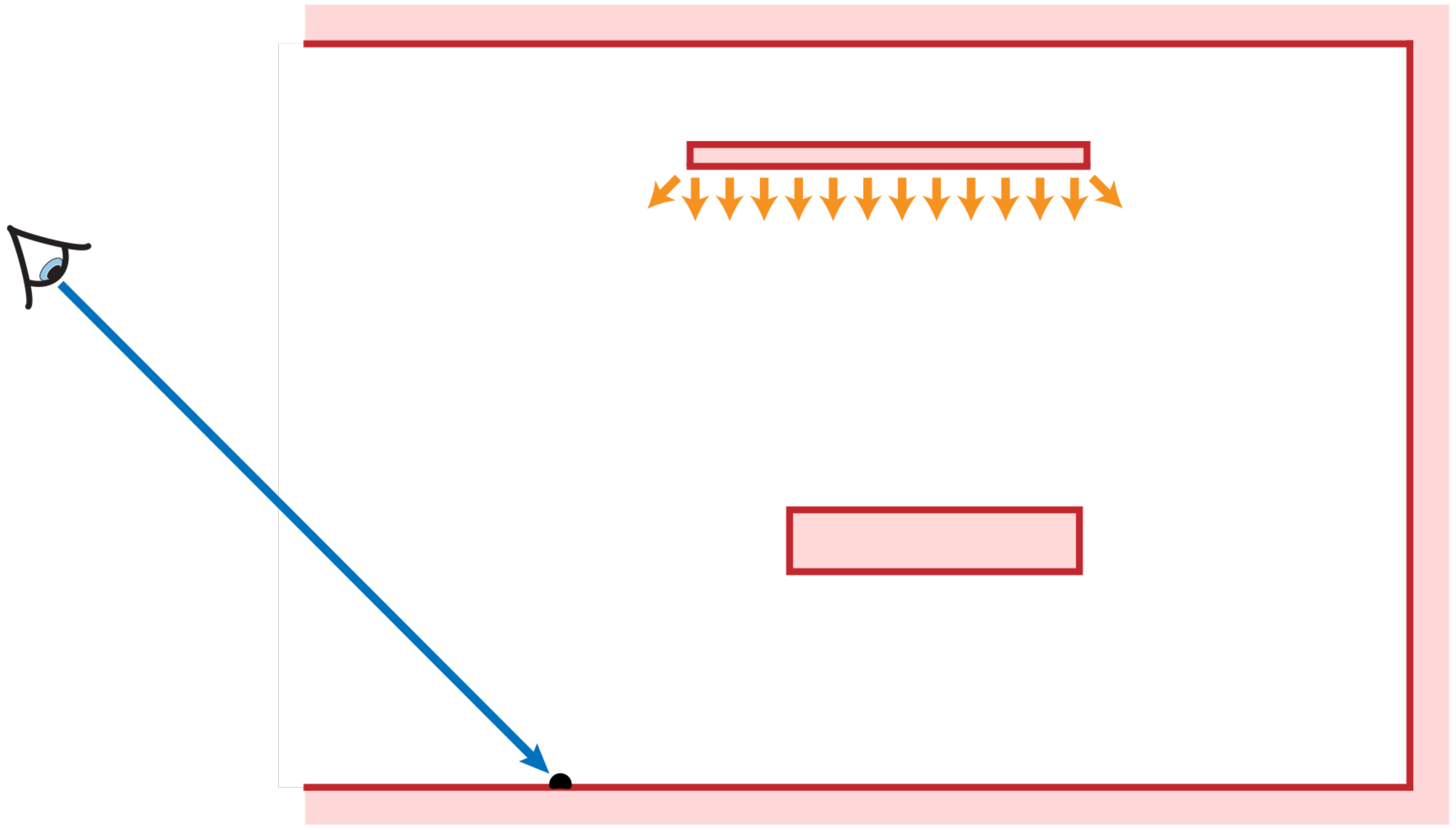
Forms of Reflection Equation

Hemispherical integration



$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

Surface Area integration



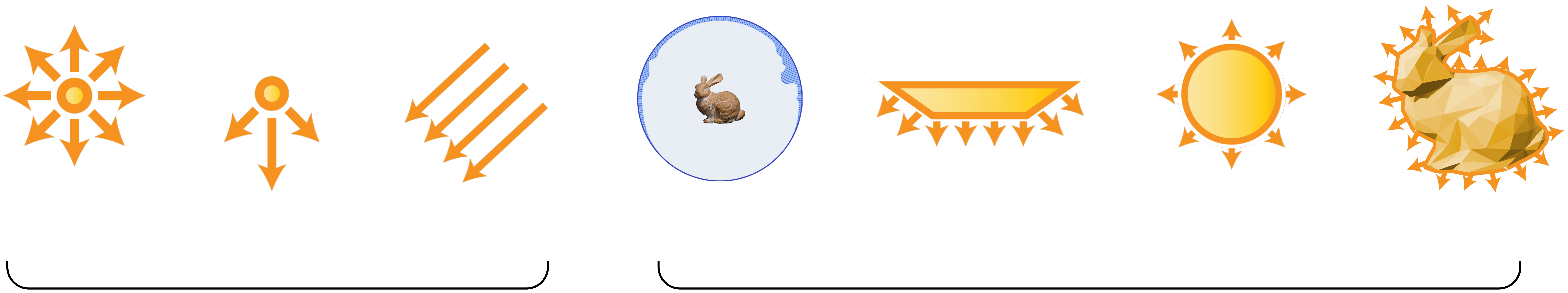
$$L_r(\mathbf{x}, \mathbf{z}) = \int_A f_r(\mathbf{x}, \mathbf{y}, \mathbf{z}) L_i(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) \, dA(\mathbf{y})$$

How do we decide which one to use for sampling direct illumination?

- The answer depends on the types of light sources and BRDFs in the scene.

Light Sources

Light Sources



Point light

Spot light

Directional light

Environment light

Quad light

Sphere light

Mesh light

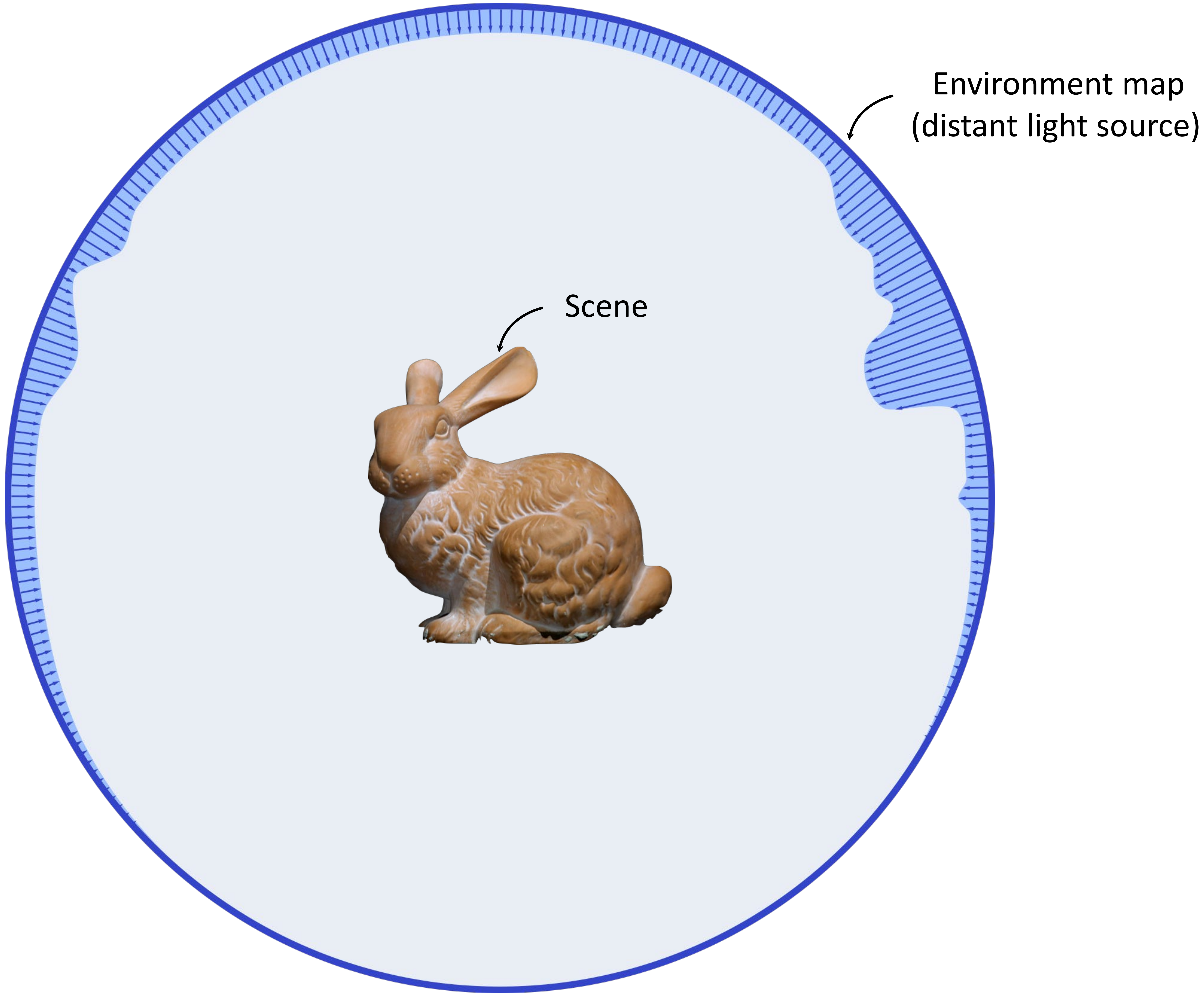
Delta lights

(create hard shadows)

Finite lights

(create soft shadows)

Environment Lighting



Environment Lighting

The image “wraps” around the virtual scene, serving as a *distant* source of illumination

Convenient to express using the *hemispherical* form of the reflectance equation

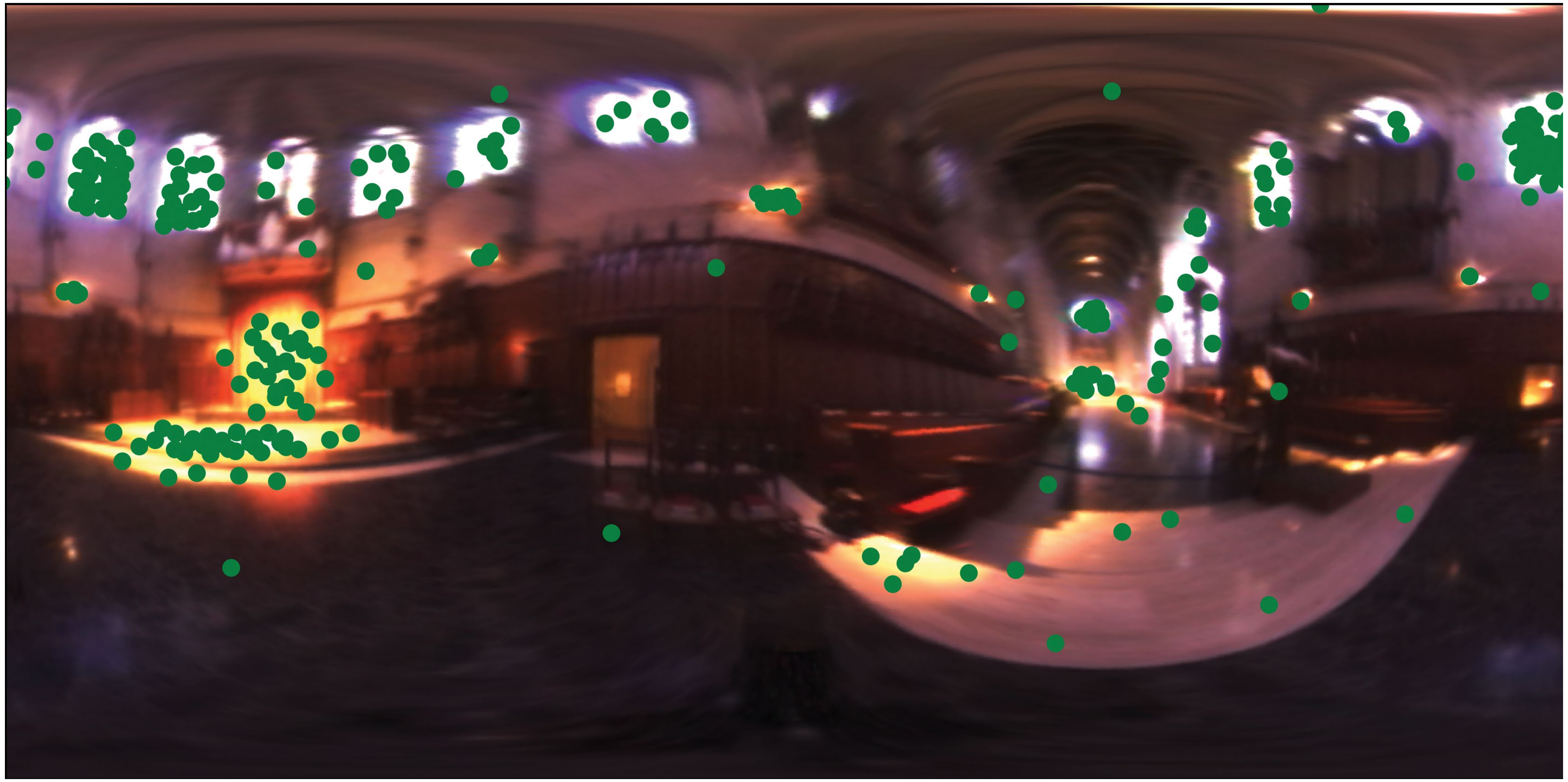
$$
\begin{aligned}
L_r(\mathbf{x}, \vec{\omega}_r) &= \int_\Omega f_r(\vec{\omega}_i, \vec{\omega}_r) L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i \\
&= \int_\Omega f_r(\vec{\omega}_i, \vec{\omega}_r) L_{\text{env}}(\vec{\omega}_i) V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i
\end{aligned}
$$

Environment Lighting

Environment Lighting

$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{\Omega} f_r(\vec{\omega}_i, \vec{\omega}_r) L_{\text{env}}(\vec{\omega}_i) V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

Importance Sampling L_{env}



Sample using the *hemispherical form* of the reflectance equation and pdf

$$p(\vec{\omega}_i) \propto L_{\text{env}}(\vec{\omega}_i)$$

Importance Sampling L_{env}

$$p(\vec{\omega}_i) \propto L_{\text{env}}(\vec{\omega}_i)$$

Several strategies exist

We'll discuss:

- Marginal/Conditional CDF method
- Hierarchical warping method

Importance Sampling

Recipe:

1. Express the desired distribution in a convenient coordinate system
    - requires computing the Jacobian
2. Compute marginal and conditional 1D PDFs
3. Sample 1D PDFs using the inversion method

Marginal/Conditional CDF

Assume the lat/long parameterization

Draw samples from joint $p(\theta, \phi) \propto L_{\mathrm{env}}(\theta, \phi) \sin\theta$

Why the Sine?

General case of integrating some $f(\vec{\omega})$ over S^2

If we set $d\vec{\omega} = \underset{\nwarrow \text{Comes from the Jacobian}}{\sin\theta} d\theta d\phi$ we want to cancel out the sine.

$$\int_{S^2} f(\vec{\omega})\, d\vec{\omega} = \int_0^{2\pi}\int_0^{\pi} f(\theta, \phi) \sin\theta \, d\theta d\phi$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} \frac{f(\theta_i, \phi_i)\sin\theta_i}{p(\theta_i, \phi_i)}$$

$$p(\theta, \phi) \propto f(\theta, \phi)\sin\theta$$

Marginal/Conditional CDF

Assume the lat/long parameterization

Draw samples from joint $p(\theta, \phi) \propto L_{\text{env}}(\theta, \phi) \sin\theta$

- Step 1: create scalar version $L'(\theta, \phi)$ of $L_{\text{env}}(\theta, \phi) \sin\theta$
- Step 2: compute marginal PDF

$$p(\theta) = \int_0^{2\pi} L'(\theta, \phi)\, d\phi$$

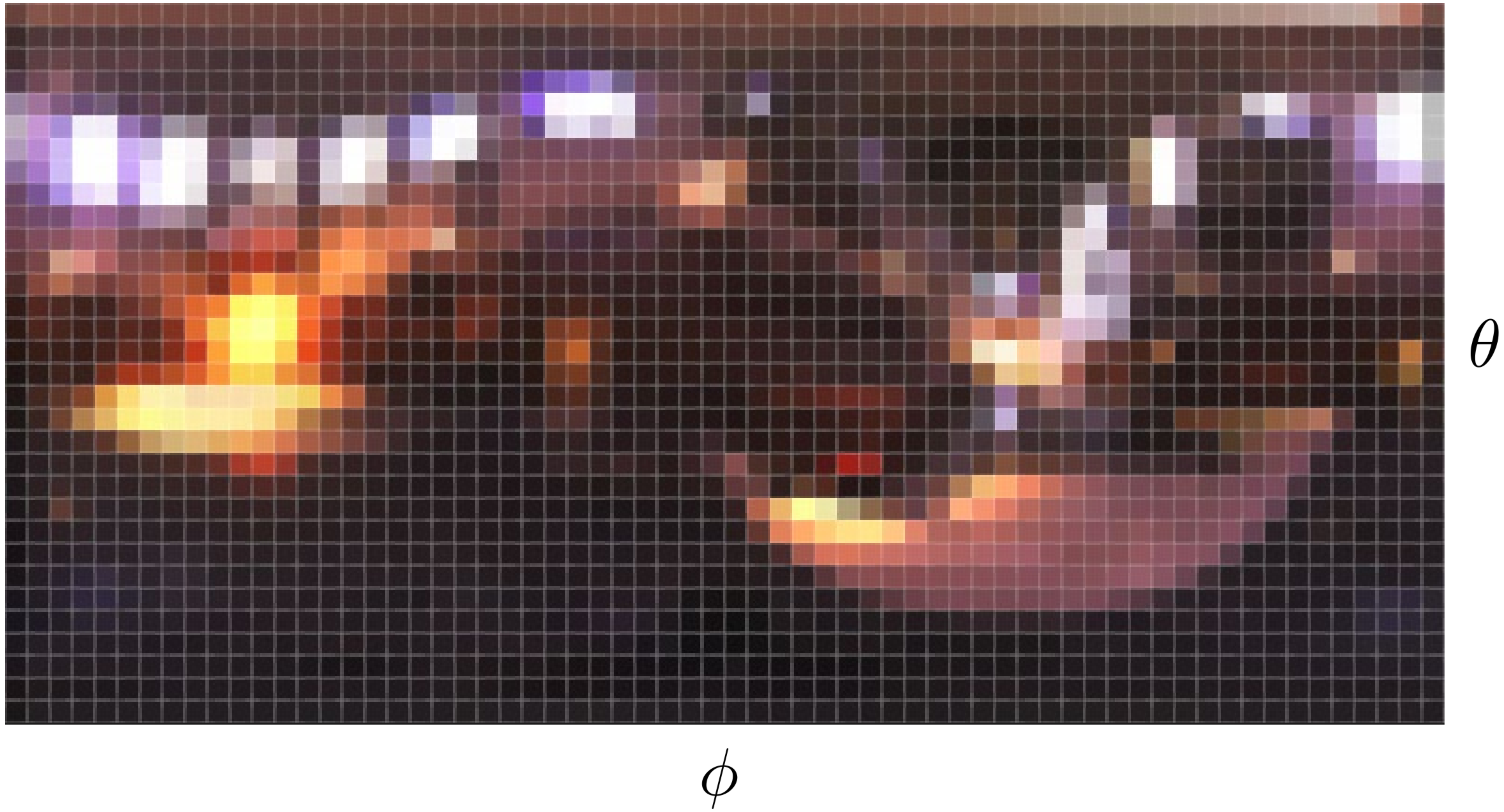
- Step 3: compute conditional PDF

$$p(\phi|\theta) = \frac{p(\theta, \phi)}{p(\theta)}$$

- Step 4: draw samples $\theta_i \sim p(\theta)$ and $\phi_i = p(\phi|\theta)$

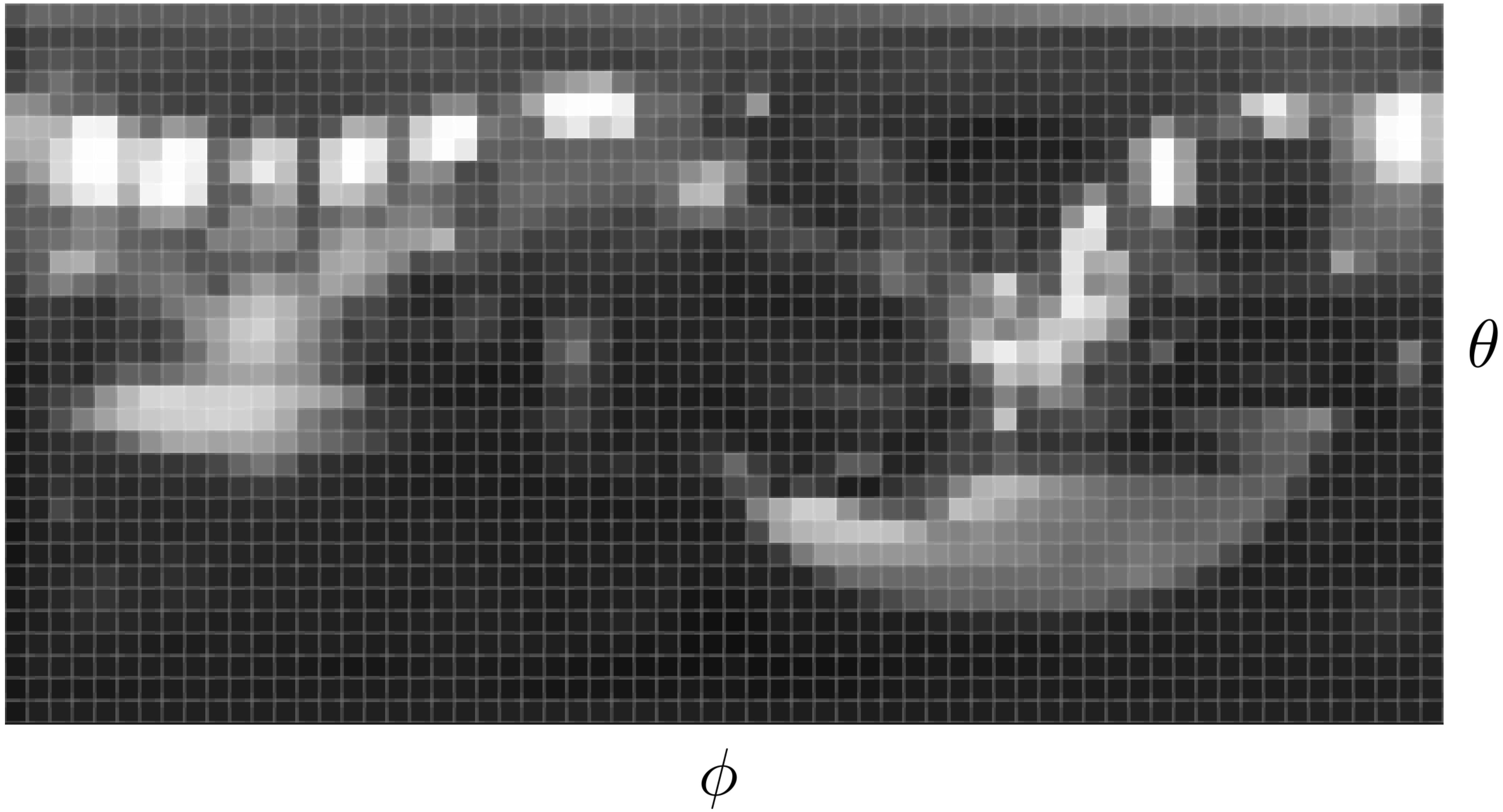
Step 1: Scalar Importance Func.

Original environment map



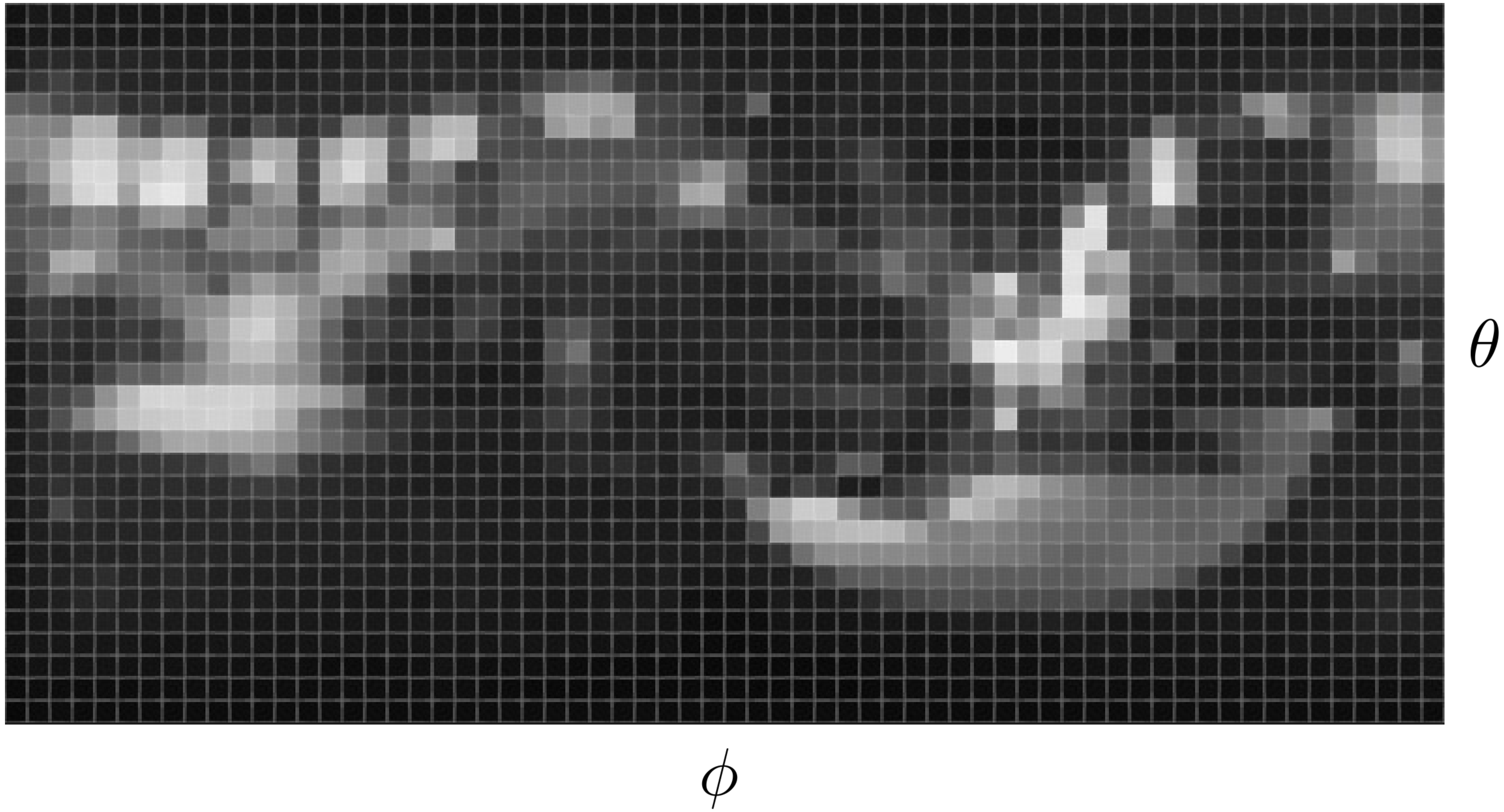
Step 1: Scalar Importance Func.

Scalar version
(average, max, or luminance of RGB channels)

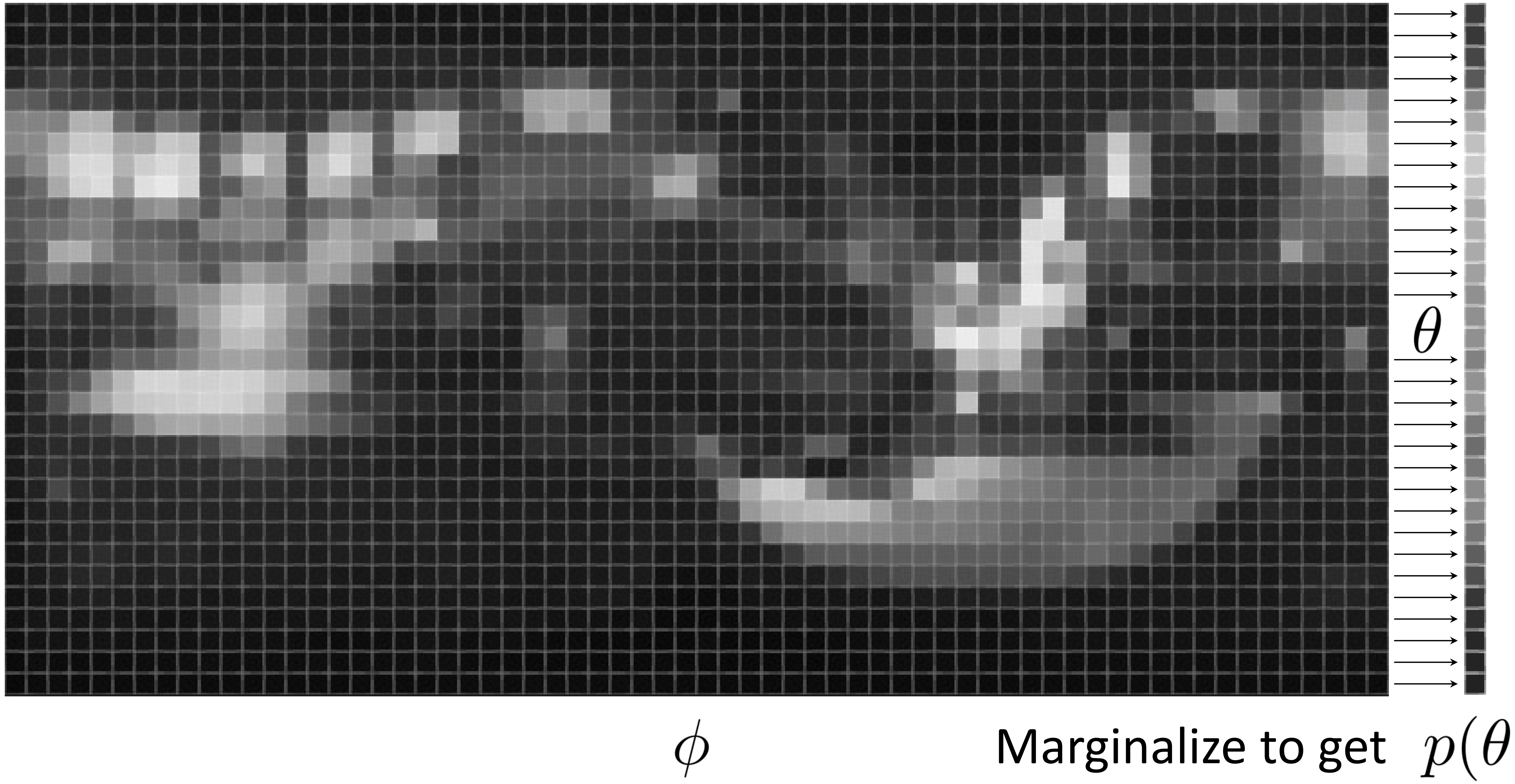


Step 1: Scalar Importance Func.

Multiplied by $\sin\theta$



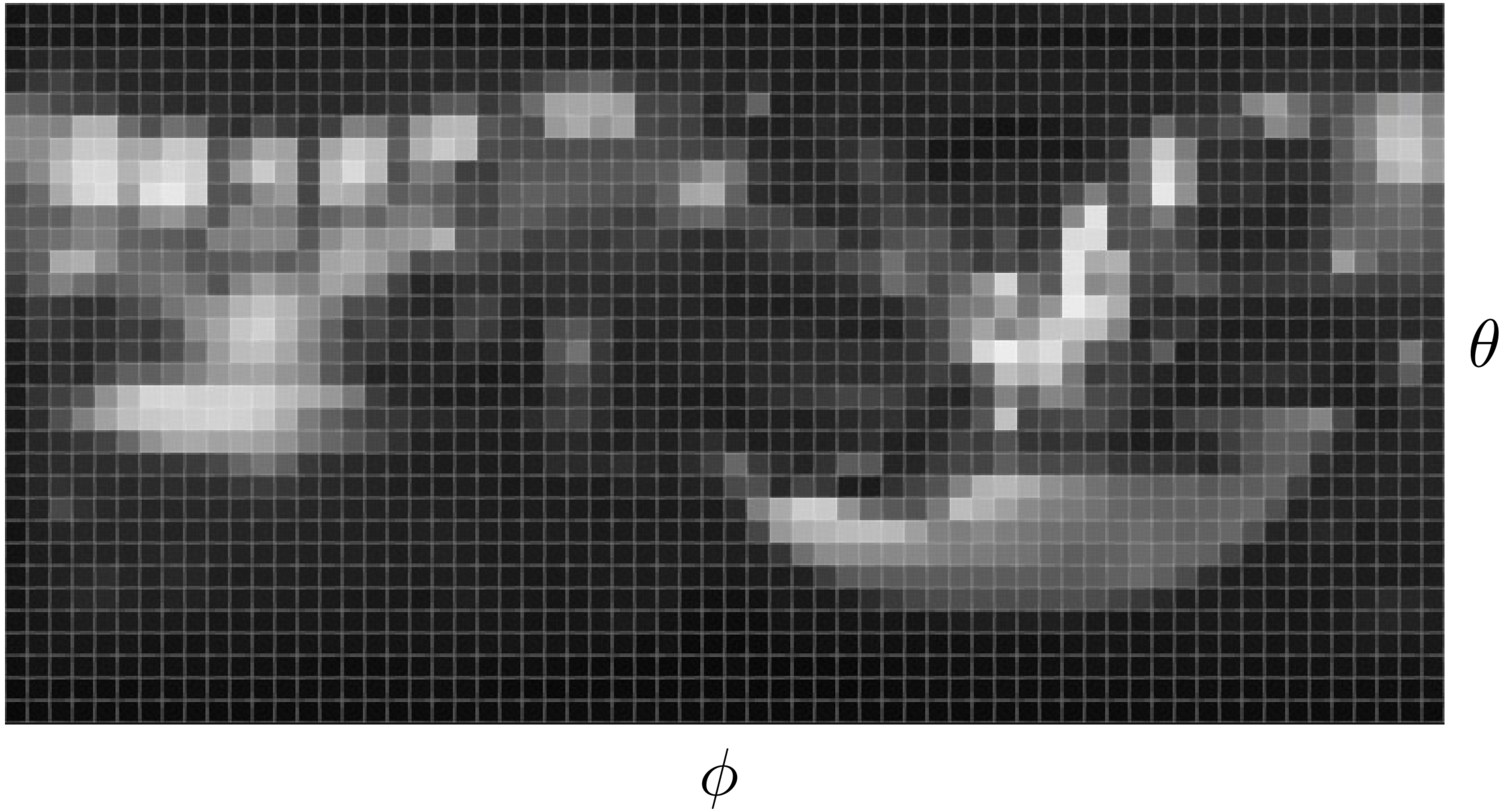
Step 2: Marginalization



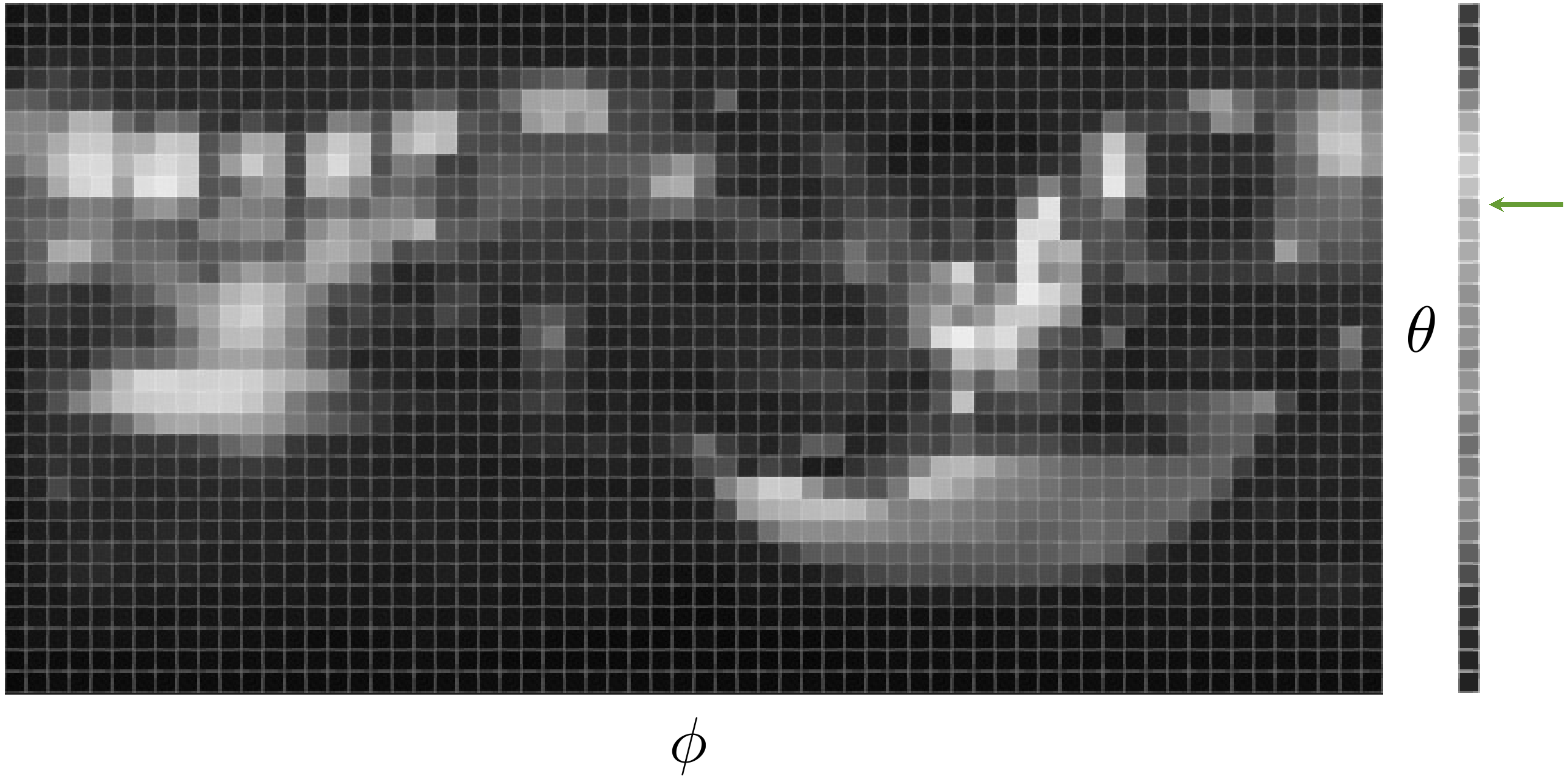
Marginalize to get $p(\theta)$

Step 3: Conditional PDFs

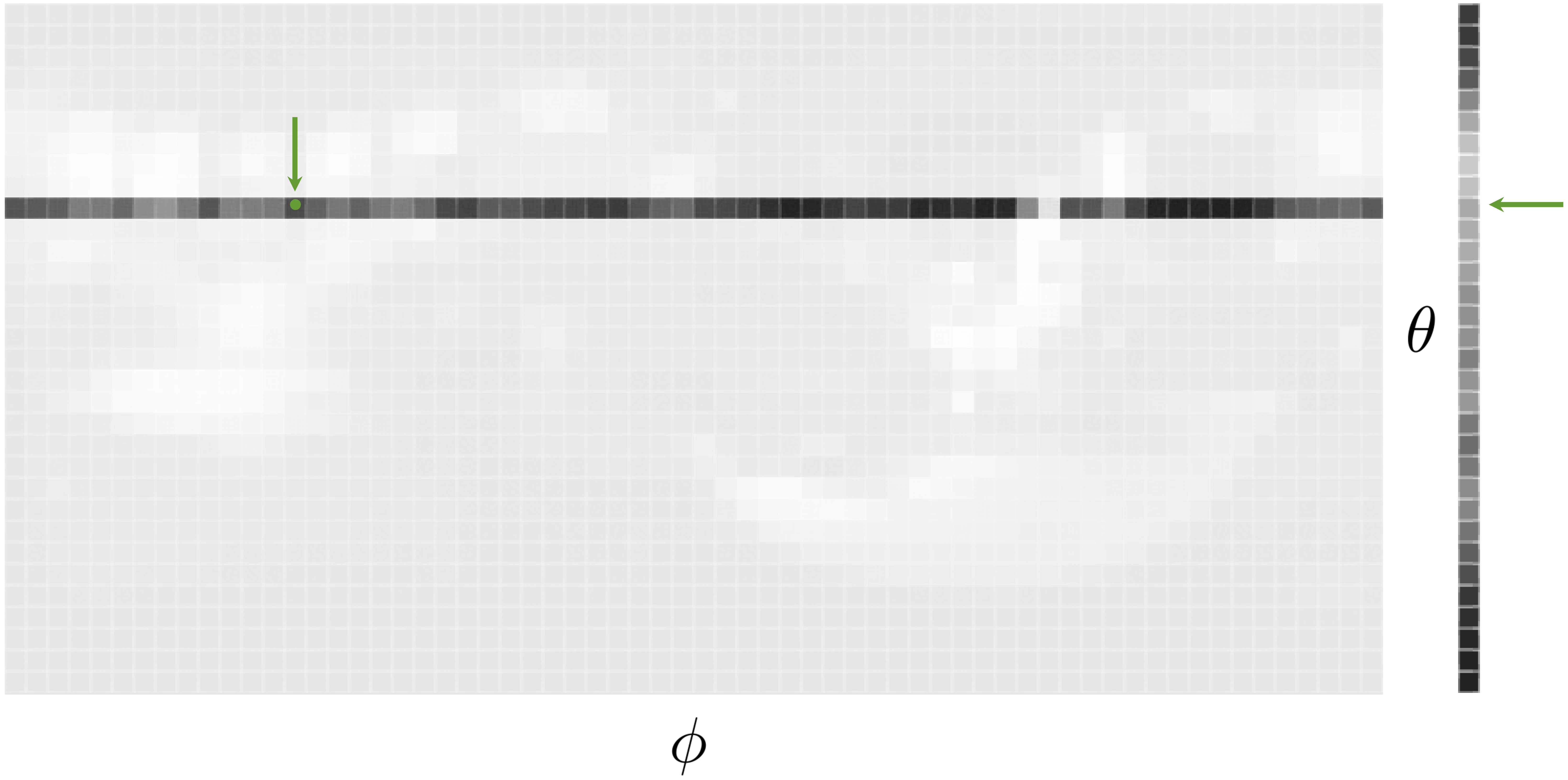
Once normalized, each row can serve as
the conditional PDF



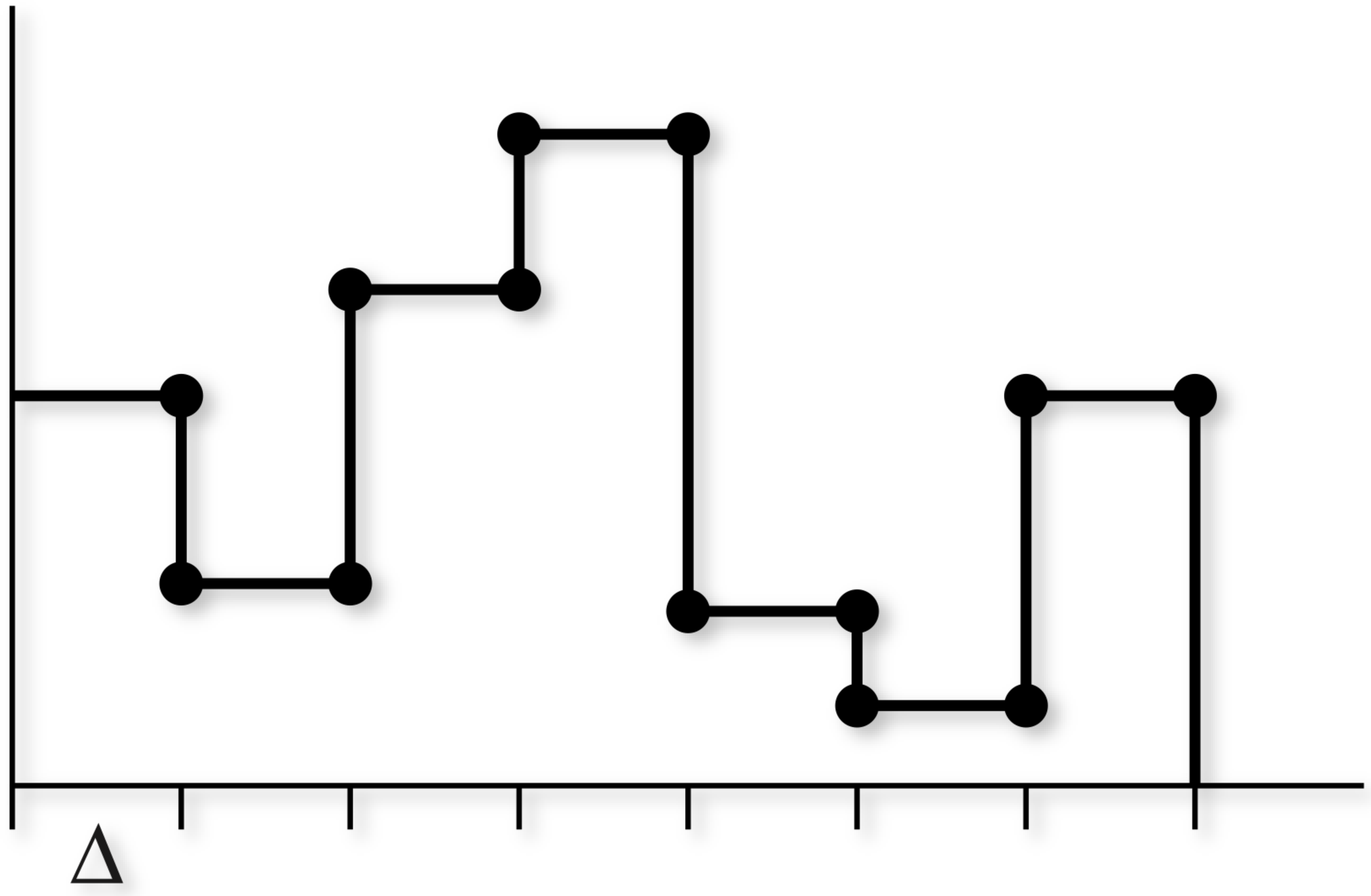
Step 4: Sampling



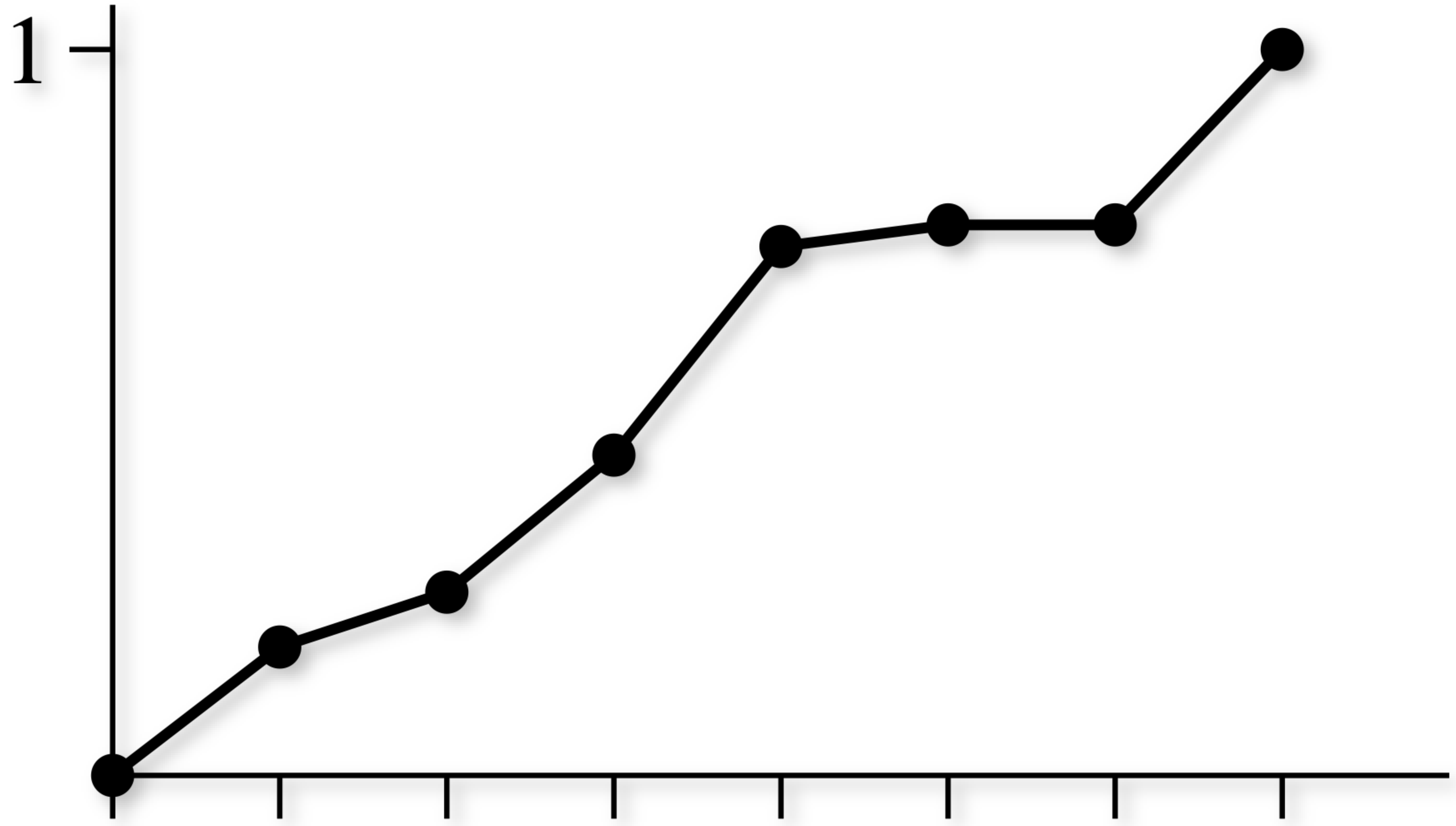
Step 4: Sampling



Sampling Discrete 1D PDFs



PDF



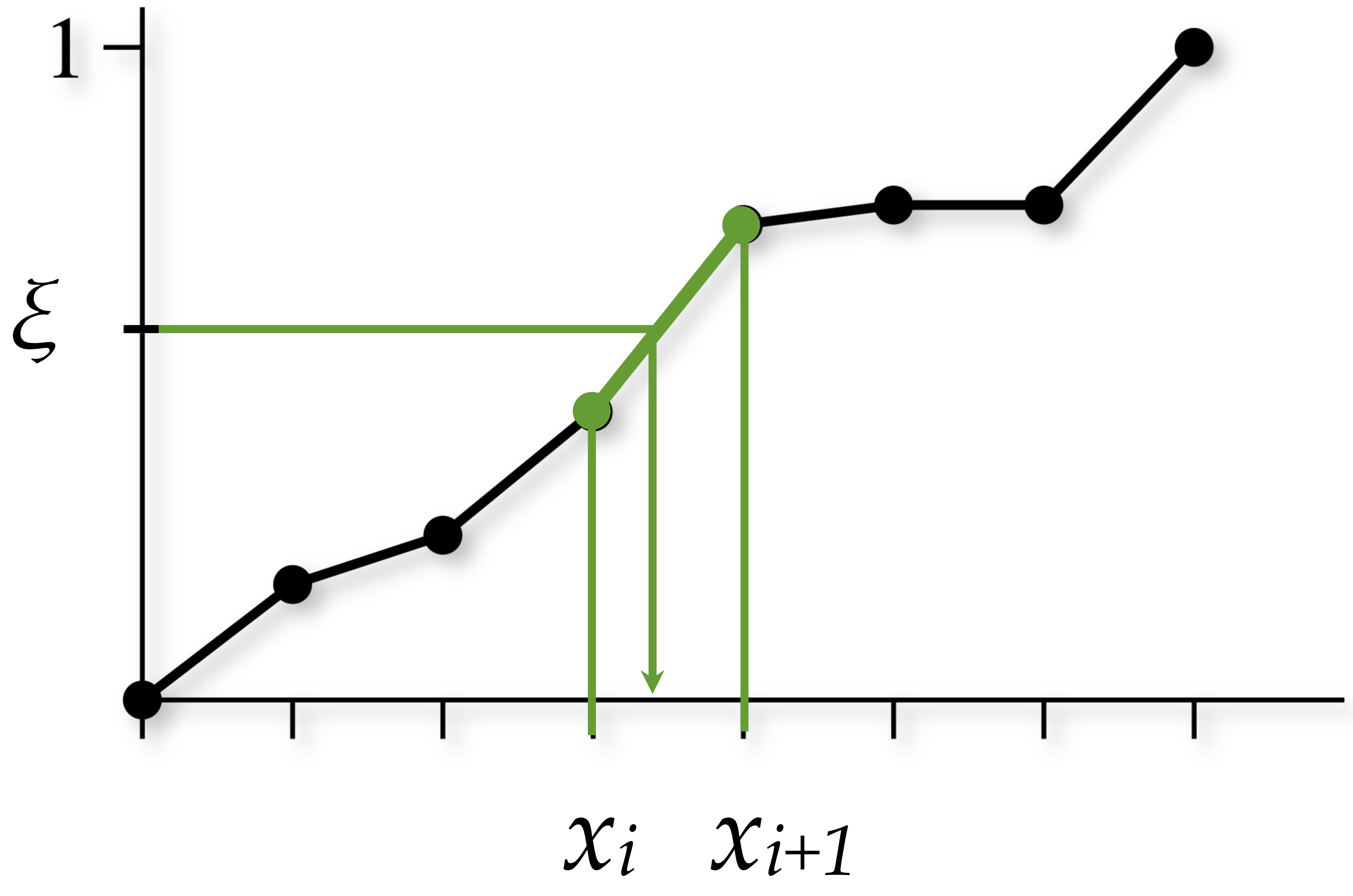
CDF

Sampling Discrete 1D PDFs

Given a uniform random value ξ

Find x_i and x_{i+1} using binary search

Linearly interpolate to find x

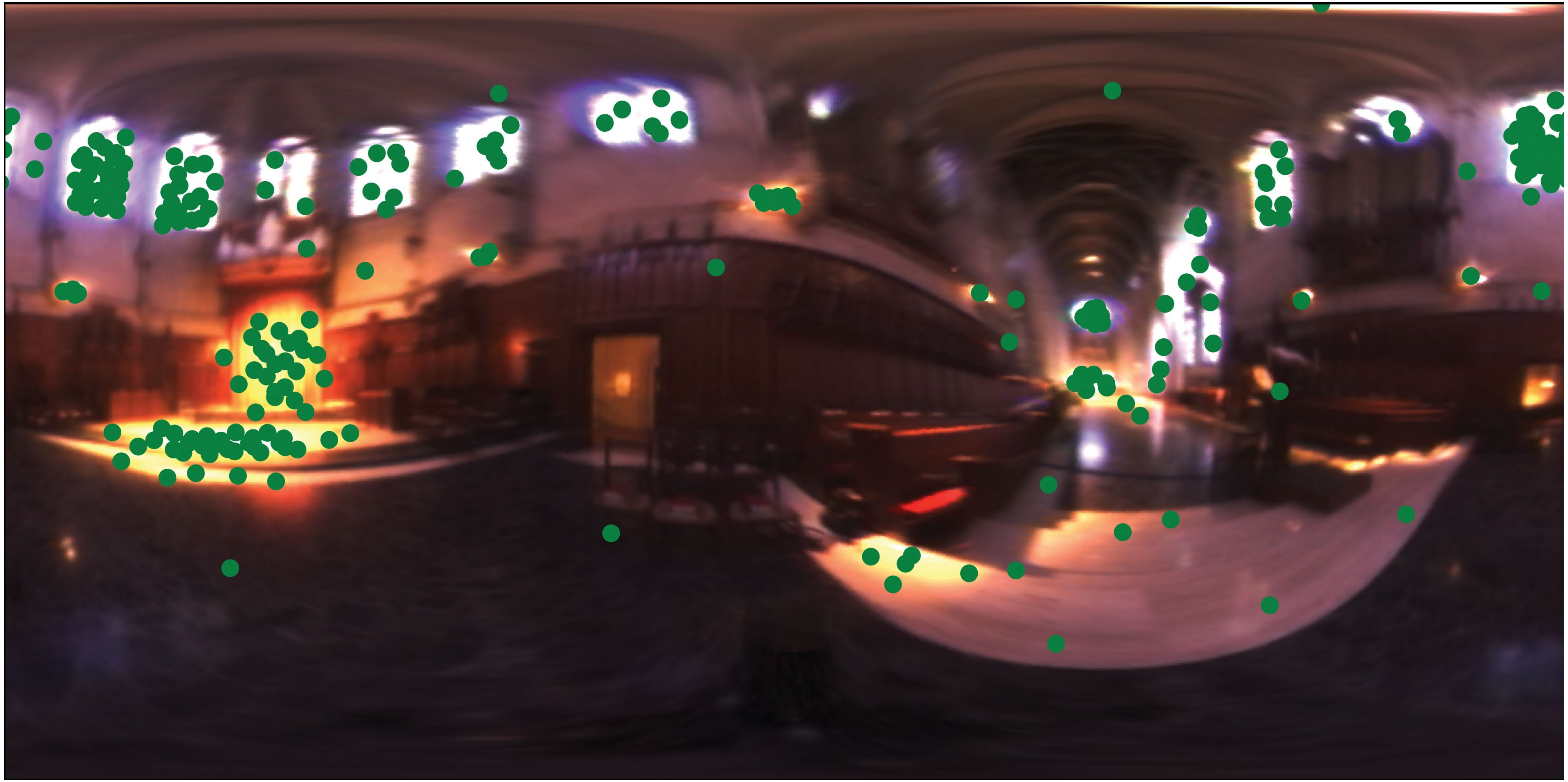


C++ details

Don’t need to implement binary search yourself!

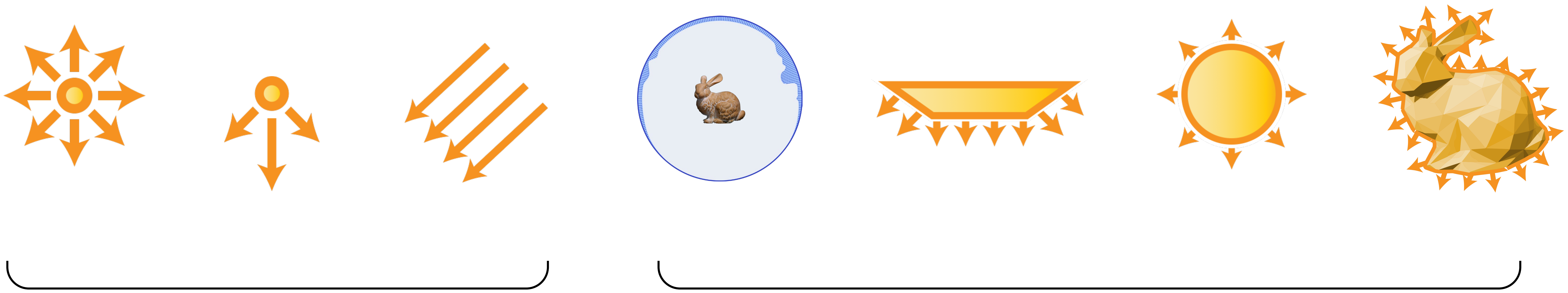
- Given sorted list, use std::lower_bound(…)
- See implementation in PBRT

Resulting Sample Distribution



hdrihaven.com

Light Sources



Point light

Spot light

Directional light

Environment light

Quad light

Sphere light

Mesh light

Delta lights

(create hard shadows)

Finite lights

(create soft shadows)

Point Light

Omnidirectional emission from a single point

Typically defined using a point $\mathbf{p}$ and emitted power Φ

- delta function with respect to which form of the reflection equation?

Point Light

Omnidirectional emission from a single point

Typically defined using a point $\mathbf{p}$ and emitted power Φ

- delta function with respect to surface integral form of the reflection equation

$$L_r(\mathbf{x},\mathbf{z}) = \int_{A_e} f_r(\mathbf{x},\mathbf{y},\mathbf{z}) \boxed{L_e(\mathbf{y},\mathbf{x})} V(\mathbf{x},\mathbf{y}) \frac{|\cos\theta_i|\,|\cos\theta_o|}{\|\mathbf{x}-\mathbf{y}\|^2} dA(\mathbf{y})$$

$$L_e(\mathbf{y},\mathbf{x}) = \frac{\Phi}{4\pi}\delta(\mathbf{y}-\mathbf{p})$$

$$L_r(\mathbf{x},\mathbf{z}) = \frac{\Phi}{4\pi} f_r(\mathbf{x},\mathbf{p},\mathbf{z}) V(\mathbf{x},\mathbf{p}) \frac{|\cos\theta_i|}{\|\mathbf{x}-\mathbf{p}\|^2}$$

Point Light

Omnidirectional emission from a single point

Typically defined using a point $\mathbf{p}$ and emitted power Φ

- delta function with respect to surface integral form of the reflection equation

$$L_r(\mathbf{x}, \mathbf{z}) = \frac{\Phi}{4\pi} f_r(\mathbf{x}, \mathbf{p}, \mathbf{z}) V(\mathbf{x}, \mathbf{p}) \frac{|\cos \theta_i|}{\|\mathbf{x} - \mathbf{p}\|^2}$$

Spot Light?

Directionally dependent emission from a single point

Typically defined using a point $\mathbf{p}$ and …

$$L_r(\mathbf{x}, \mathbf{z}) = \boxed{\frac{\Phi}{4\pi}} f_r(\mathbf{x}, \mathbf{p}, \mathbf{z}) V(\mathbf{x}, \mathbf{p}) \frac{|\cos\theta_i|}{\|\mathbf{x} - \mathbf{p}\|^2}$$

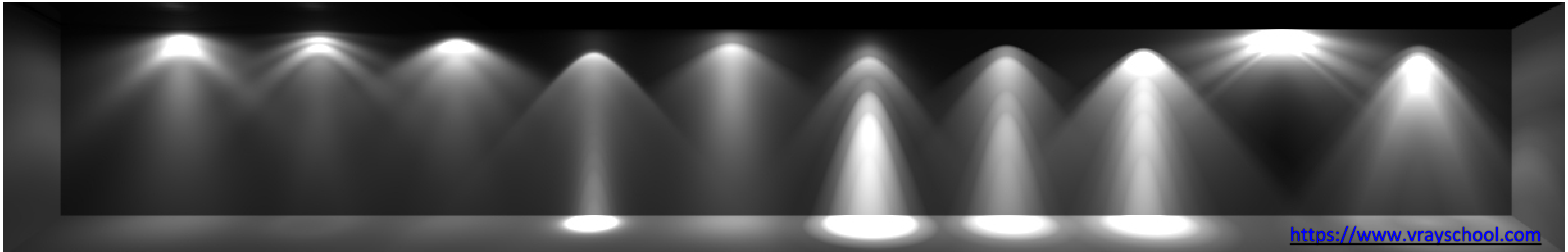
Spot Light

Directionally dependent emission from a single point

Typically defined using a point $\mathbf{p}$ and a directionally dependent radiant intensity function I

$$L_r(\mathbf{x},\mathbf{z}) = \boxed{I(\mathbf{p},\mathbf{x})} f_r(\mathbf{x},\mathbf{p},\mathbf{z}) V(\mathbf{x},\mathbf{p}) \frac{|\cos\theta_i|}{\|\mathbf{x}-\mathbf{p}\|^2}$$

The intensity can be defined using IES profiles:



https://www.vrayschool.com

Directional Light

Far-away emission from single direction (delta environment map)

Typically defined using a direction $\vec{\omega}_d$ and radiance L_d

- delta function with respect to which form of the reflection equation?

Directional Light

Far-away emission from single direction (delta environment map)

Typically defined using a direction $\vec{\omega}_d$ and radiance L_d

- delta function with respect to hemispherical integral form of the reflection equation

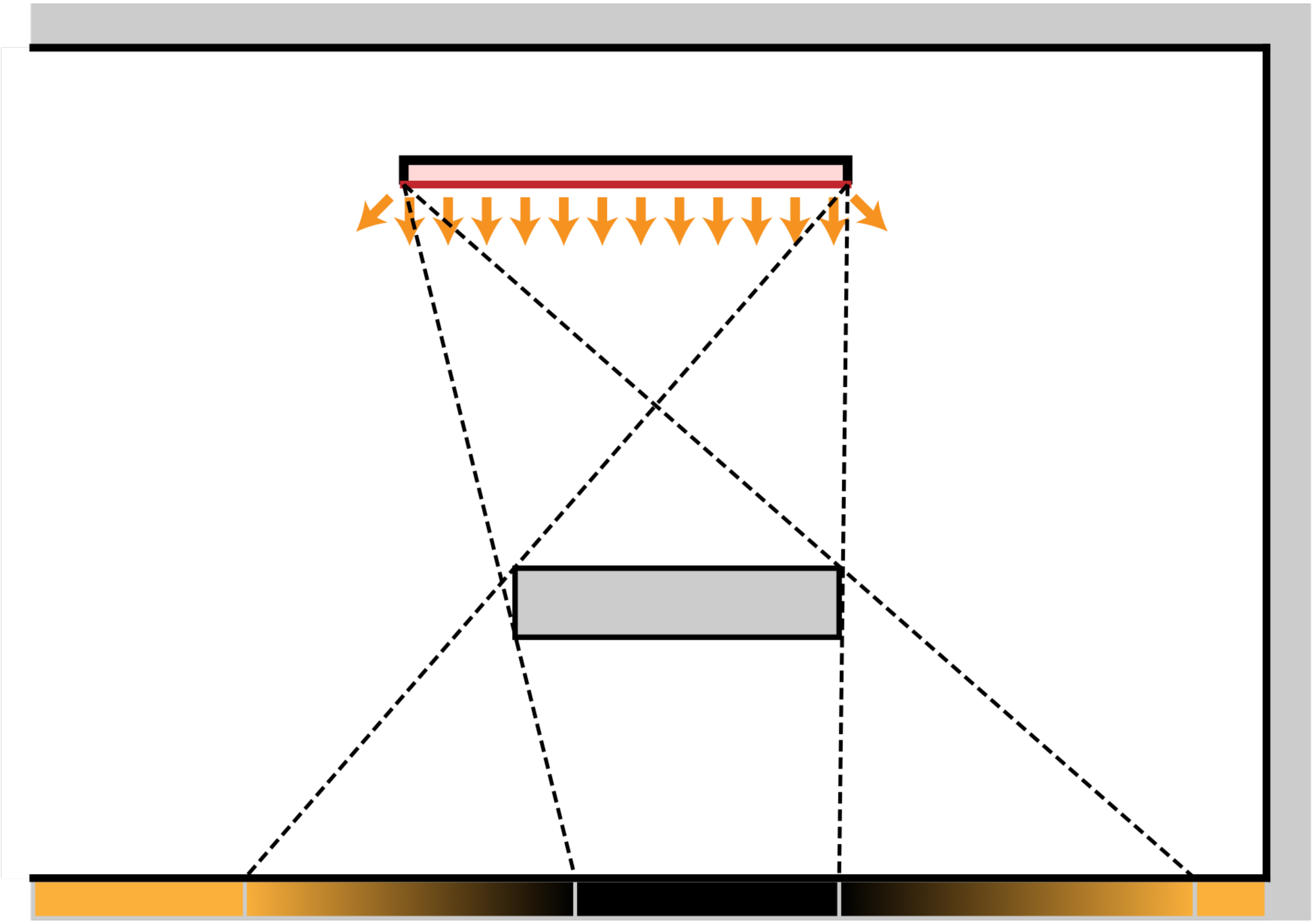
$$L_r(\mathbf{x}, \vec{\omega}_r) = {}_e\int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) \boxed{L_e(r(\mathbf{x}, \vec{\omega}_i), -\vec{\omega}_i)} \cos\theta_i \, d\vec{\omega}_i$$

$$L_e(\mathbf{y}, \vec{\omega}) = V(\mathbf{y}, \vec{\omega}_d) L_d \delta(\vec{\omega}_d - \vec{\omega})$$

$$L(\mathbf{x}, \vec{\omega}_r) = f_r(\mathbf{x}, \vec{\omega}_d, \vec{\omega}_r) V(\mathbf{x}, \vec{\omega}_d) L_d \cos\theta_d$$

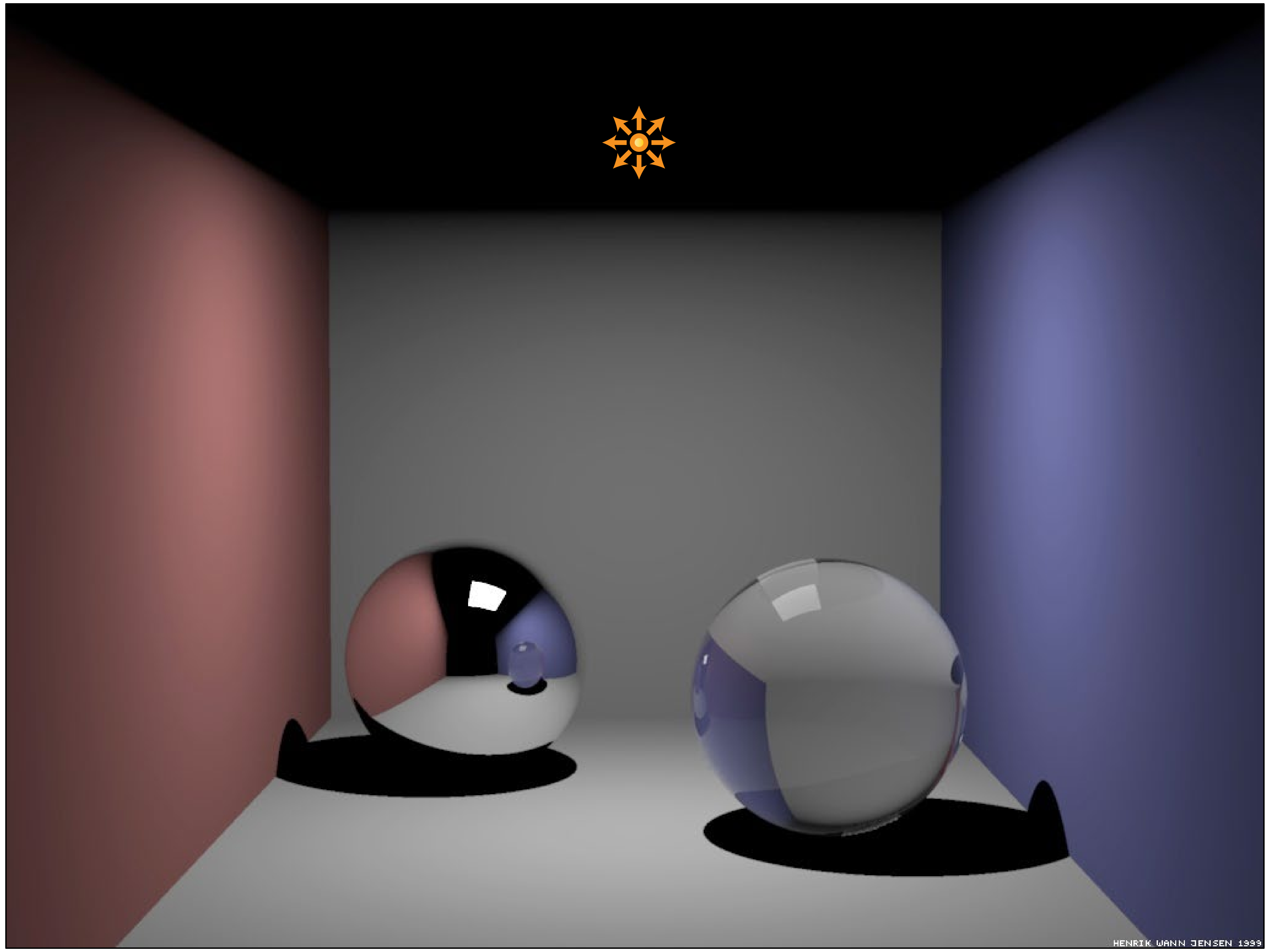
Quad Light

Has finite area… creates soft shadows

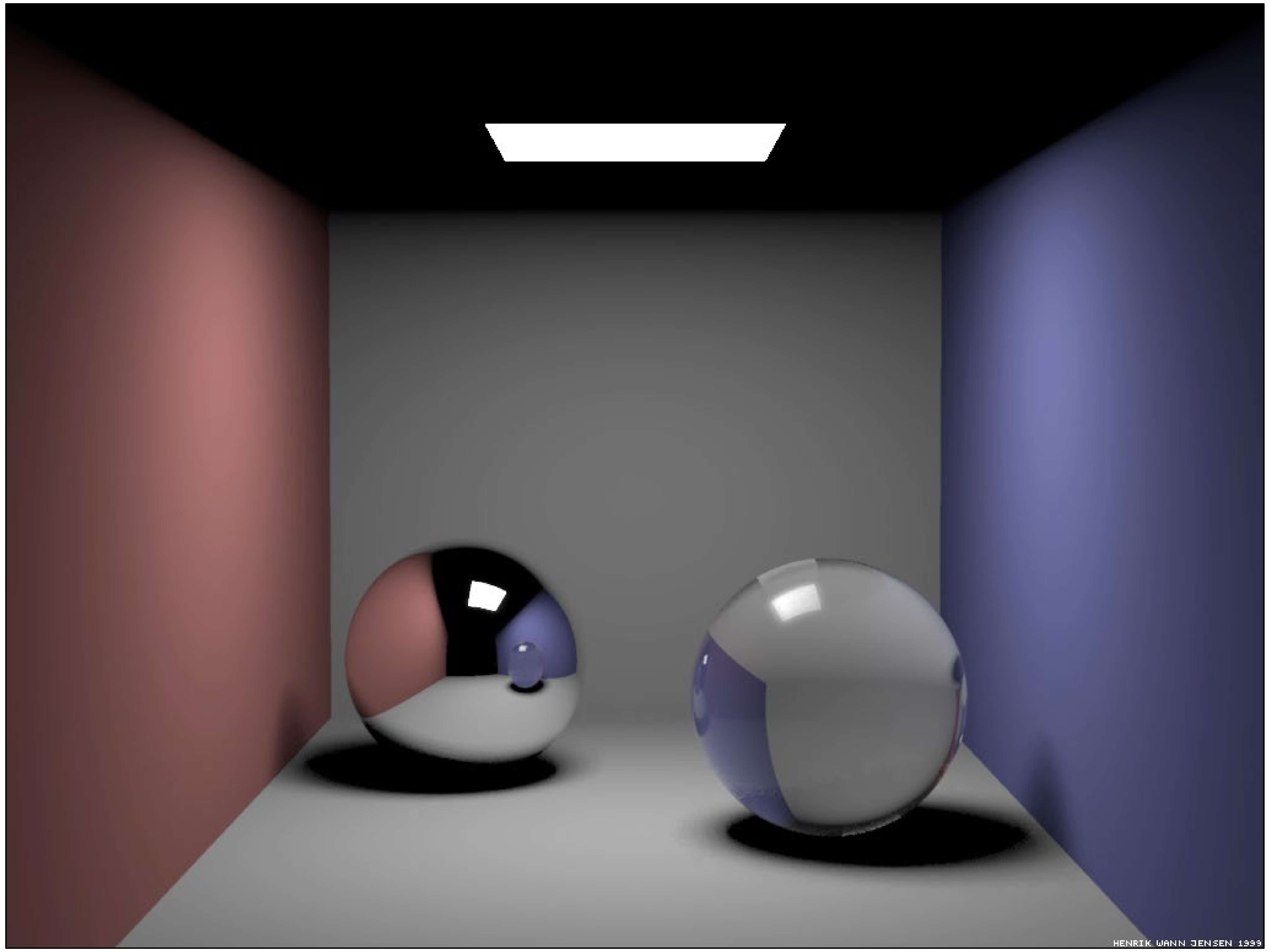


Quad Light

Point light



Quad light



Sphere Light

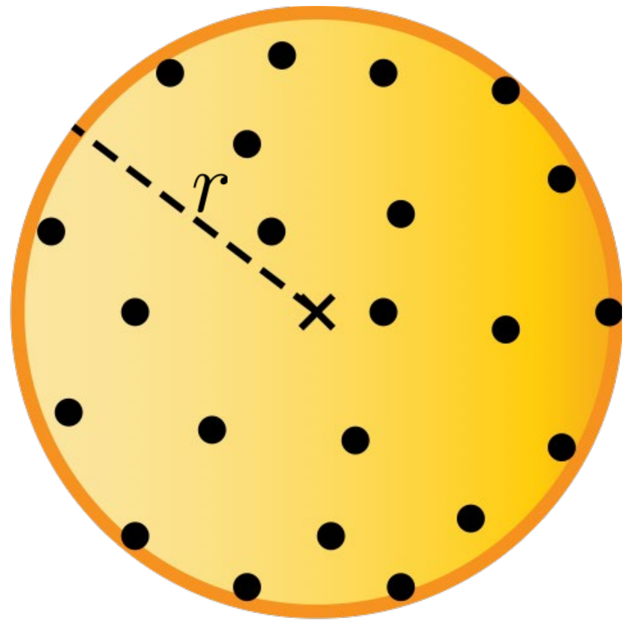
Typically defined using a center $\mathbf{p}$, radius r, and emitted power Φ (or emitted radiance L_e)

Has finite surface area $4\pi r^2$

Sphere Light

How to sample points on the sphere light?

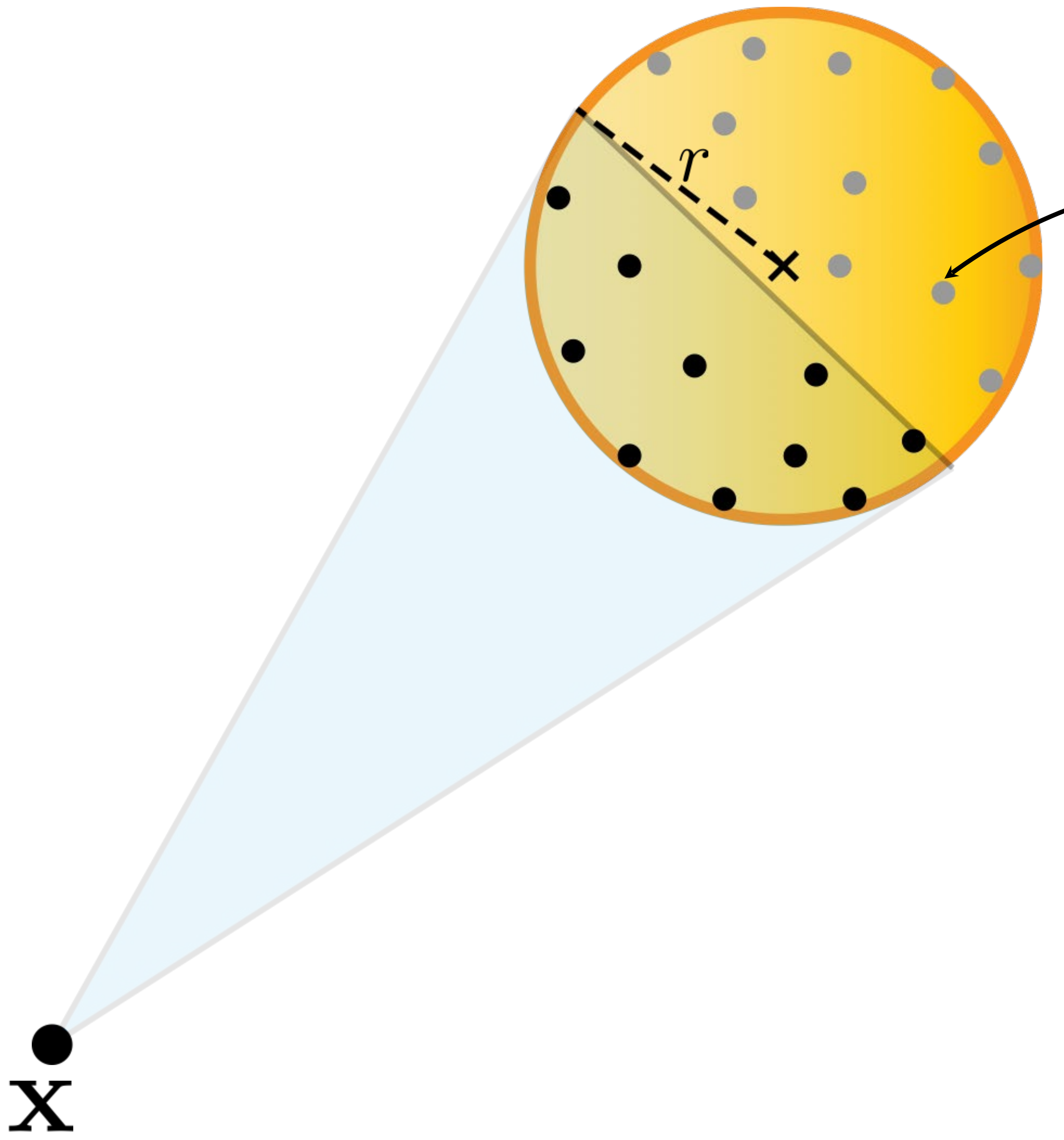
Approach 1: uniformly sample *sphere* *area*



Sphere Light

How to sample points on the sphere light?

Approach 1: uniformly sample *sphere* *area*

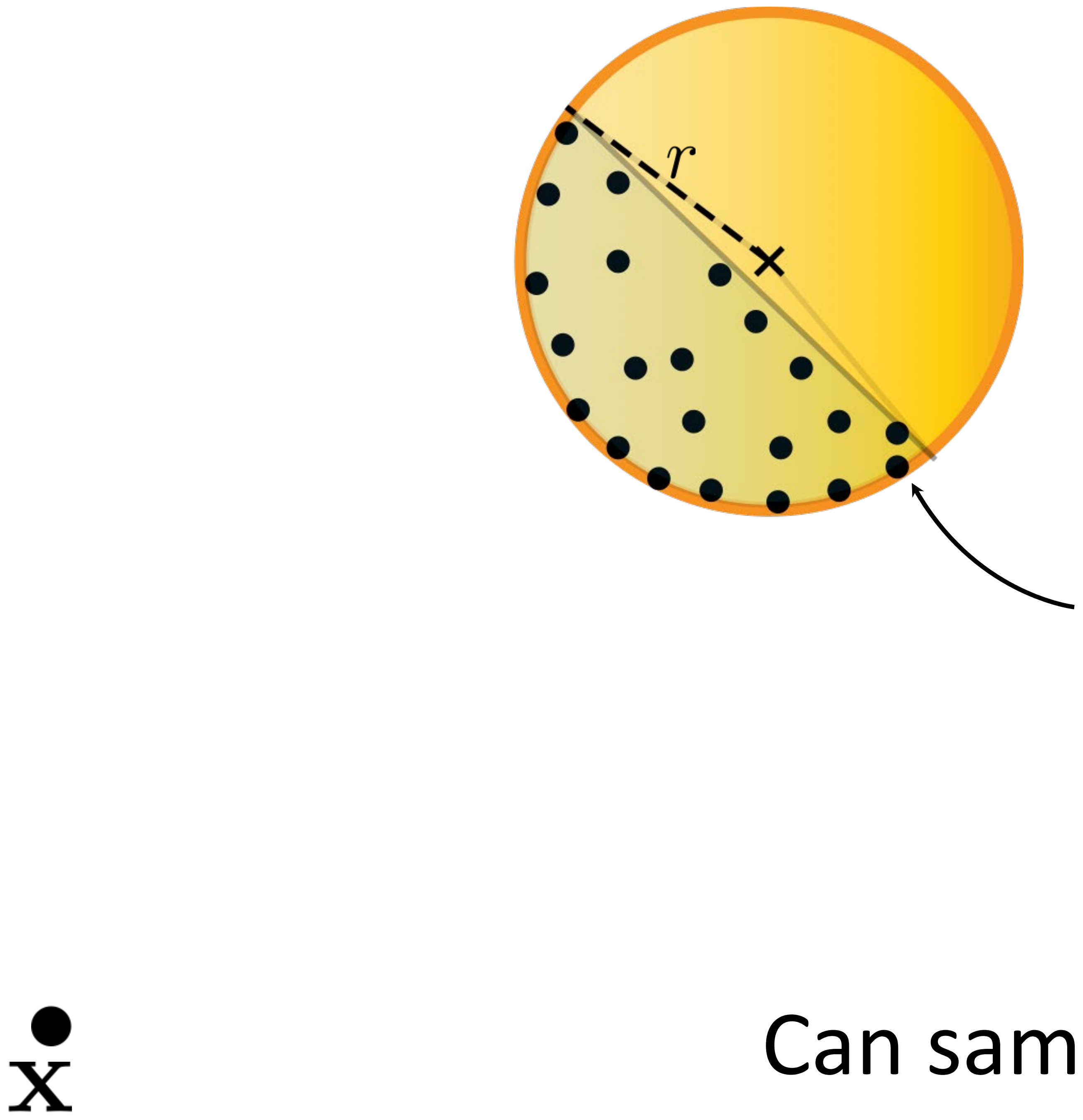


Many samples are not visible from the shading point!

Sphere Light

How to sample points on the sphere light?

Approach 2 (better): uniformly sample *area* of the *visible spherical cap*



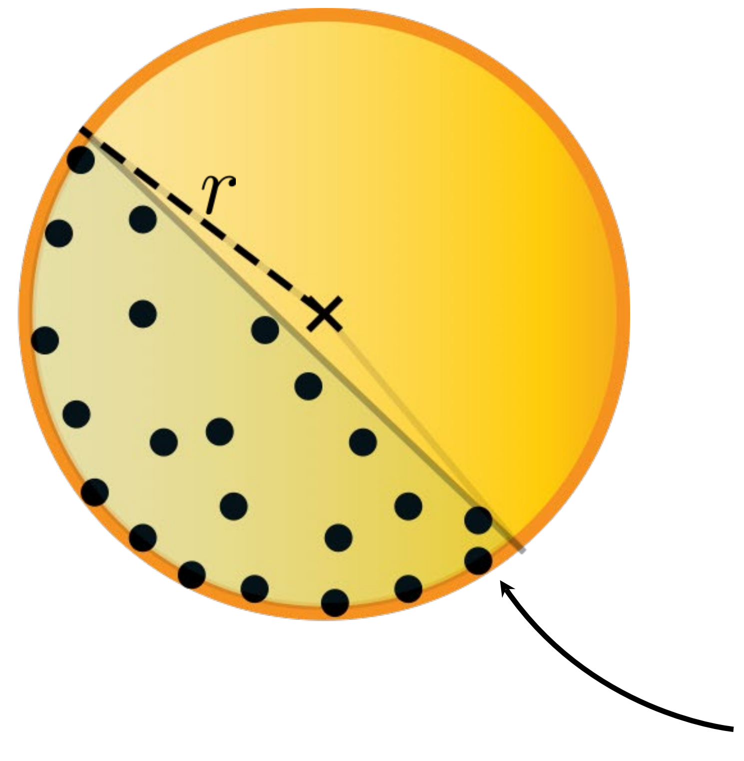
spherical cap on light *area*

Can sample a spherical cap using Hat-Box theorem!

Sphere Light

How to sample points on the sphere light?

Approach 2 (better): uniformly sample *__area__* of the *visible spherical cap*

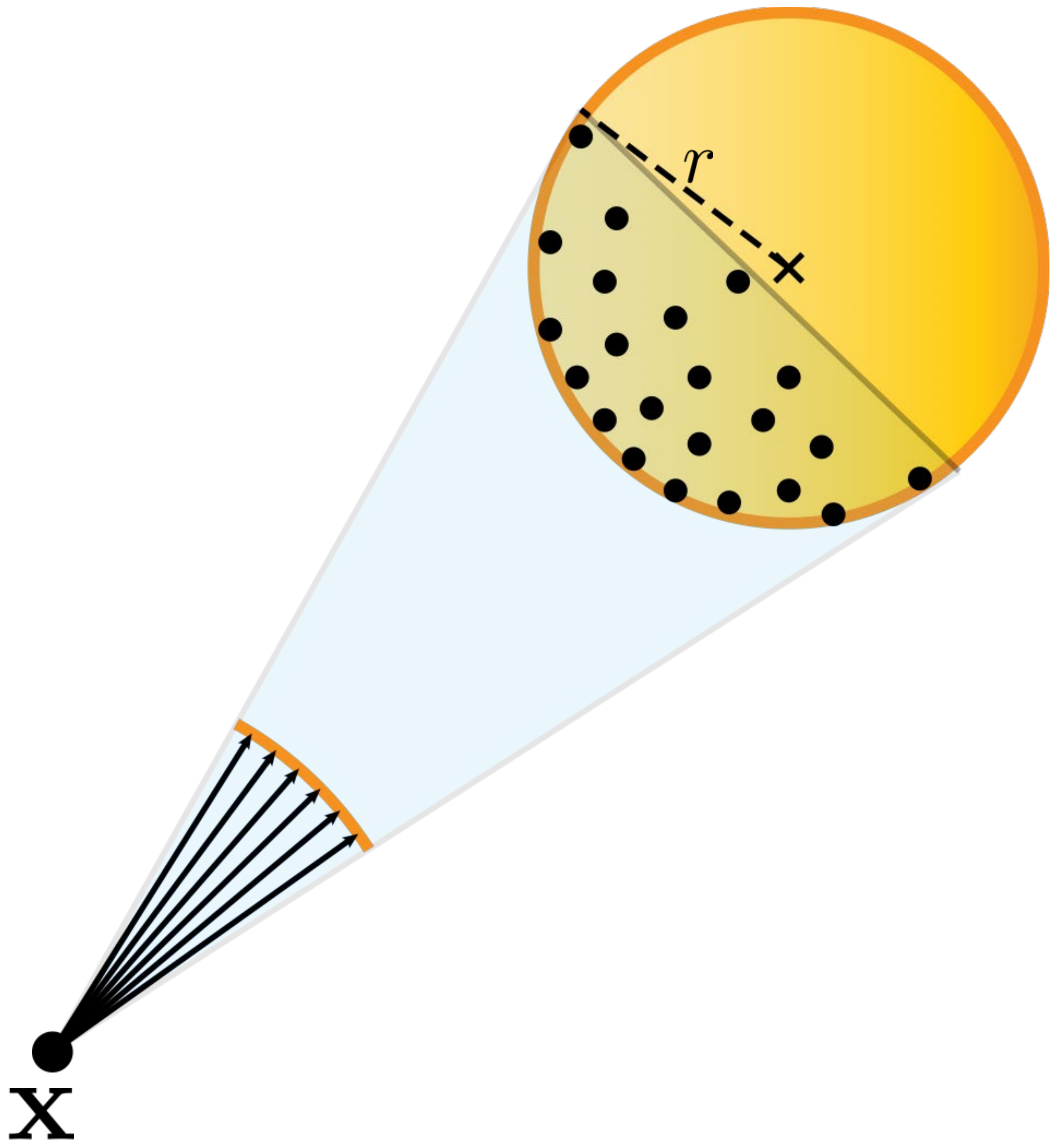


Uniform area-density is not ideal as emitted radiance is weighted by the cosine term (recall the form factor in the G term)

Sphere Light

How to sample points on the sphere light?

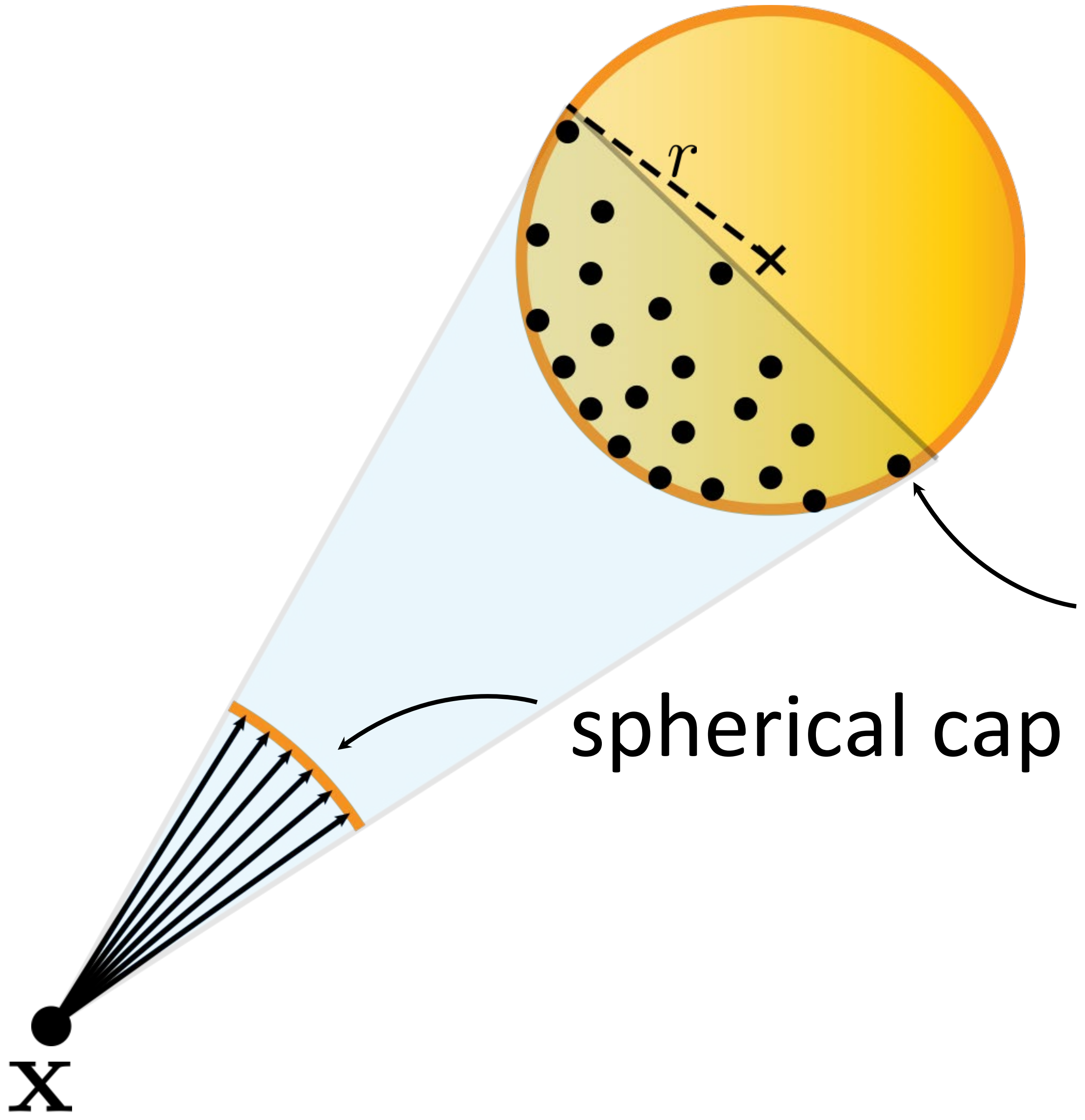
Approach 3 (even better): uniformly sample *solid angle subtended* by the sphere



Sphere Light

How to sample points on the sphere light?

Approach 3 (even better): uniformly sample *solid angle subtended* by the sphere



resulting points on area of light

spherical cap of *directions*

Sphere Light

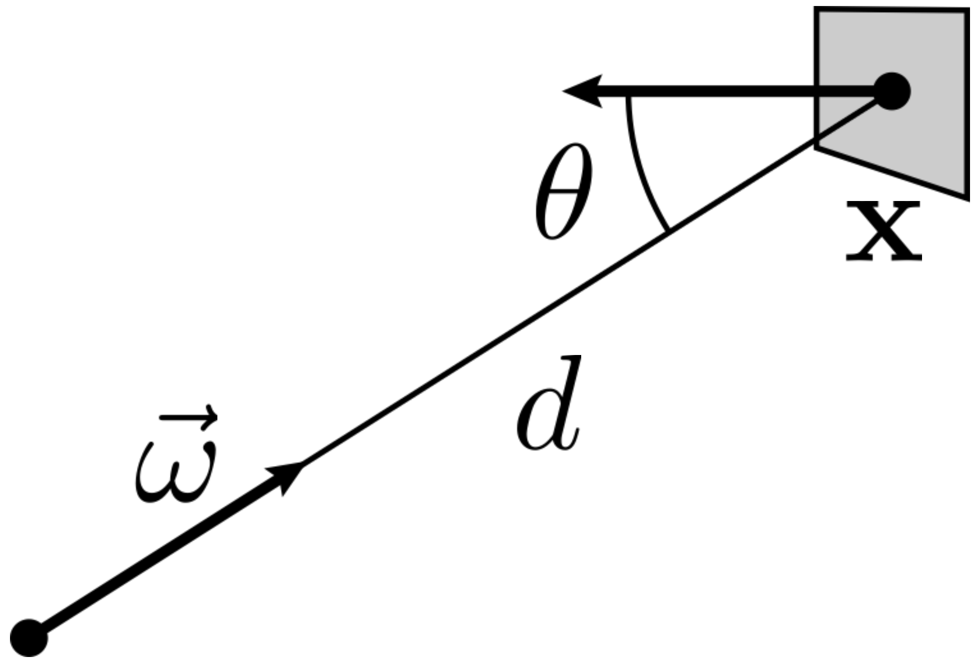
How to sample points on the sphere light?

Caution!

- Approaches use PDFs defined wrt different measures
- Make sure to convert the PDF into the measure of the integral!

$$p_A(\mathbf{x}) = \frac{\cos\theta}{d^2} p_\Omega(\vec{\omega})$$

$$p_\Omega(\vec{\omega}) = \frac{d^2}{\cos\theta} p_A(\mathbf{x})$$

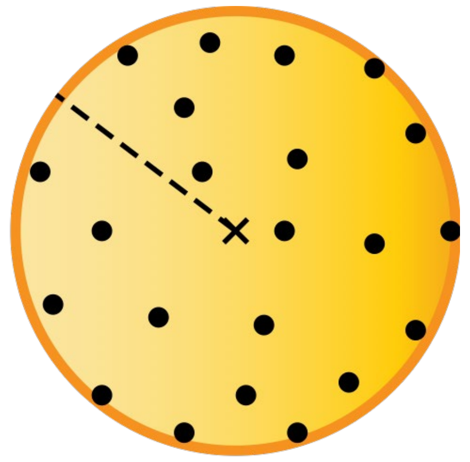


Sphere Light

How to sample points on the sphere light?

Caution!

- Approaches use PDFs defined wrt different measures
- Make sure to convert the PDF into the measure of the integral!
- Example: using approach 1 for MC integration of the hemispherical formulation of the reflection eq.

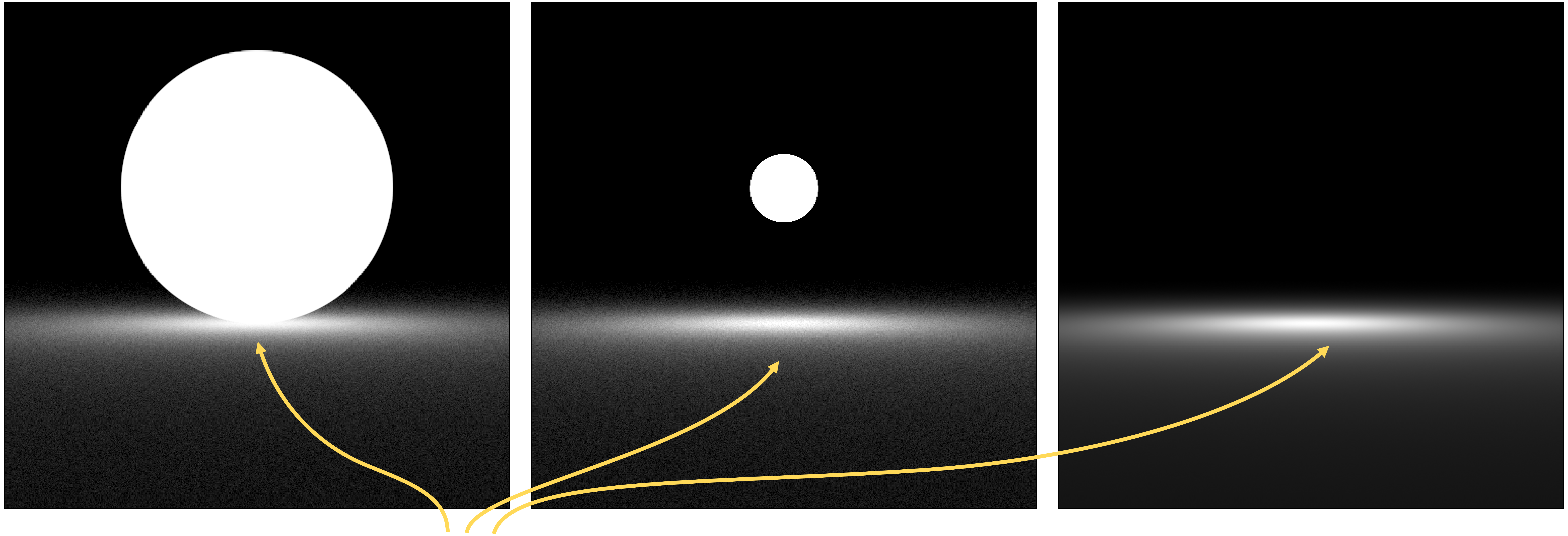


$$\langle L_r(\mathbf{x},\vec{\omega}_r)\rangle = \frac{1}{N}\sum_{k=1}^{N}\frac{f_r(\mathbf{x},\vec{\omega}_{i,k},\vec{\omega}_r)L_i(\mathbf{x},\vec{\omega}_{i,k})\cos\theta_{i,k}}{p_\Omega(\vec{\omega}_{i,k})}$$

$$p_A(\mathbf{y}) = \frac{1}{4\pi r^2} \qquad p_\Omega(\vec{\omega}_i) = \frac{\|\mathbf{x}-\mathbf{y}\|^2}{\left|-\omega_i\cdot\mathbf{n}_\mathbf{y}\right|4\pi r^2}$$

Sphere Light

Validation: irradiance is independent of radius
(assuming it emits always the same power & no occluders)



A sphere light

A smaller sphere light

A point light

Identical irradiance profiles

Mesh Light

An emissive mesh where every surface point emits given radiance L_{e}

Total area: $\sum A(k)$

Mesh Light

How to importance sample?

Preprocess:

- build a discrete PDF, p_Δ, for choosing polygons (triangles) *proportional to their area:*

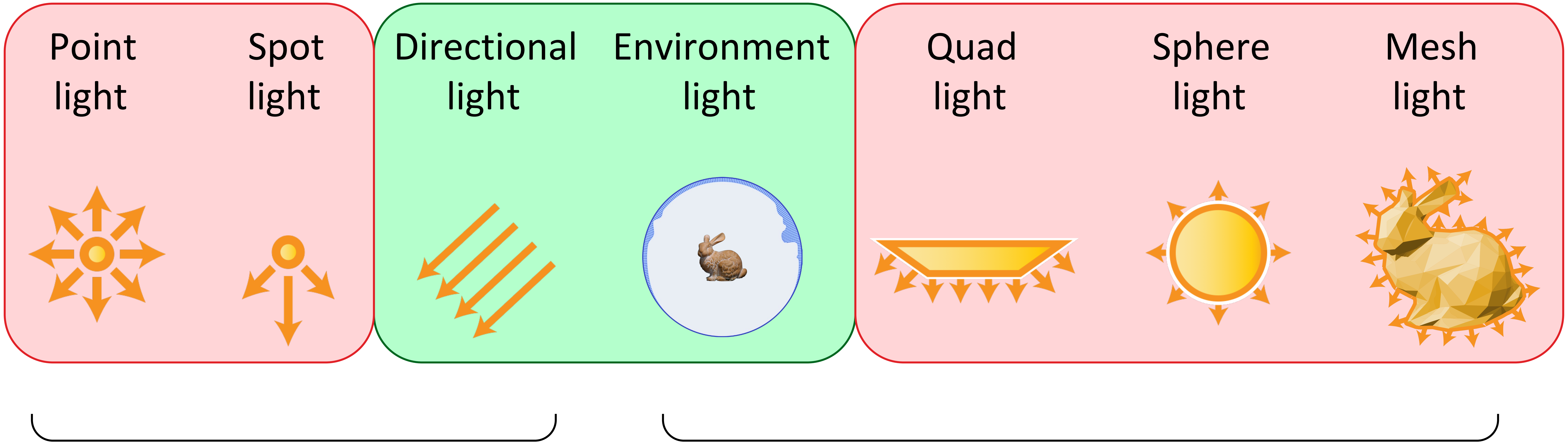
$$p_\Delta(i) = \frac{A(i)}{\sum_k A(k)}$$

Run-time:

- sample a polygon i and a point $\mathbf{x}$ on i
- compute the PDF of choosing the point:

$$p_A(\mathbf{x}) = p_\Delta(i) p_A(\mathbf{x}|i) = \frac{1}{\sum A(k)}$$

Light Sources



Point light

Spot light

Directional light

Environment light

Quad light

Sphere light

Mesh light

Delta lights

(create hard shadows)

Finite lights

(create soft shadows)

- sample using surface integral form
- sample using hemispherical integral form

typically, but not always

Reflection Equation

$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

What terms can we importance sample?

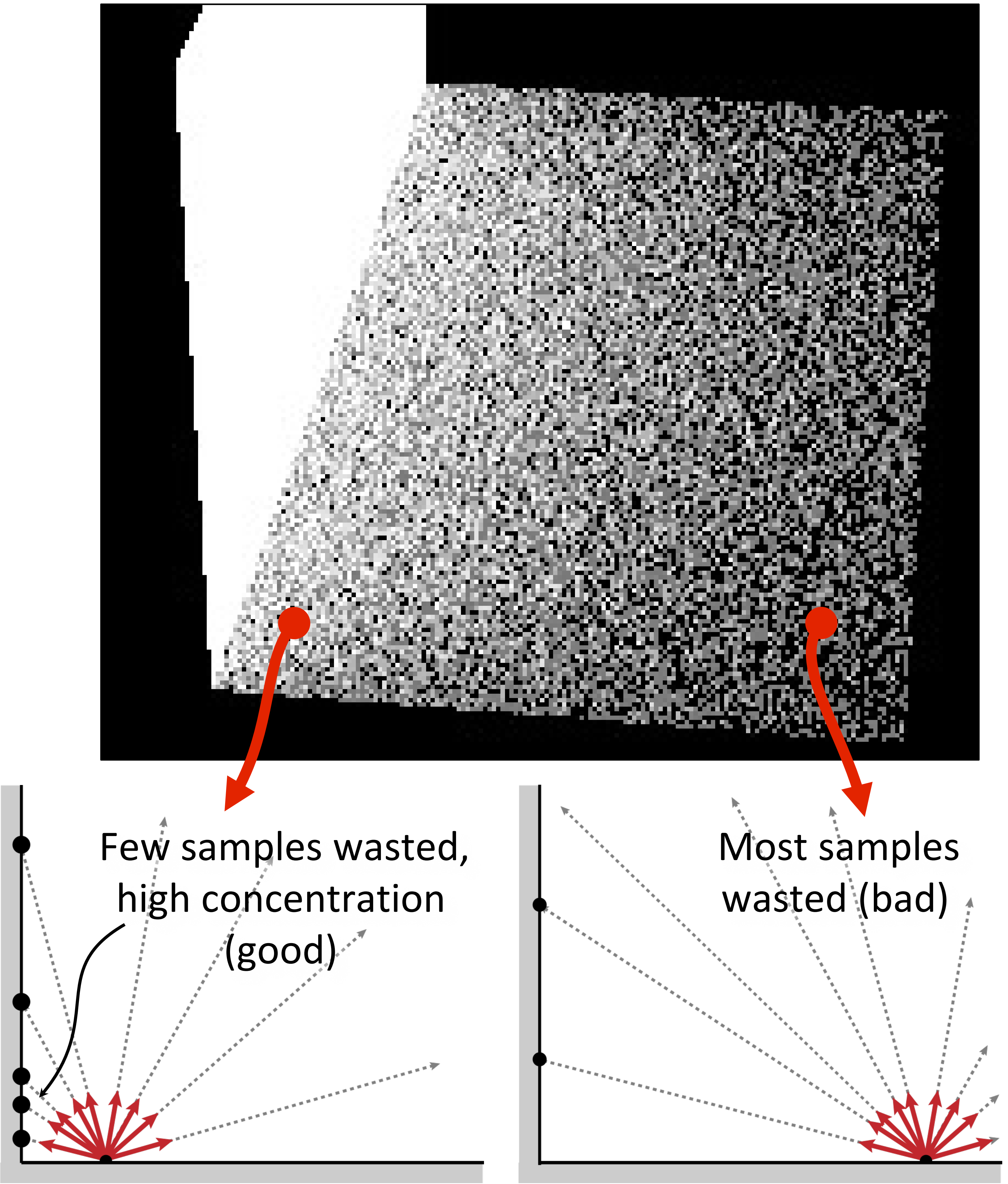
- BRDF
- incident radiance
- cosine term

What terms **should** we importance sample?

- depends on the context, hard to make a general statement

Multiple Strategies

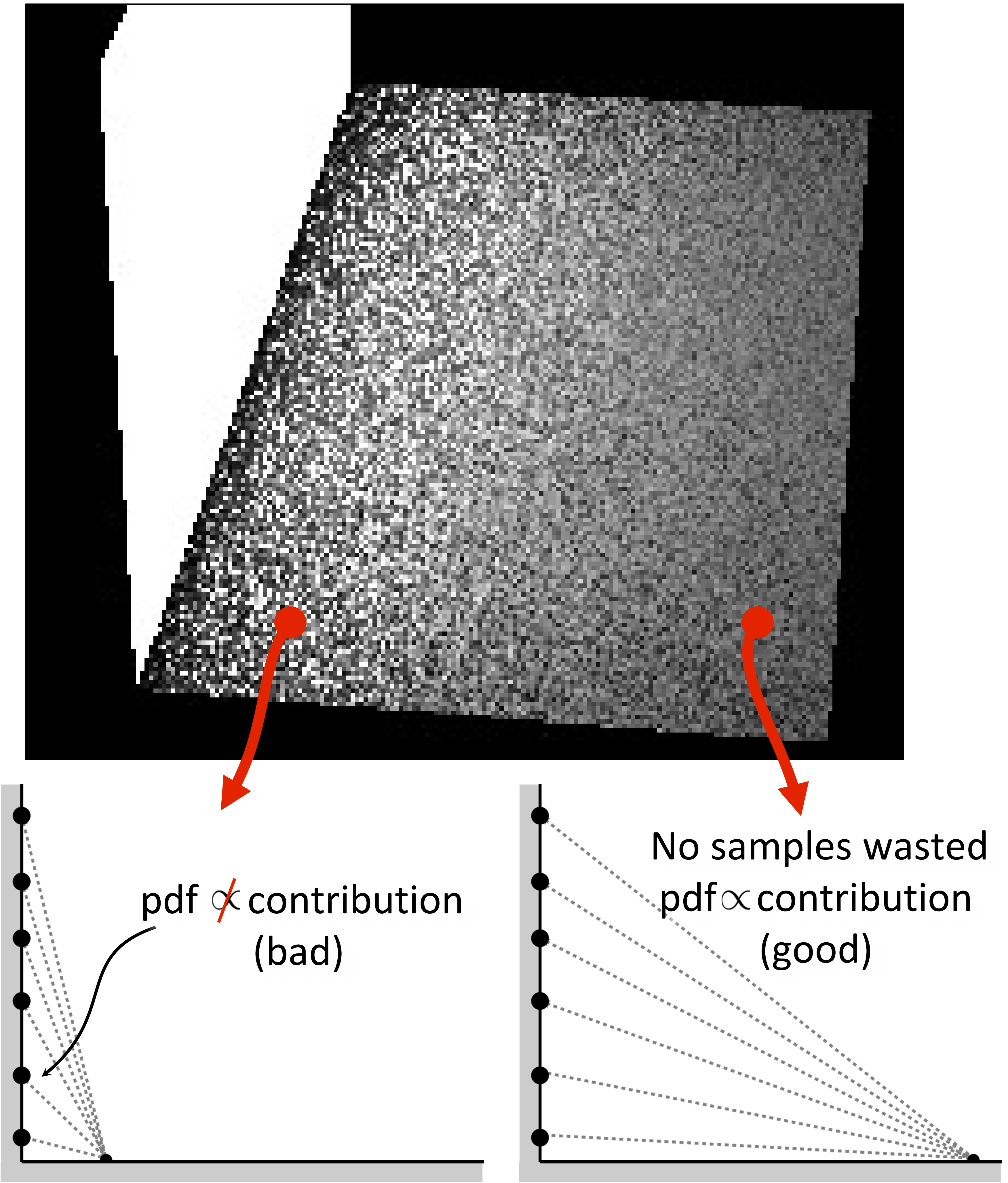
Cosine-weighted hemisphere



Few samples wasted, high concentration (good)

Most samples wasted (bad)

Uniform surface area

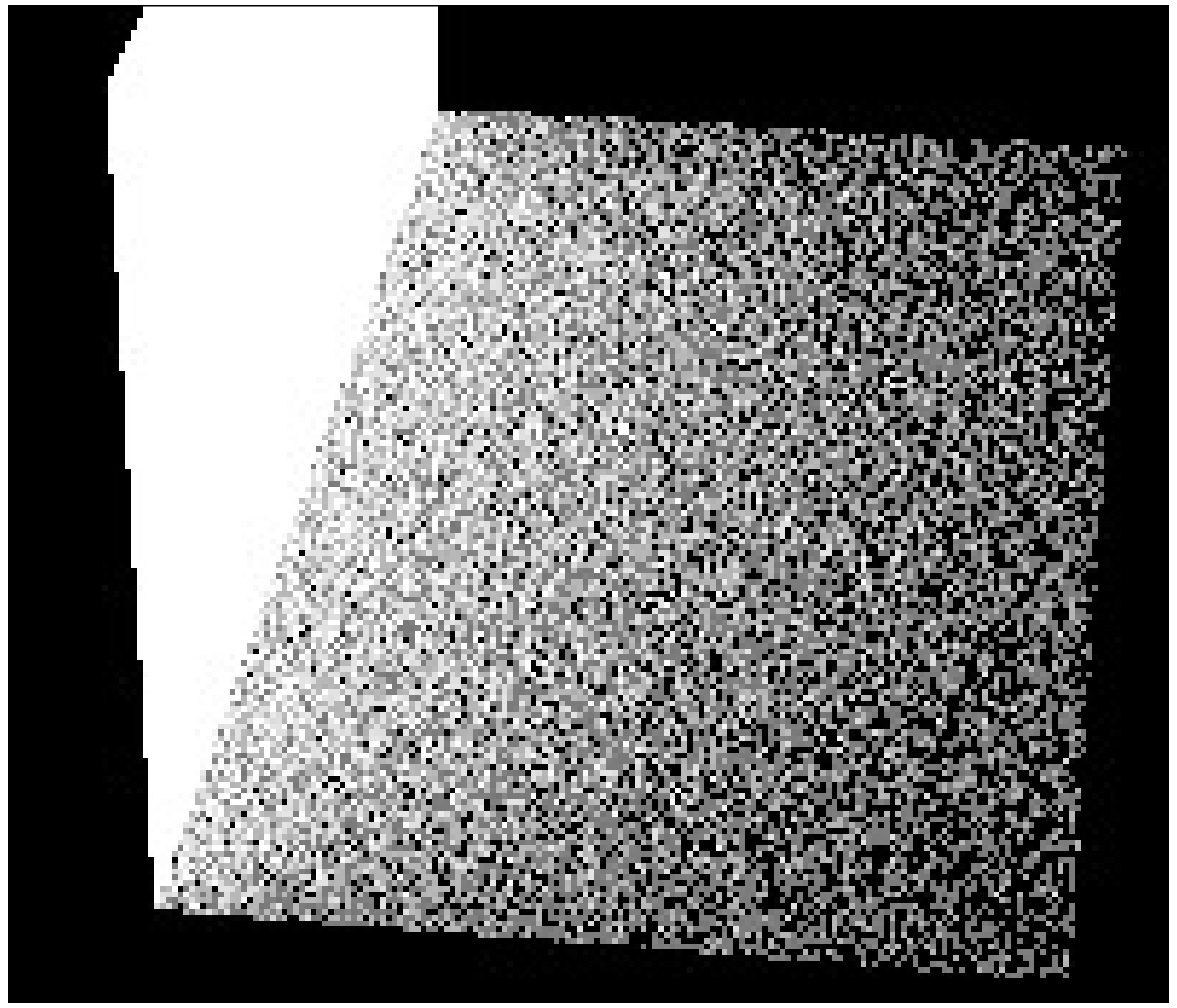


pdf $\not\propto$ contribution (bad)

No samples wasted pdf $\propto$ contribution (good)

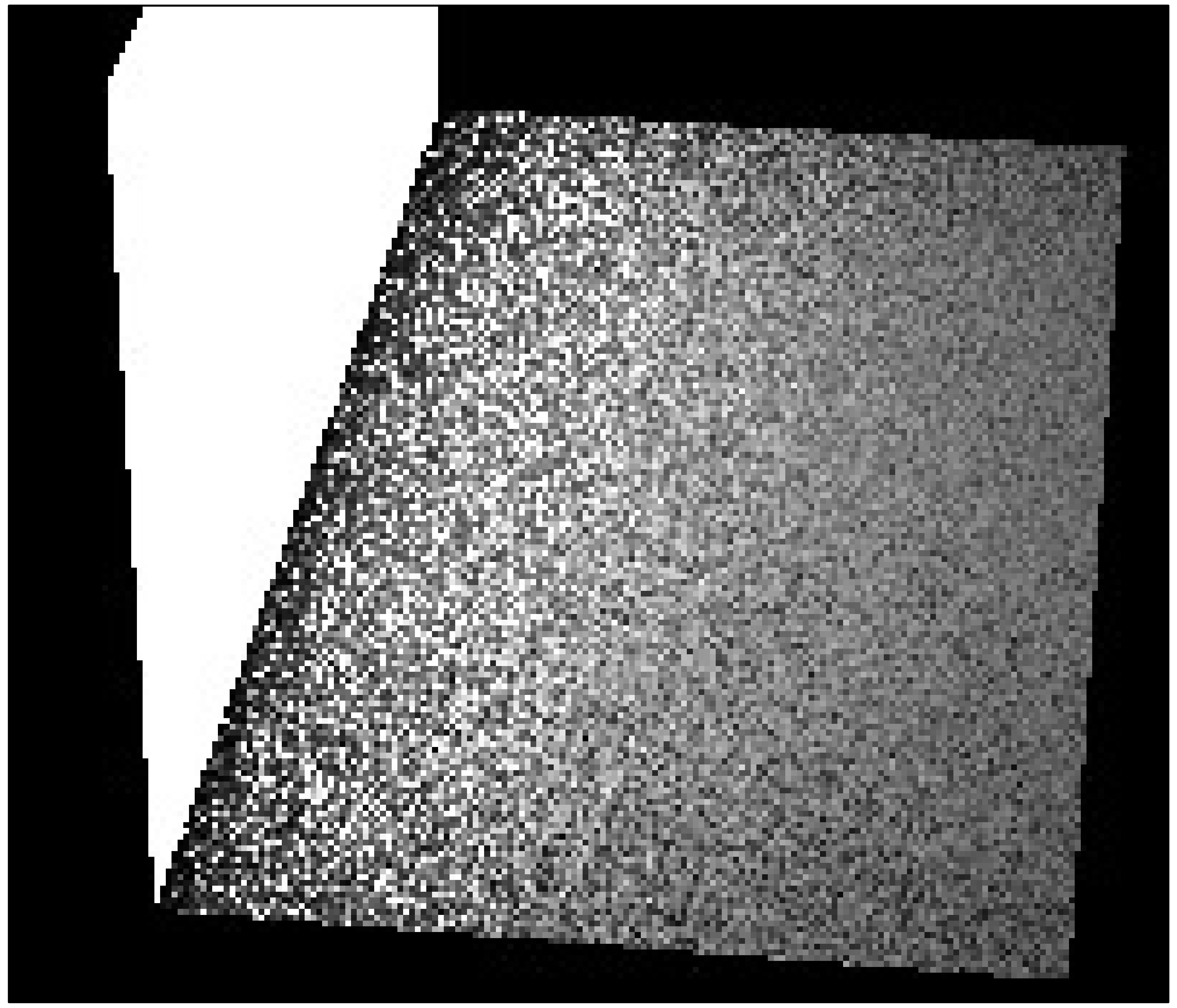
Combining Multiple Strategies

Cosine-weighted hemisphere



$$p_1(\vec{\omega}) = \frac{\cos\theta}{\pi}$$

Uniform surface area

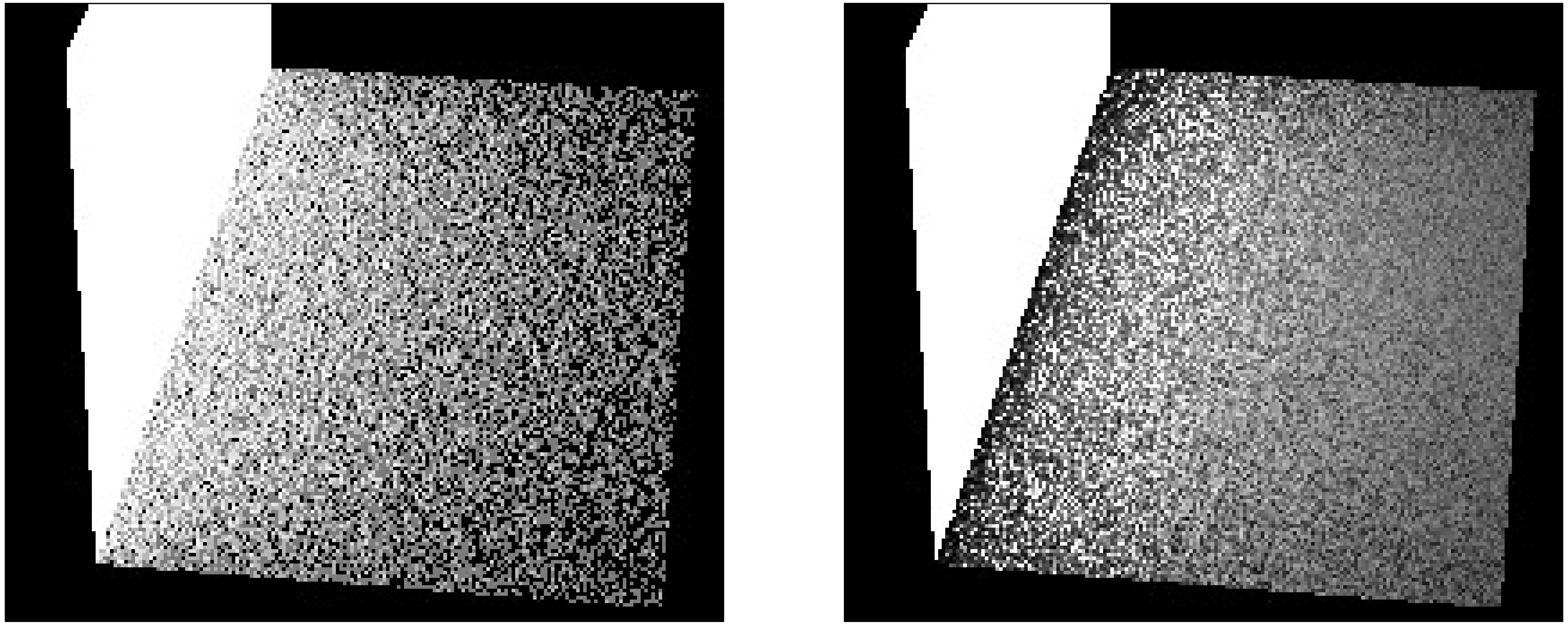


$$p_2(\mathbf{x}) = \frac{1}{A}$$

Combining Multiple Strategies

Cosine-weighted hemisphere

Uniform surface area



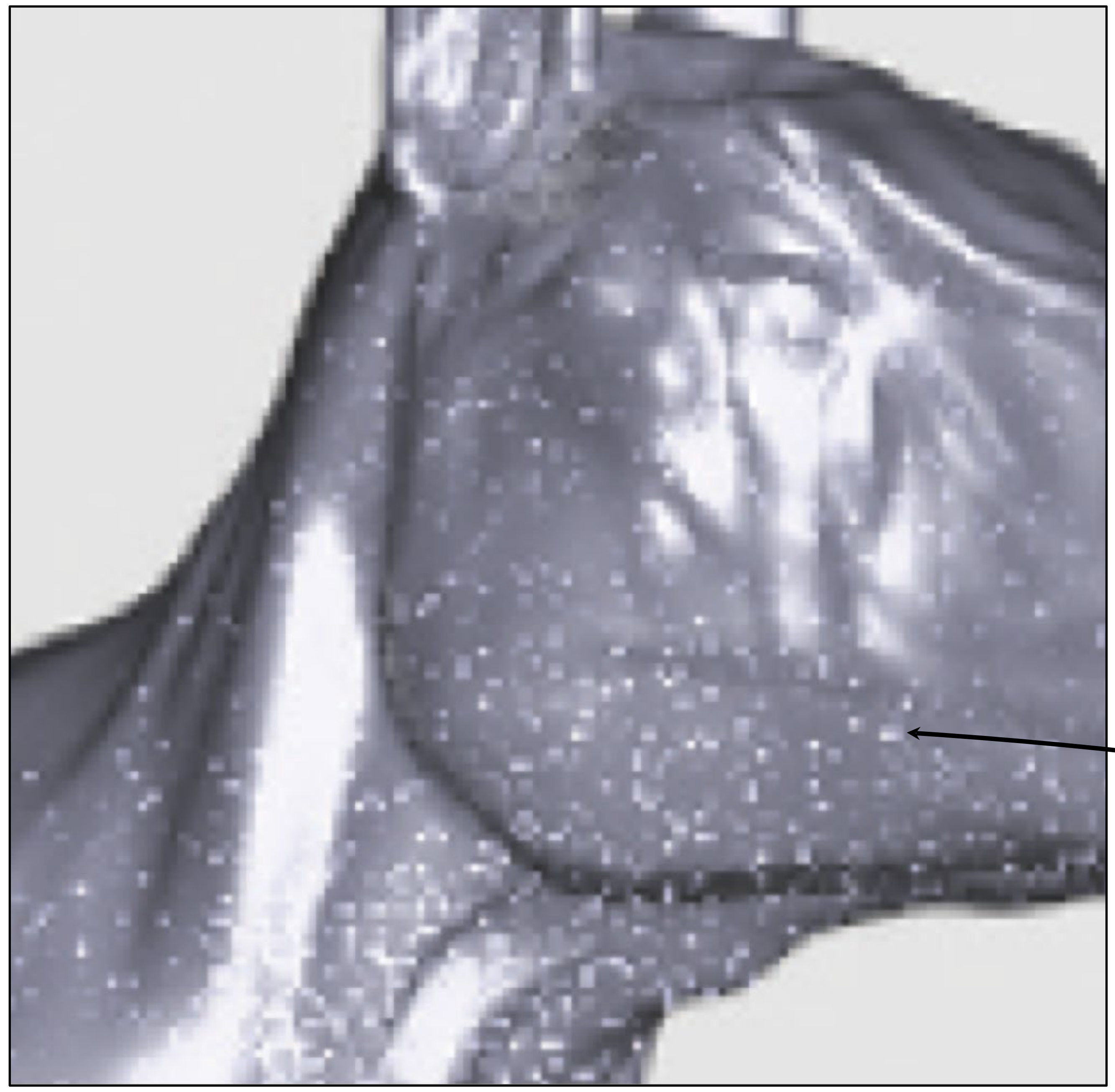
$$p_1(\vec{\omega}) = \frac{\cos\theta}{\pi}$$

$$p_2(\mathbf{x}) = \frac{1}{A} \quad p_2(\vec{\omega}) = \frac{1}{A}\frac{d^2}{\cos\theta}$$

Fireflies

In MC integration, variance is high when the PDF is not proportional to the integrand

Worst case: *rare* samples with *huge* contributions



$$\langle F^N \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

large value

small value

“fireflies”

Motivation

In MC integration, variance is high when the PDF is not proportional to the integrand

Worst case: *rare* samples with *huge* contributions

$$\langle F^N \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i) \longleftarrow \text{large value}}{p(x_i) \longleftarrow \text{small value}}$$

We often have multiple sampling strategies

If at least one covers each part of the integrand well, then combining them should reduce fireflies

Mixture sampling and multiple importance sampling (MIS)

Combining Multiple Strategies

Could just average two different estimators:

$$\frac{0.5}{N_1}\sum_{i=1}^{N_1}\frac{f(x_i)}{p_1(x_i)}+\frac{0.5}{N_2}\sum_{i=1}^{N_2}\frac{f(x_i)}{p_2(x_i)}$$

– doesn’t really help if weights independent of sample: *variance is additive*

Mixture sampling

Instead of averaging multiple estimators

$$\frac{0.5}{N_1}\sum_{i=1}^{N_1} w_1(x_i)\frac{f(x_i)}{p_1(x_i)} + \frac{0.5}{N_2}\sum_{i=1}^{N_2} w_2(x_i)\frac{f(x_i)}{p_2(x_i)}, \qquad N_1 + N_2 = N$$

sample from the average PDF

$$\frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{0.5(p_1(x_i)+p_2(x_i))}$$

Sample from Average PDF (mixture sampling)

You are given two sampling functions and their corresponding pdfs:

```
float sample1(float rnd); float pdf1(float x);
```

```
float sample2(float rnd); float pdf2(float x);
```

Create a new function:

```
float sampleAvg(float rnd);
```

which has the corresponding pdf:

```
float pdfAvg(float x)
{
    return 0.5 * (pdf1(x) + pdf2(x));
}
```

Sample from Average PDF (mixture sampling)

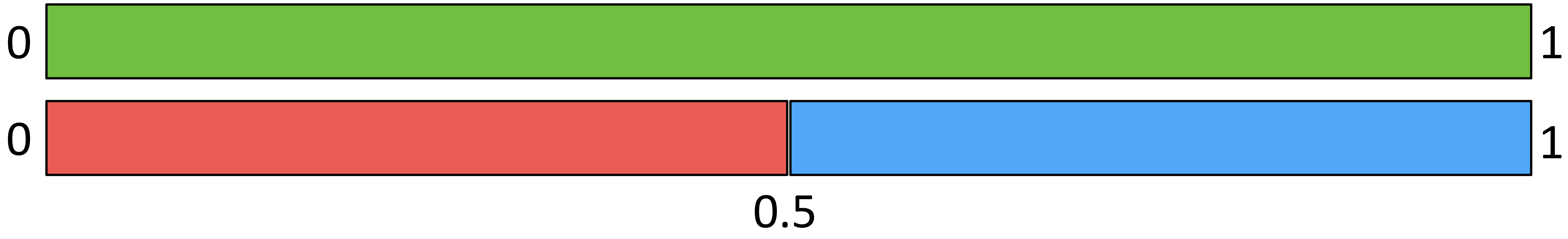
```
float sampleAvg(float rnd)
{
float Prob1 = 0.5;
if (rand.nextFloat() < Prob1)
return sample1(rnd);
else
return sample2(rnd);
}
```

Requires extra random number (can be avoided)

Sample from Average PDF (mixture sampling)

```
float sampleAvg(float rnd)
{
float Prob1 = 0.5;
if (rnd < Prob1)
return sample1(rnd);
else
return sample2(rnd);
}
```

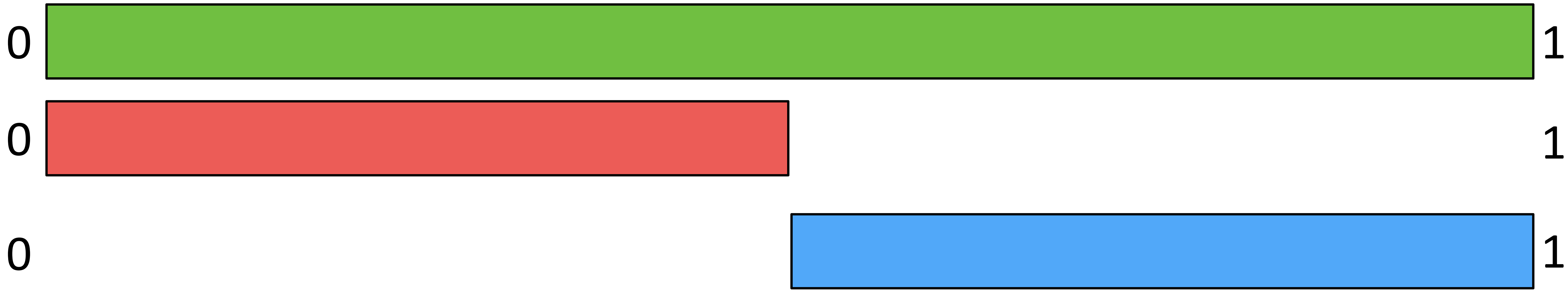
These need to be uniform random numbers in [0..1)



Sample from Average PDF (mixture sampling)

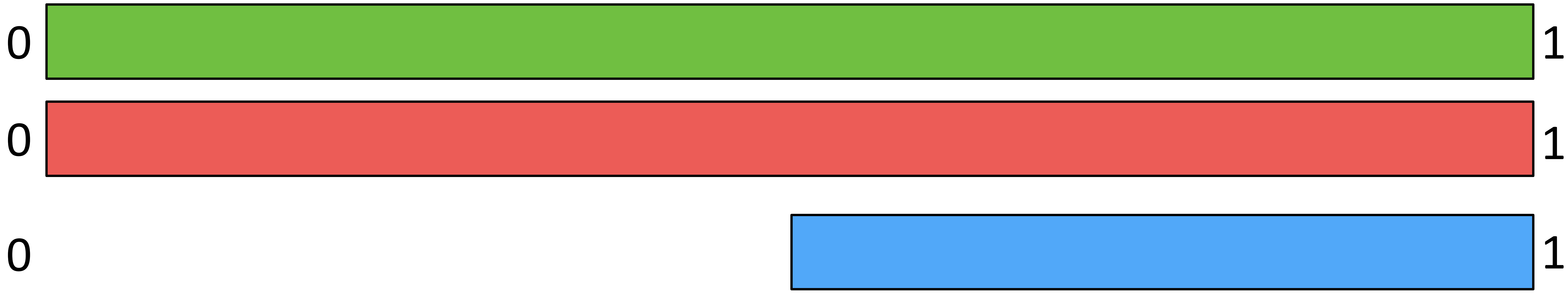
```
float sampleAvg(float rnd)
{
float Prob1 = 0.5;
if (rnd < Prob1)
return sample1(rnd);
else
return sample2(rnd);
}
```

These need to be uniform random numbers in [0..1)



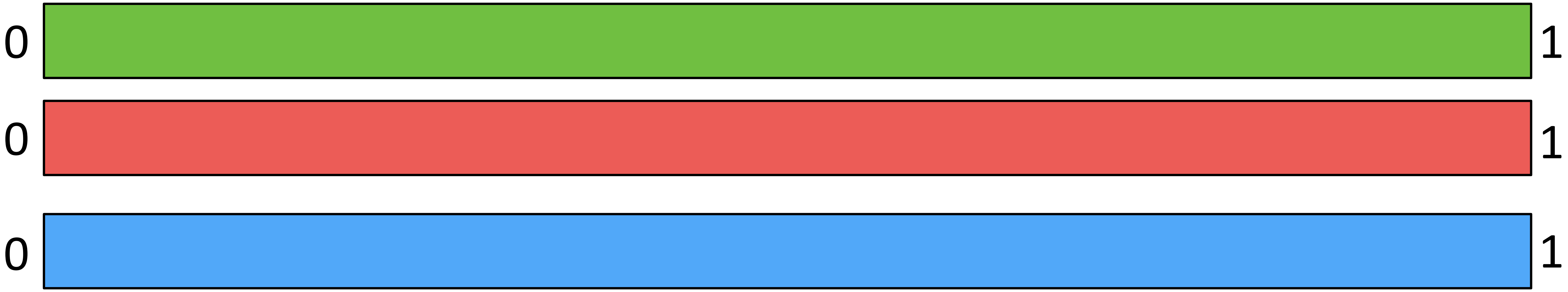
Sample from Average PDF (mixture sampling)

```
float sampleAvg(float rnd)
{
float Prob1 = 0.5;
if (rnd < Prob1)
return sample1(rnd / Prob1);
else
return sample2(rnd);
}
```



Sample from Average PDF (mixture sampling)

```
float sampleAvg(float rnd)
{
float Prob1 = 0.5;
if (rnd < Prob1)
return sample1(rnd / Prob1);
else
return sample2((rnd-Prob1) / (1-Prob1));
}
```



Sample from Weighted Average

```
float sampleWeightedAvg(float rnd)
{
float Prob1 = 0.25;
if (rnd < Prob1)
return sample1(rnd / Prob1);
else
return sample2((rnd-Prob1) / (1-Prob1));
}

float pdfWeightedAvg(float x)
{
    return 0.25 * pdf1(x) + 0.75 * pdf2(x);
}
```

Still works, just change Prob1

Multiple Importance Sampling

Combination of 2 strategies using *sample-dependent* weights:

$$\langle F^{\mathrm{MIS}} \rangle = \boxed{w_1(x_1)} \frac{f(x_1)}{p_1(x_1)} + \boxed{w_2(x_2)} \frac{f(x_2)}{p_2(x_2)}$$

- where:

$$w_1(x) + w_2(x) = 1$$

Multiple Importance Sampling

Combination of *M* strategies with *sample-dependent* weights:

$$\langle F^{\sum N_s} \rangle = \sum_{s=1}^{M} \frac{1}{N_s} \sum_{i=1}^{N_s} w_s(x_i) \frac{f(x_i)}{p_s(x_i)}$$

– where:

$$\sum_{s=1}^{M} w_s(x) = 1$$

How to choose the weights?

Multiple Importance Sampling

Balance heuristic (provably good):

$$w_s(x) = \frac{p_s(x)}{\sum_j p_j(x)}$$

Power heuristic (more aggressive, can be better):

$$w_s(x) = \frac{p_s(x)^\beta}{\sum_j p_j(x)^\beta}$$

Other heuristics exist

- e.g. cutoff heuristic, maximum heuristic, …

Multiple Importance Sampling

Multi-sample model: *deterministically* allocate N_s samples to s-th strategy

$$\langle F^{\sum N_s} \rangle = \sum_{s=1}^{M} \frac{1}{N_s} \sum_{i=1}^{N_s} w_s(x_i) \frac{f(x_i)}{p_s(x_i)}$$

What if we want to draw just ***one*** sample?

One-sample model: *randomly* select to use s-th strategy

$$\langle F^1 \rangle = w_s(x) \frac{f(x)}{q_s \, p_s(x)}$$

where q_s is the probability of using strategy s, and $\sum_{s=1}^{N} q_s = 1$

Interpreting the Balance Heuristic

Balance heuristic for the one-sample model:

$$w_s(x) = \frac{q_s\, p_s(x)}{\sum_j q_j\, p_j(x)}$$

Plugged into the one-sample model:

$$\langle F^1 \rangle = w_s(x) \frac{f(x)}{q_s\, p_s(x)} = \frac{q_s\, p_s(x)}{\sum_j q_j\, p_j(x)} \frac{f(x)}{q_s\, p_s(x)} = \frac{f(x)}{\sum_j q_j\, p_j(x)}$$

One-sample model with balance heuristic samples from average PDF (*mixture sampling*)

Multiple Importance Sampling with Balance Heuristic

Multi-sample model: Equivalent to mixture sampling with *stratification* (deterministic allocation of samples per strategy).

$$\langle F^{\sum N_s} \rangle = \sum_{s=1}^{M} \frac{1}{N_s} \sum_{i=1}^{N_s} w_s(x_i) \frac{f(x_i)}{p_s(x_i)}$$

One-sample model: Equivalent to mixture sampling.

$$\langle F^1 \rangle = w_s(x) \frac{f(x)}{q_s \, p_s(x)}$$

where q_s is the probability of using strategy s, and $\sum_{s=1}^{N} q_s = 1$

Why Does it Work?

Using a single strategy:

$$\langle F^N \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

large value

small value

Combining multiple strategies using balance heuristic (MIS or mixture sampling):

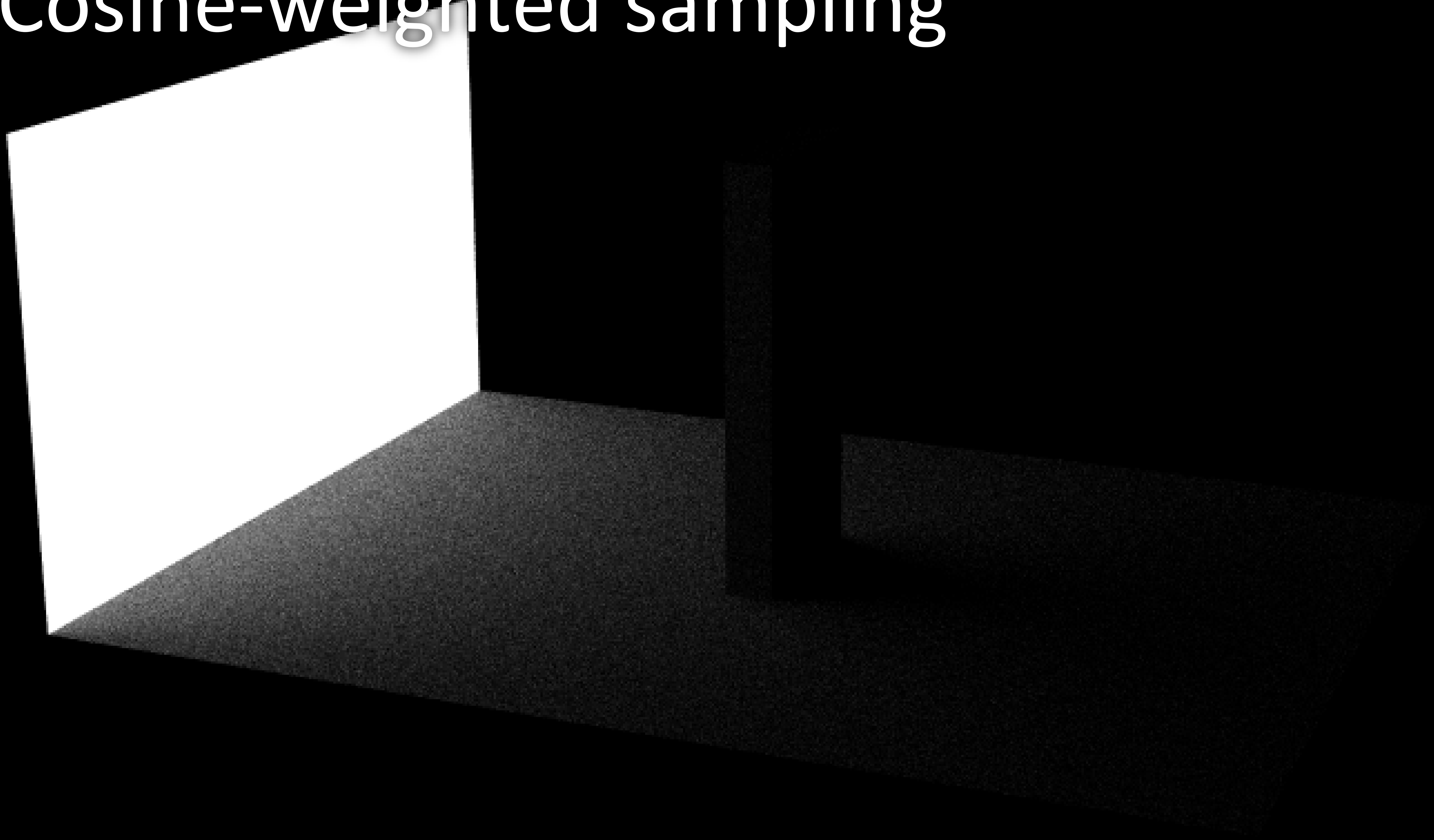
$$\langle F^N \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{\sum_j q_j p_j(x_i)}$$

large value

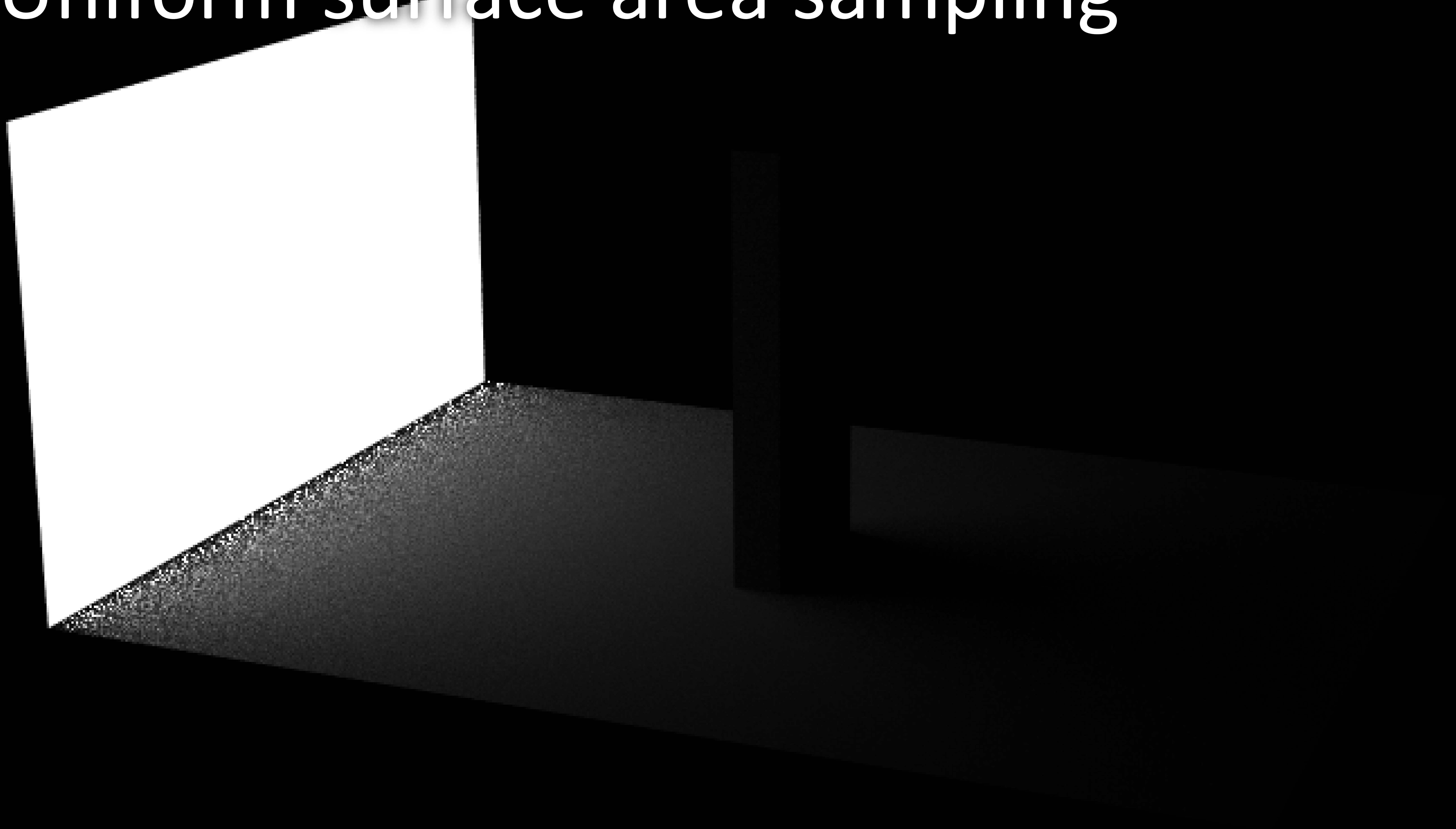
relatively large value

(as long as at least one PDF is large)

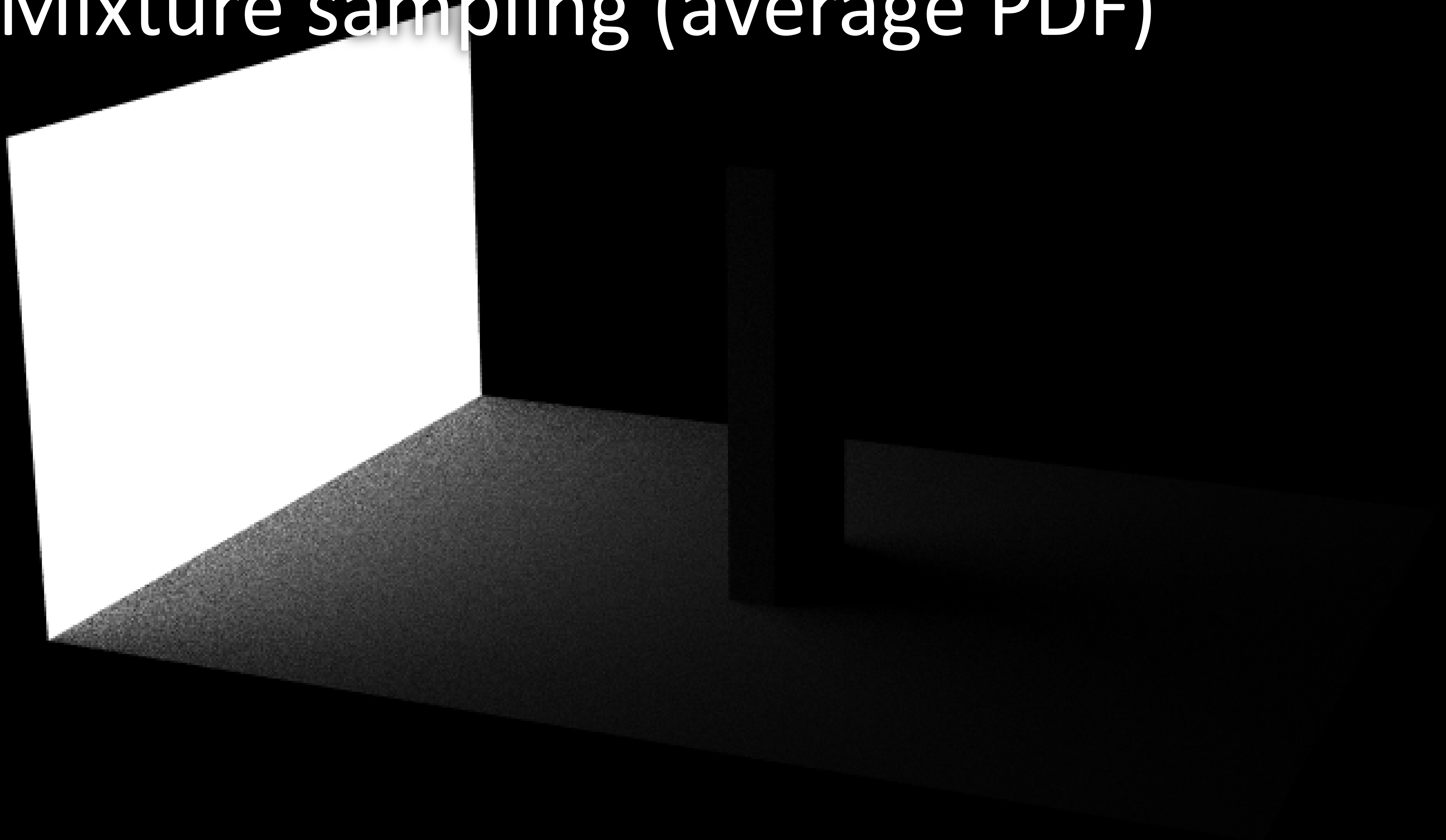
Cosine-weighted sampling



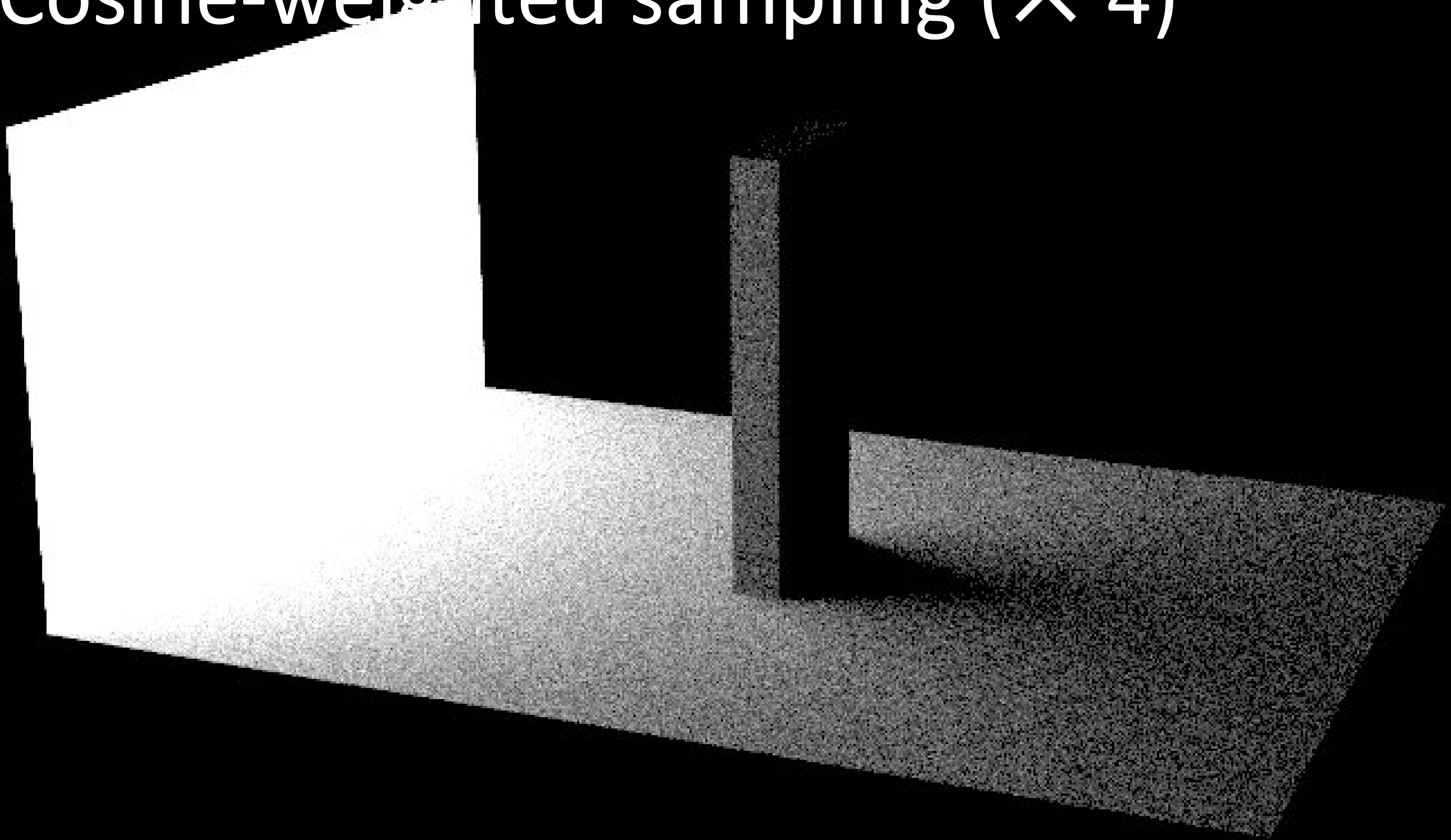
Uniform surface area sampling



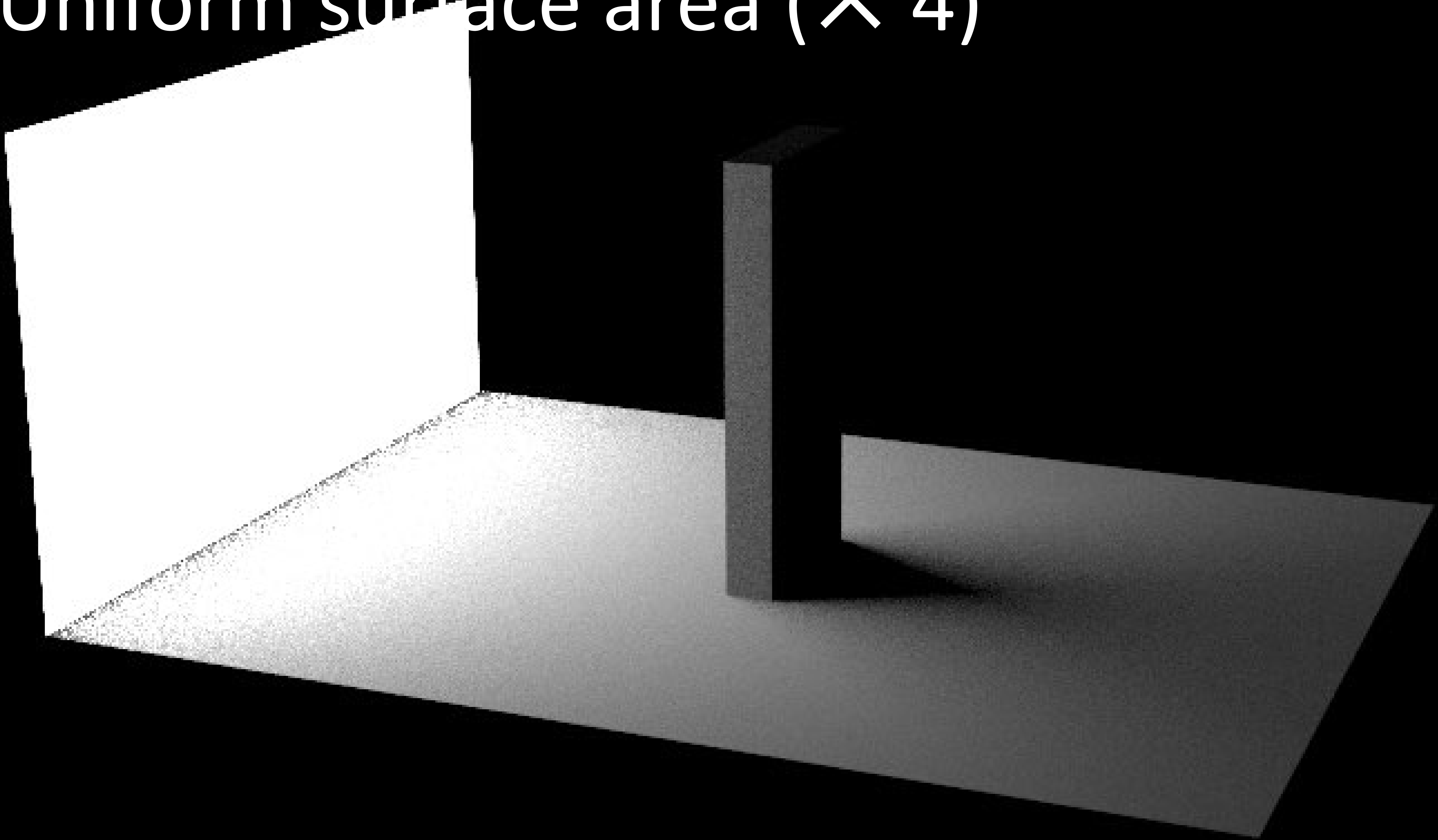
Mixture sampling (average PDF)



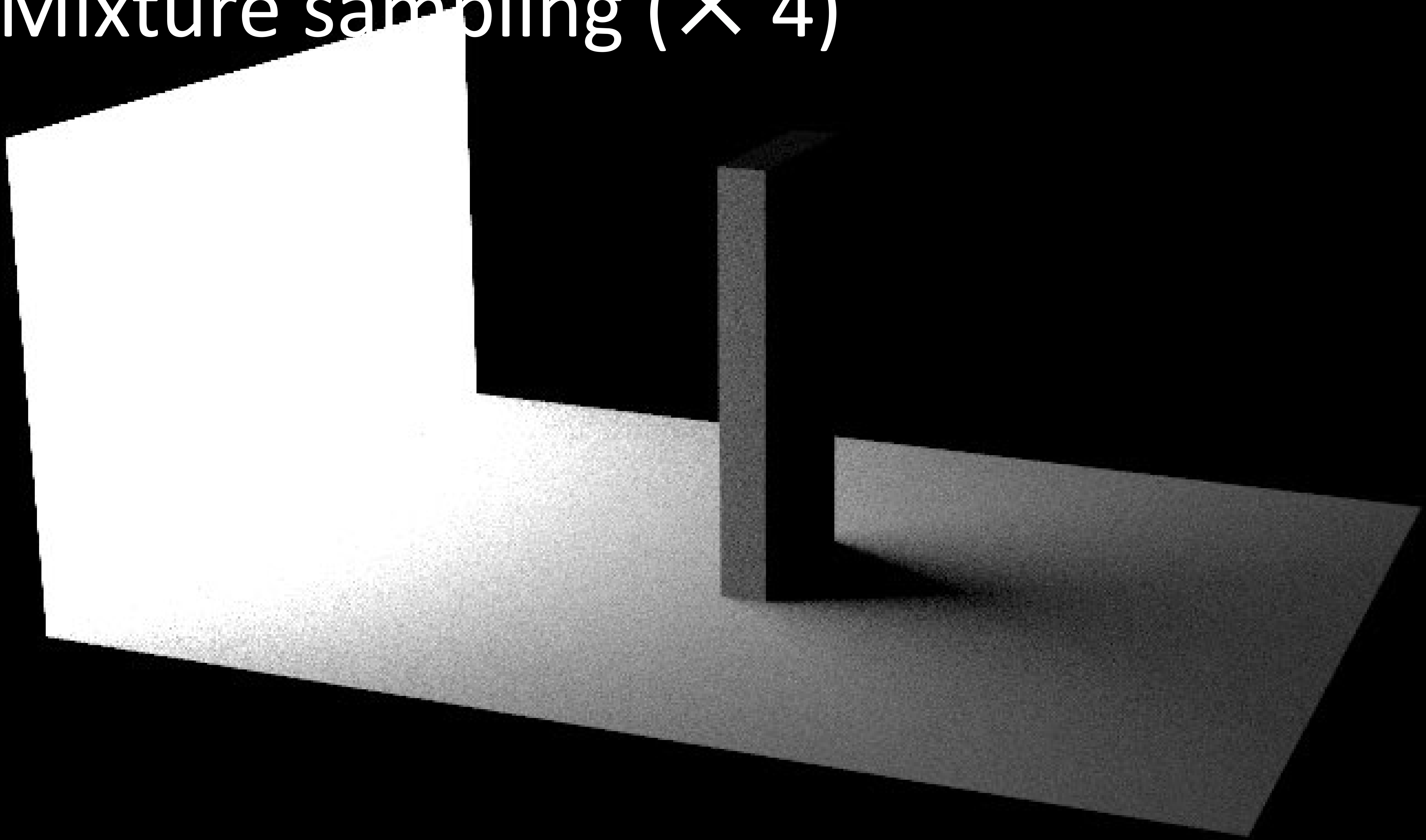
Cosine-weighted sampling (✕ 4)



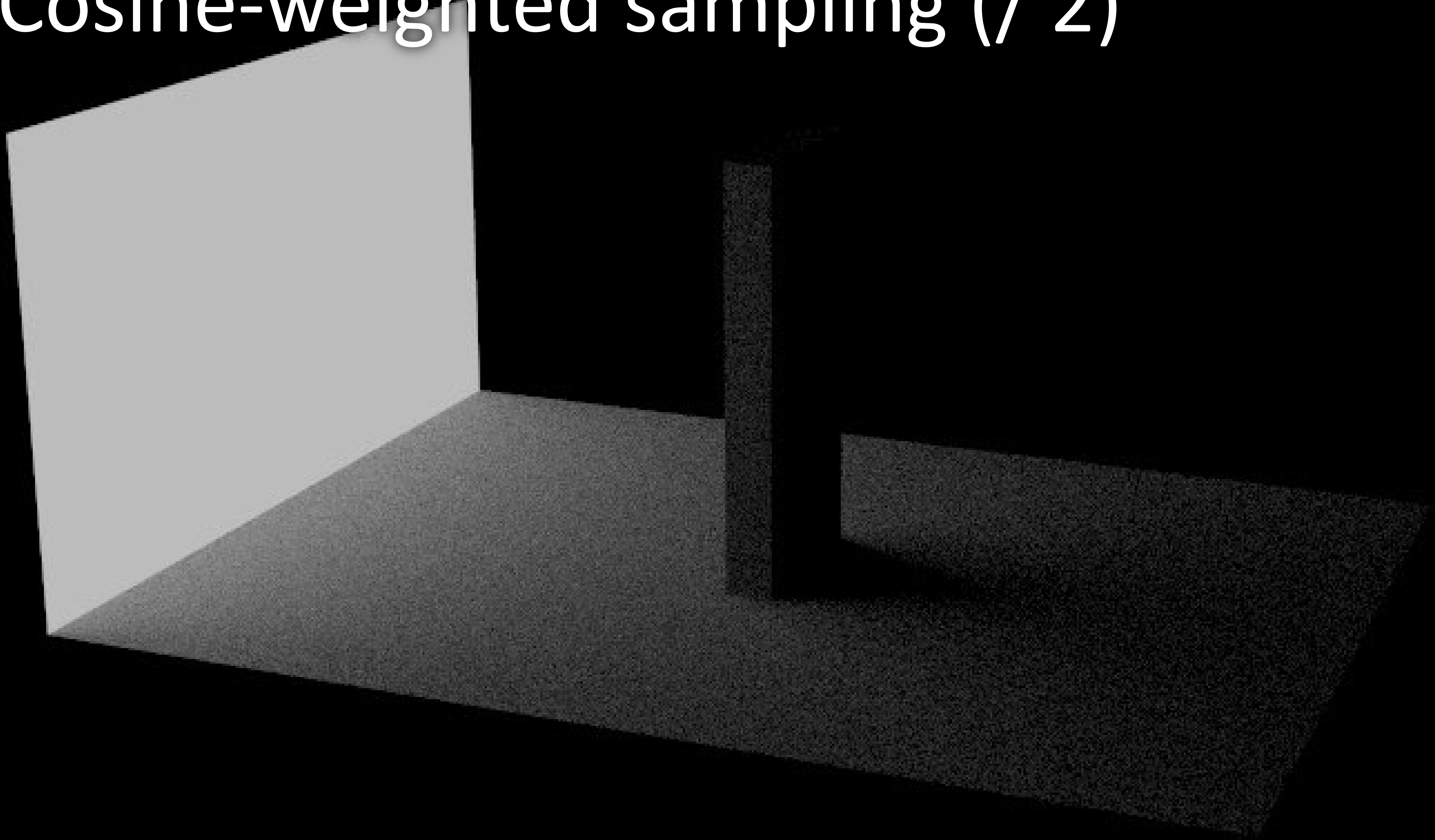
Uniform surface area (× 4)



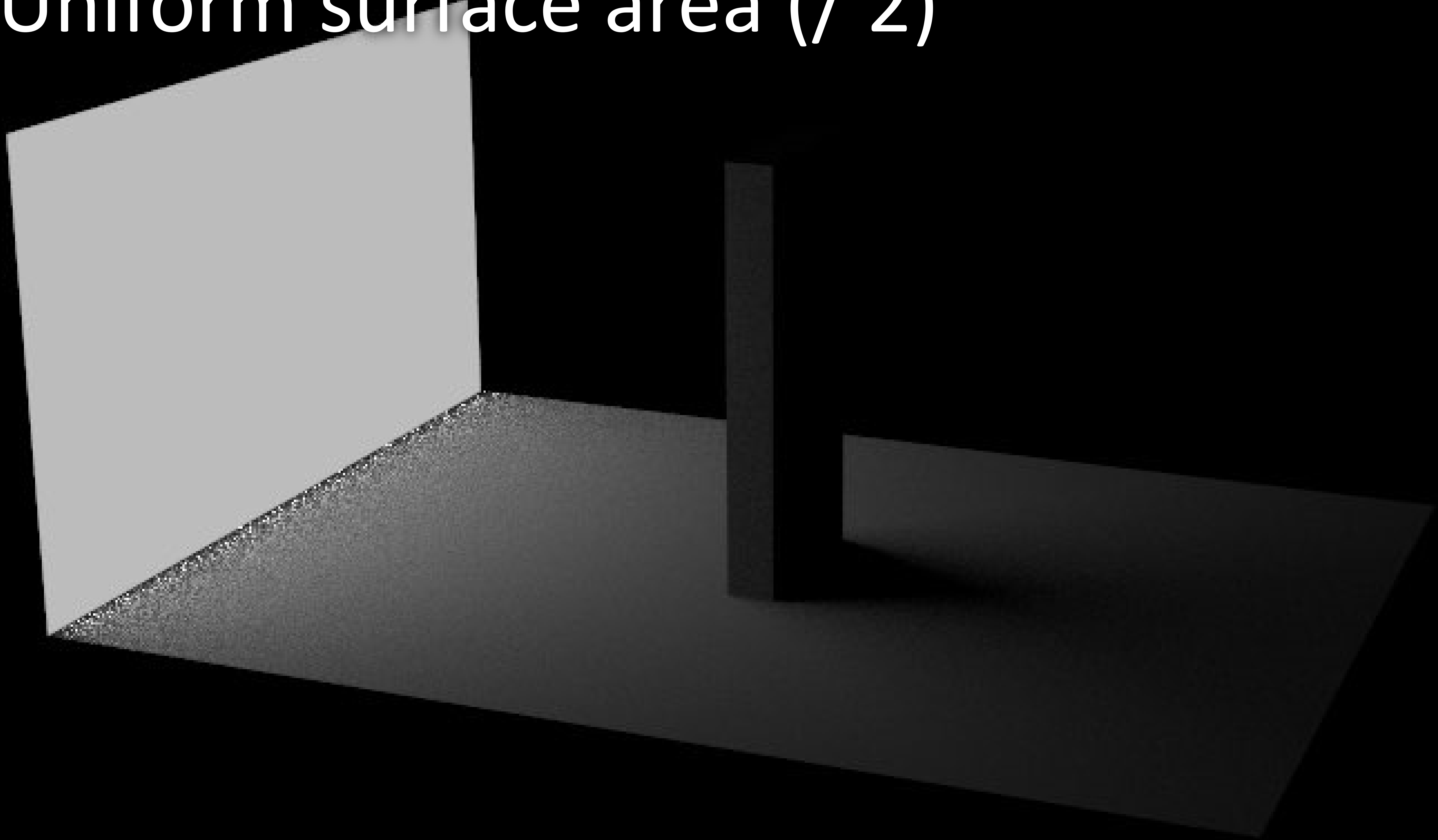
Mixture sampling (× 4)



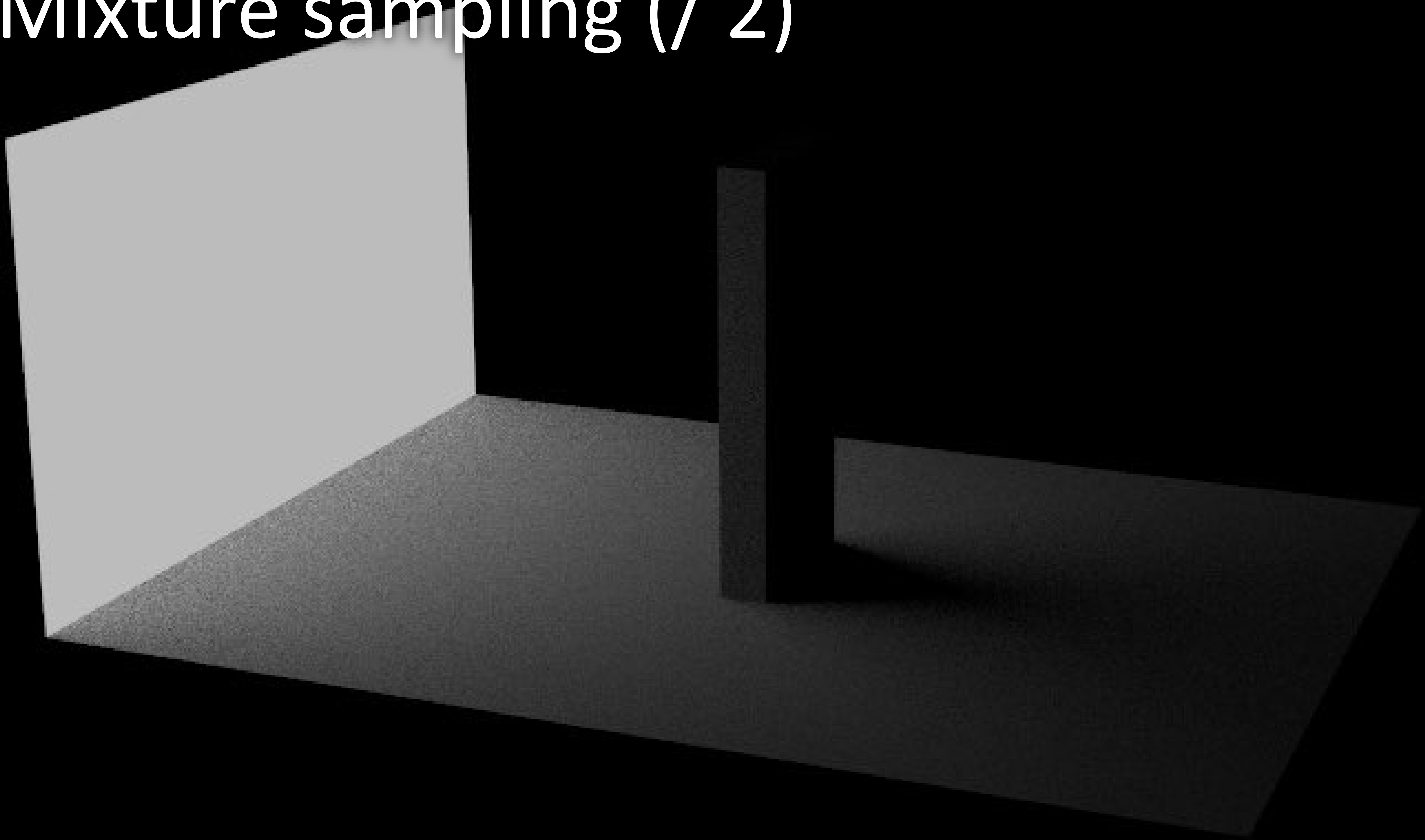
Cosine-weighted sampling (/ 2)

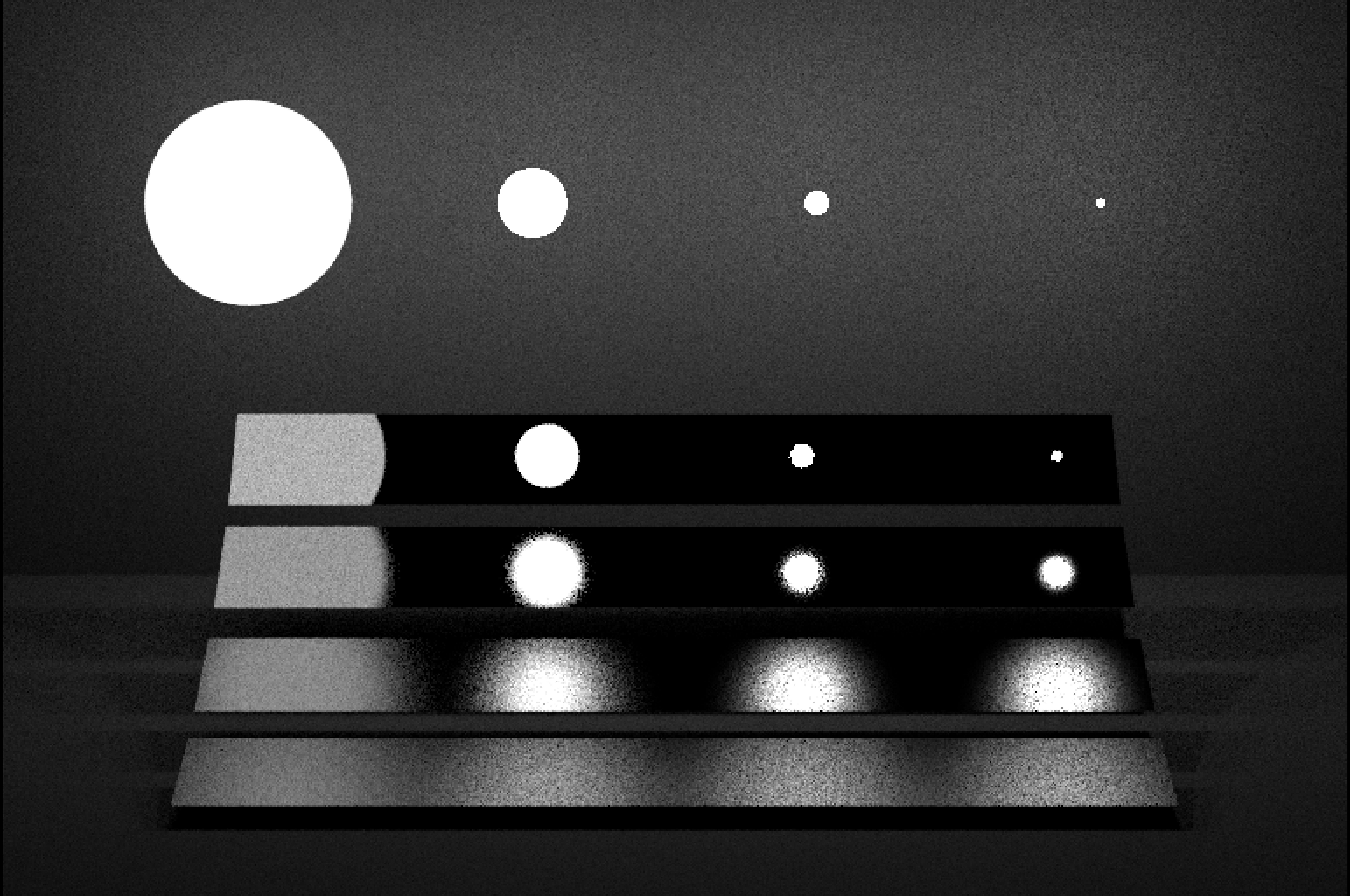


Uniform surface area (/ 2)

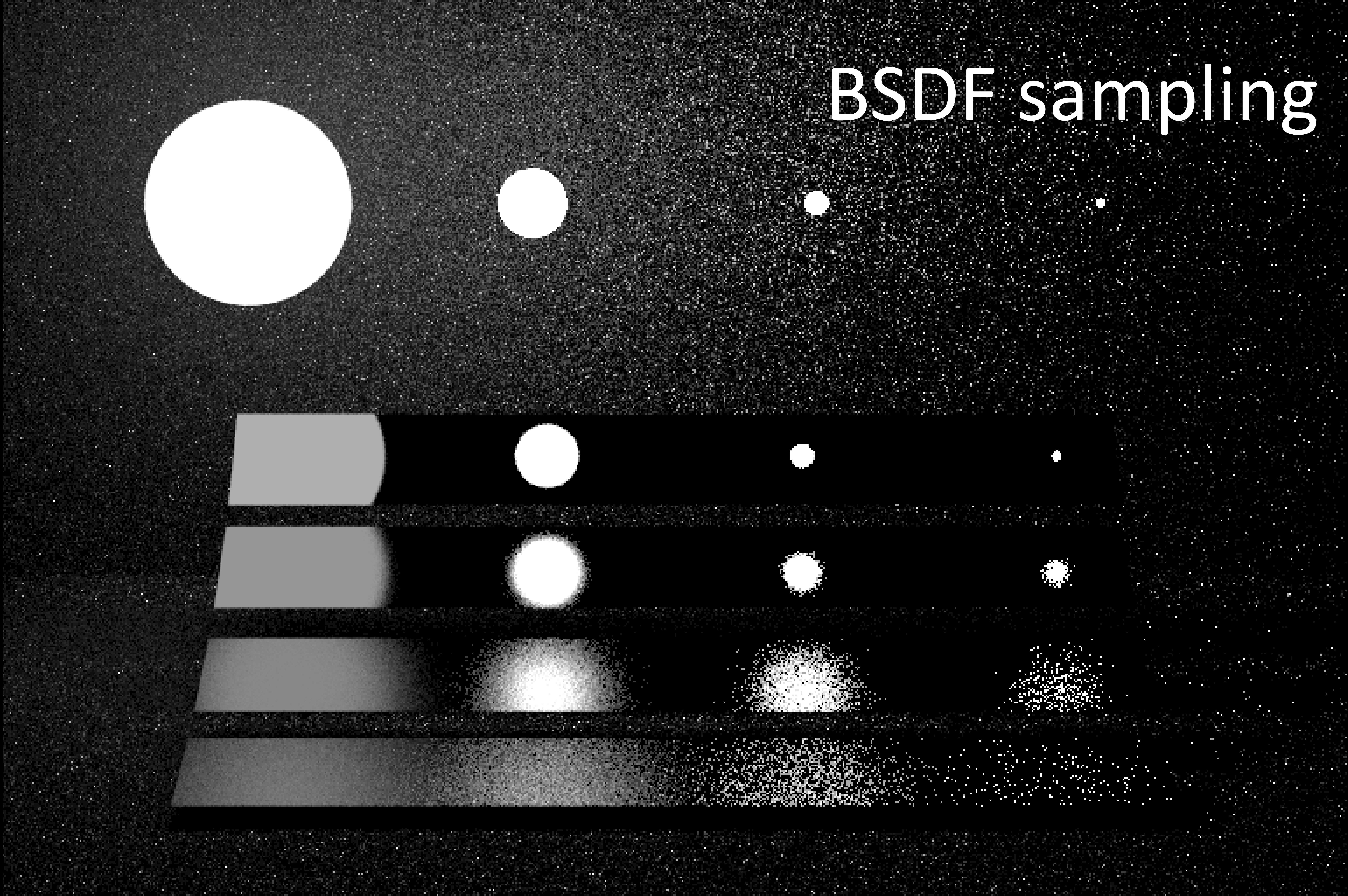


Mixture sampling (/ 2)

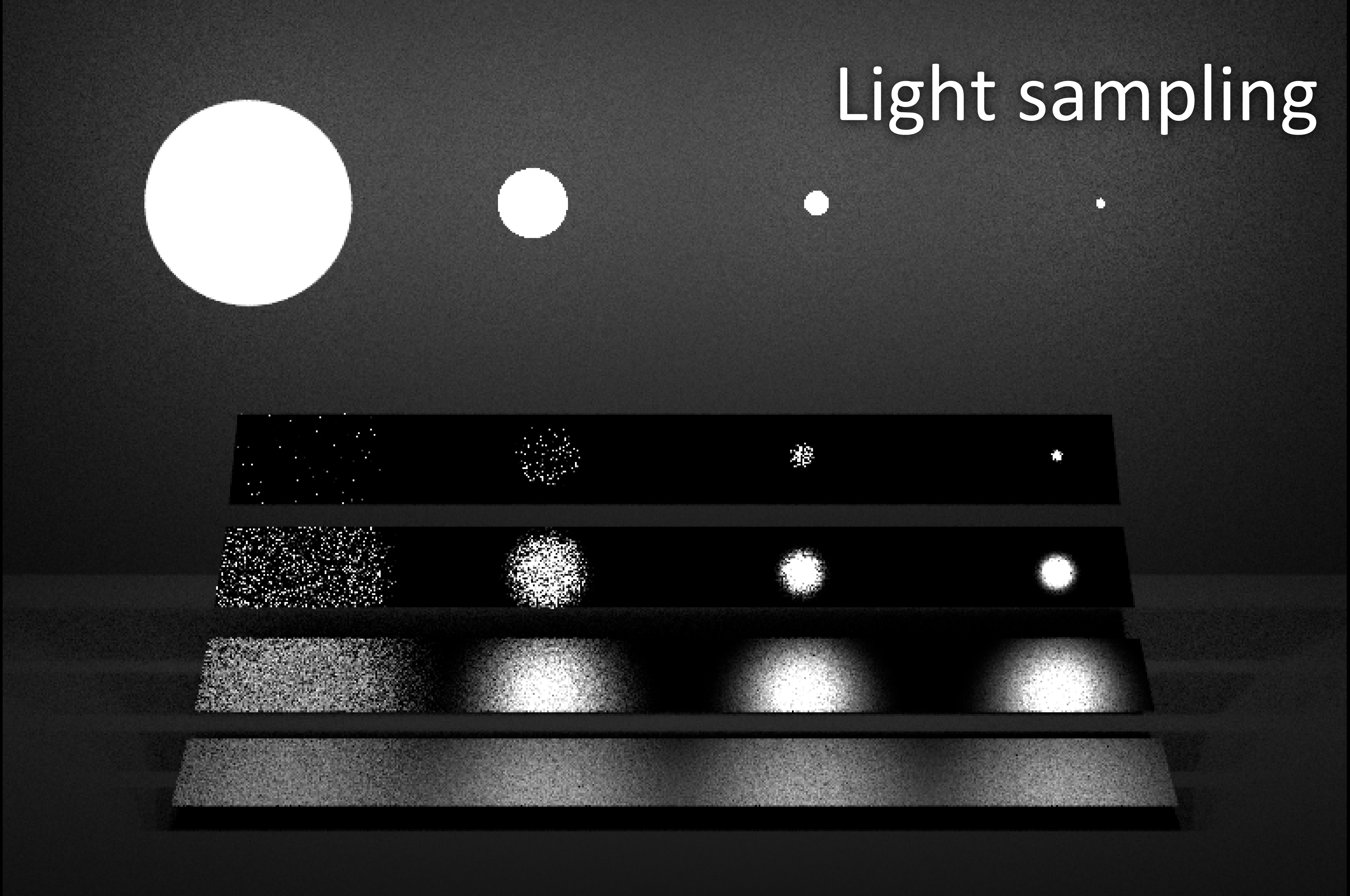




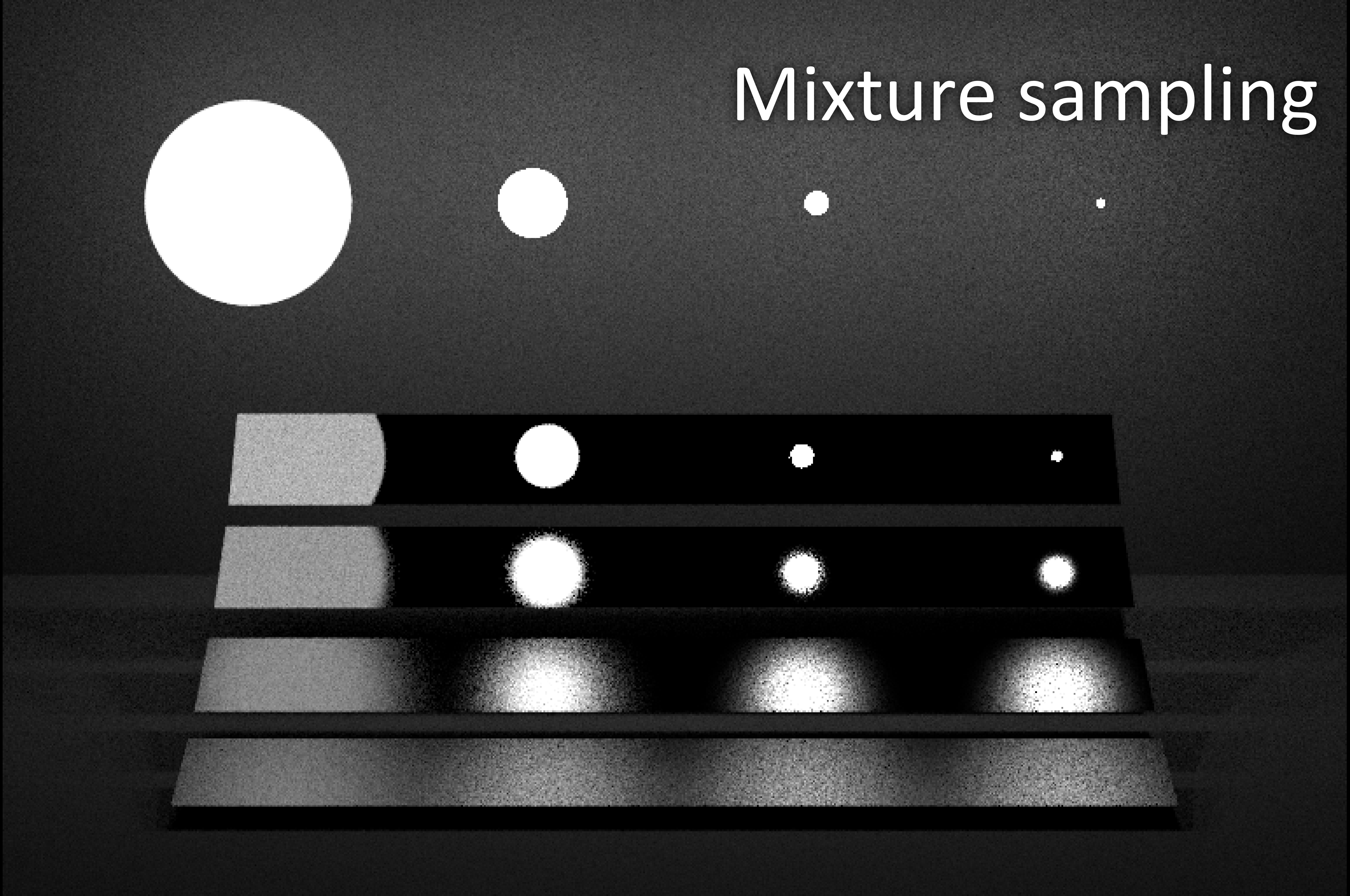
BSDF sampling



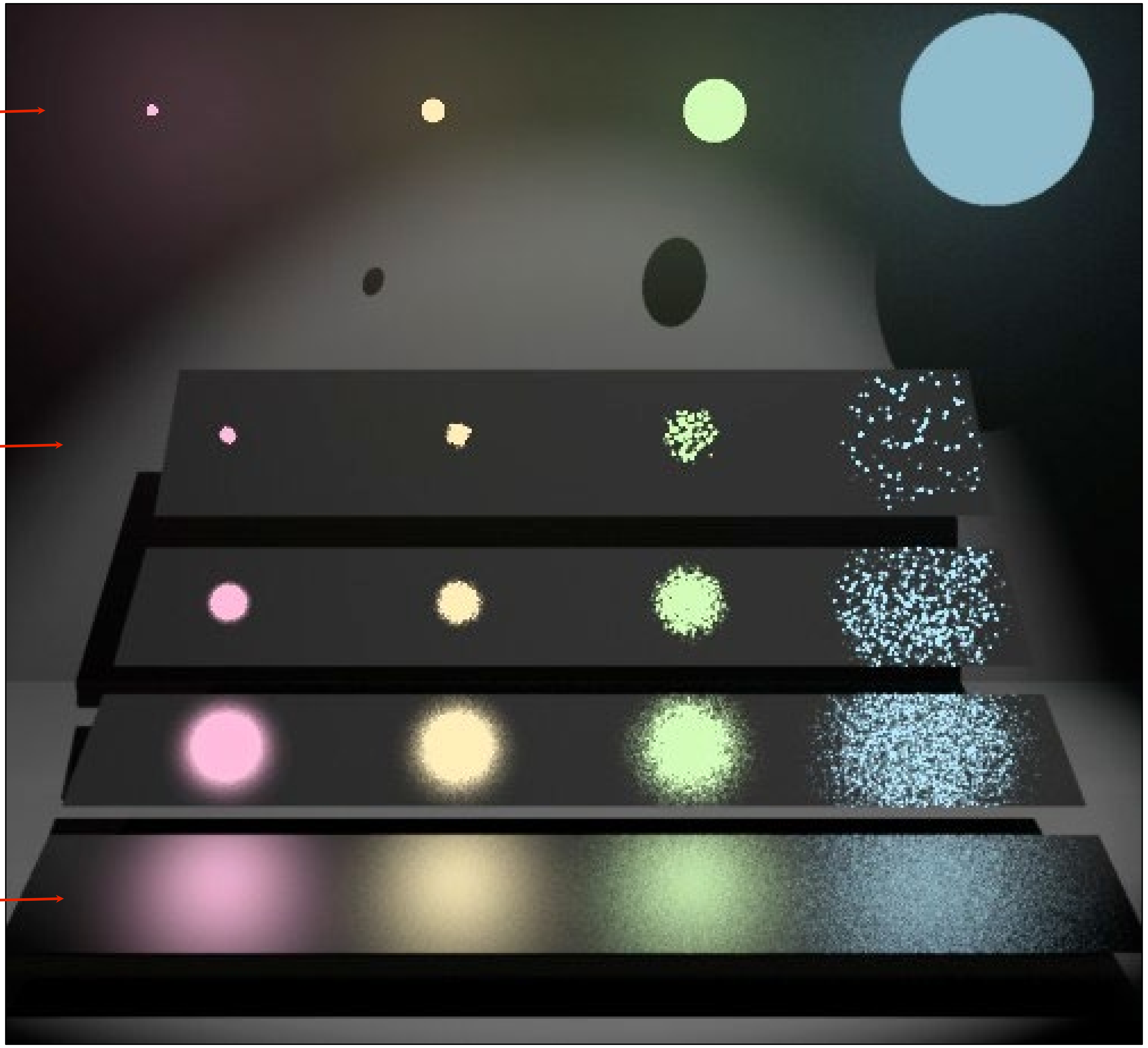
Light sampling



Mixture sampling



Sampling the Light



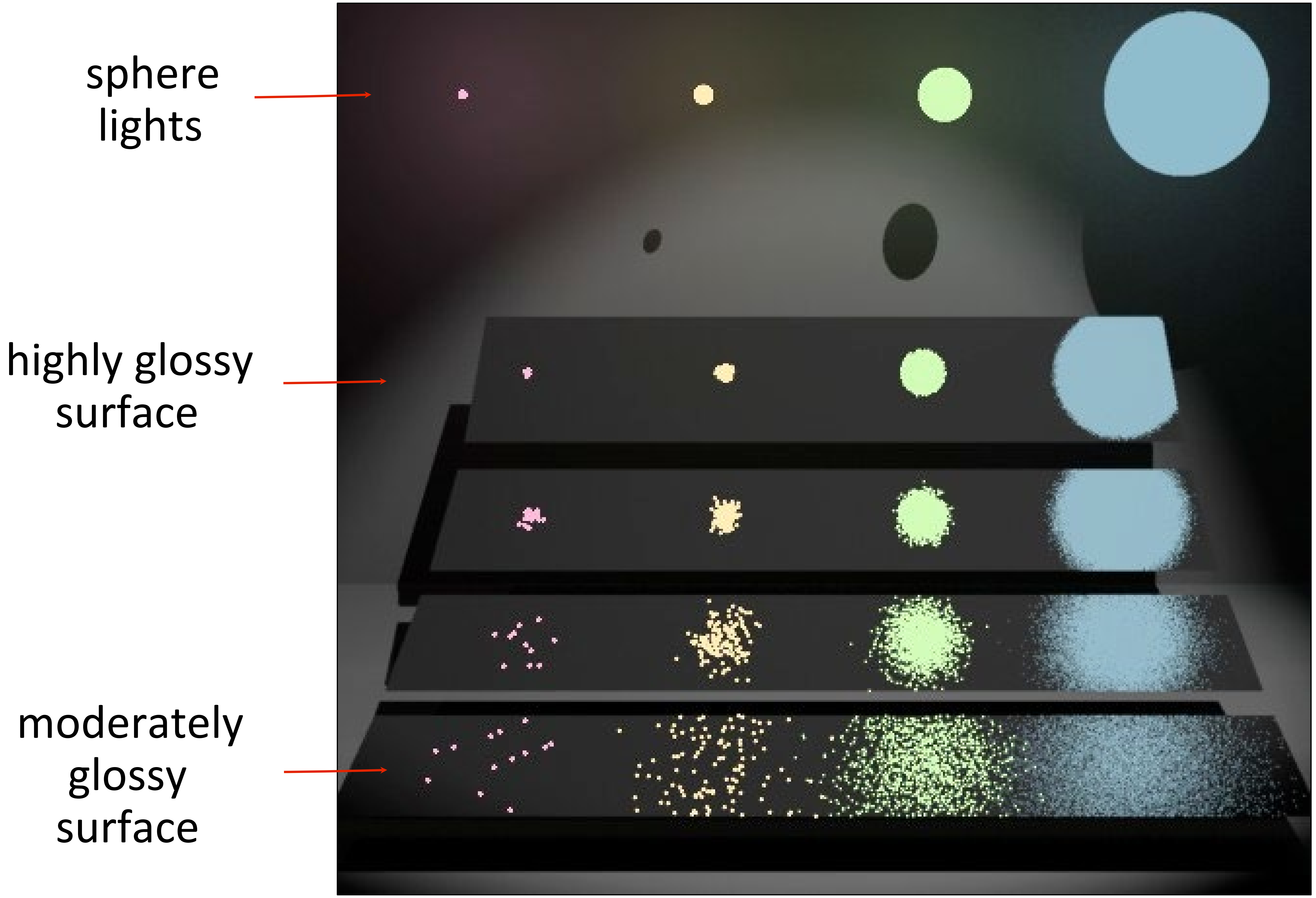
sphere lights

highly glossy surface

moderately glossy surface

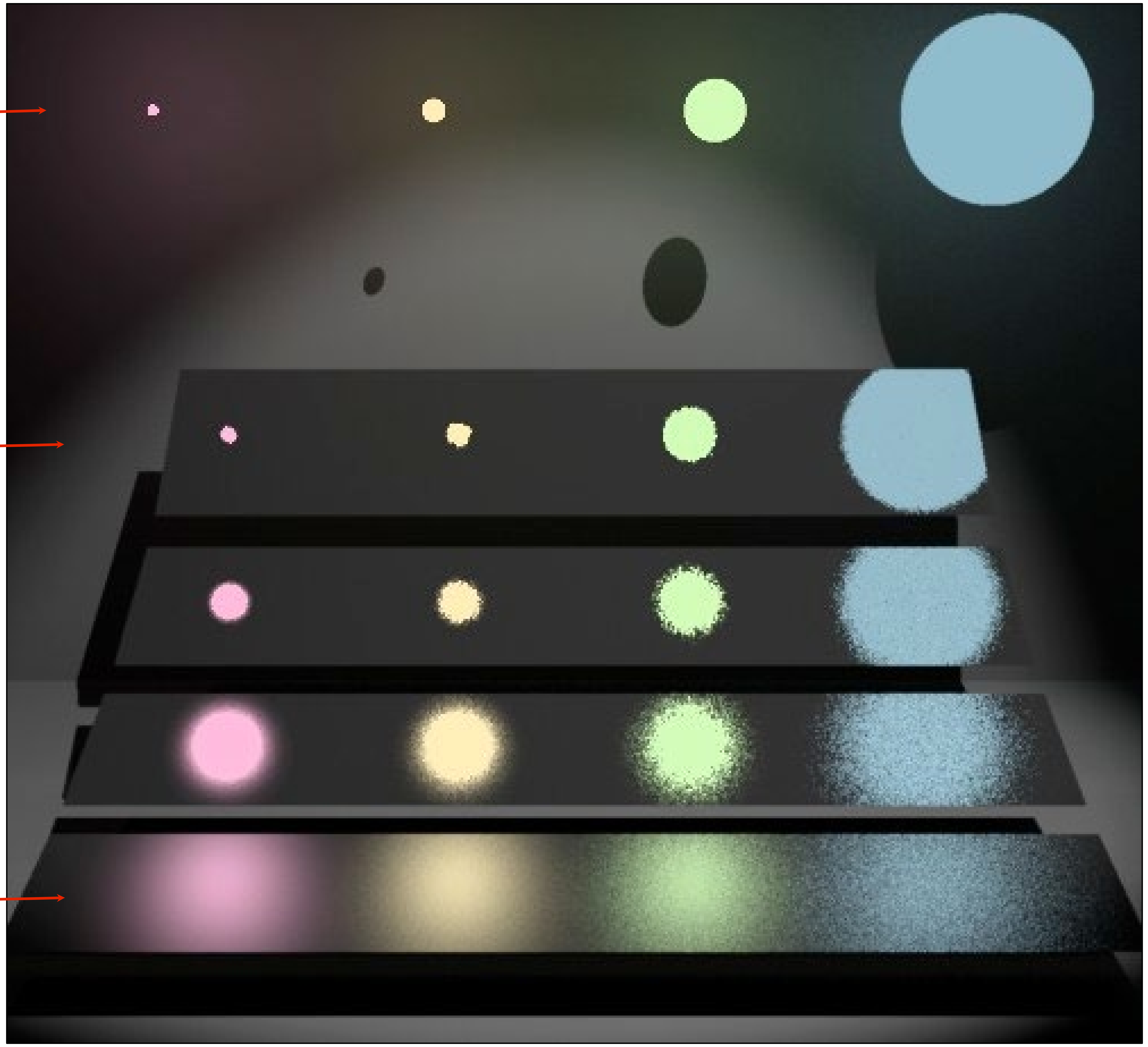
Eric Veach and Leonidas J. Guibas 1995.

Sampling the BRDF



Eric Veach and Leonidas J. Guibas 1995.

Multiple Importance Sampling



sphere lights

highly glossy surface

moderately glossy surface

Eric Veach and Leonidas J. Guibas 1995.

Multiple Importance Sampling

See PBRe3 13.10.1 for more details

Eric Veach and Leonidas J. Guibas 1995.