Monte Carlo integration
Course announcements

• Programming assignment 2 posted, due Friday 2/24 at 23:59.
  - How many of you have looked at/started/finished it?
  - Any questions?

• Take-home quiz 3 due tonight.

• Take-home quiz 4 will be posted tonight.
Overview of today’s lecture

• Leftover from BRDFs.
• Monte Carlo integration.
• Sampling techniques.
• Importance sampling.
• Ambient occlusion.
Most of these slides were directly adapted from:

- Wojciech Jarosz (Dartmouth).
Numerical Integration - Motivation

For very, very simple integrals, we can compute the solution analytically

\[ \int_0^1 \frac{1}{3} x^2 \, dx = \left[ x^3 \right]_0^1 = 1 \]

But ours are a bit more complicated:

\[ L_r(x, \bar{\omega}_r) = \int_{H^2} f_r(x, \bar{\omega}_i, \bar{\omega}_r) L_i(x, \bar{\omega}_i) \cos \theta_i \, d\bar{\omega}_i \]
Typical quadrature: Trapezoid rule

**Approximate** integral of $f(x)$ by assuming function is piecewise linear

For equal length segments: $h = \frac{b-a}{n-1}$
Typical quadrature: Trapezoid rule

Consider cost and accuracy as $n \to \infty$ (or $h \to 0$)

Work: $O(n)$

Error can be shown to be: $O(h^2) = O\left(\frac{1}{n^2}\right)$ (for $f(x)$ with continuous second derivative)
What about a 2D function?

How should we approximate the area (volume) underneath?
Multidimensional integrals & Fubini’s theorem

\[ \int_{X \times Y \times Z} f(x, y, z) \, d(x, y, z) = \int_X \left( \int_Y \left( \int_Z f(x, y, z) \, dx \right) \, dy \right) \, dz \]

Apply the trapezoid rule repeatedly
Multidimensional integrals & Fubini’s theorem

\[ \int_{X \times Y \times Z} f(x, y, z) \, d(x, y, z) = \int_X \left( \int_Y \left( \int_Z f(x, y, z) \, dx \right) \, dy \right) \, dz \]

Apply the trapezoid rule repeatedly

Can show that:

- Errors add, so error still: \( O(h^2) \)
- But work is now: \( O(n^2) \) (n x n set of measurements)
Multidimensional integrals & Fubini’s theorem

\[ \int_{X \times Y \times Z} f(x, y, z) \, dx \, dy \, dz = \int_X \left( \int_Y \left( \int_Z f(x, y, z) \, dx \right) \, dy \right) \, dz \]

Apply the trapezoid rule repeatedly

Can show that:

- Errors add, so error still: \( O(h^2) \)
- But work is now: \( O(n^2) \) \((n \times n \text{ set of measurements})\)

Must perform much more work in 2D to get same error bound!
Curse of Dimensionality

How much does it cost to apply the trapezoid rule as we go up in dimension?

- 1D: $O(n)$
- 2D: $O(n^2)$
- ... 
- kD: $O(n^k)$
Curse of Dimensionality

How much does it cost to apply the trapezoid rule as we go up in dimension?

- 1D: $O(n)$
- 2D: $O(n^2)$
- ...  
- kD: $O(n^k)$

Deterministic quadrature does not scale to higher dimensions!

Need a fundamentally different approach...
Monte Carlo Integration
Monte Carlo vs Las Vegas

Random variation creeps into the results

Always gives the correct answer, e.g., a randomized sorting algorithm
Monte Carlo History

Use random numbers to solve numerical problems

Early use during development of atomic bomb

Von Neumann, Ulam, Metropolis

Named after the casino in Monte Carlo
Playing Solitaire

Lose  Win  Win  Lose

What’s the chance of winning with a properly shuffled deck?
Playing Solitaire

\[ P_n = \frac{1}{n} \sum_{i=1}^{n} \left\{ \begin{array}{ll} 1, & \text{game } i \text{ is won}, \\ 0, & \text{game } i \text{ is lost} \end{array} \right\} \]

\[ P = \lim_{n \to \infty} P_n \]
Monte Carlo Integration

Estimate value of integral using *random* sampling of function

- Value of estimate depends on random samples used
- But algorithm gives the correct value “on average”
Monte Carlo Integration Advantages

Only requires function to be evaluated at random points on its domain

- Applicable to functions with discontinuities, functions that are impossible to integrate directly

Error is independent of dimensionality of integral!

- $O(n^{-0.5})$
Review: random variables

\( X \): random variable. Represents a distribution of potential outcomes. Assigns a value of each outcome.

Two types: discrete vs. continuous
Discrete Random Variables

Discrete Random Variable: countable set of outcomes
Discrete Random Variables

Discrete Random Variable: countable set of outcomes

Probability mass function (pmf) of $X$:

- $p_X(x_i) = P(X = x_i)$, or simply $p_i = p(x_i) = P(X = x_i)$
- $p(x_i) \geq 0$
- Sums to one: $\sum_a p(a) = 1$
Probability mass function

\[ p_X(x_i) \]

\( x_i \)

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
\frac{1}{6} & \frac{5}{6} & \frac{4}{6} & \frac{3}{6} & \frac{2}{6} & \frac{1}{6} & \frac{1}{6}
\end{array}
\]
Probability mass function

$p_{Z}(z_i)$
Cumulative distribution function (CDF)

Cumulative pmf: \[ P(j) = \sum_{i=1}^{j} p(i) \]

where: \[ 0 \leq P(i) \leq 1 \]

\[ P_n = 1 \]
Continuous Random Variables

**Probability density function (pdf) of** \( X: p(x) \)

- \( p(x) \geq 0 \)
- No restriction that \( p(x) < 1 \) (Not a probability!)
Continuous Random Variables

Probability density function (pdf) of \( X: p(x) \)

- \( p(x) \geq 0 \)
- No restriction that \( p(x) < 1 \) (Not a probability!)

Cumulative distribution function (cdf): \( P(x) \)

\[
P(x) = \int_0^x p(x') \, dx'
\]

\[
P(x) = \Pr(X < x)
\]

\[
\Pr(a \leq X \leq b) = \int_a^b p(x') \, dx'
\]

\[
= P(b) - P(a)
\]
Continuous Random Variables

Canonical uniform random variable

\[ p(x) = \begin{cases} 
1 & x \in [0, 1], \\
0 & \text{otherwise}. 
\end{cases} \]
Ingredient: Uniform variates

Need: realizations of a uniformly distributed variable on the interval [0.0, 1.0]

Desired properties:

- sequence should pass statistical tests of randomness
- sequence should have a long period
- efficient to compute, requires only little storage
- repeatability: always produce the same sequence (different compilers, operating systems, processors)
Sources of randomness
A Million Random Digits
A modern example: PCG32

```c
struct pcg32_random_t { uint64_t state; uint64_t inc; }

uint32_t pcg32_random_r(pcg32_random_t* rng) {
    uint64_t oldstate = rng->state;
    rng->state = oldstate * 6364136223846793005ULL + (rng->inc | 1);
    uint32_t xorshifted = ((oldstate >> 18u) ^ oldstate) >> 27u;
    uint32_t rot = oldstate >> 59u;
    return (xorshifted >> rot) | (xorshifted << ((-rot) & 31));
}
```

[http://www.pcg-random.org/]

Expected value

Intuition: what value does the random variable take, on average?
Expected value

Intuition: what value does the random variable take, on average?

- e.g., consider a fair coin where heads = 1, tails = 0
- Equal probability (1/2 both)
- Expected value is then $(1/2) \times 1 + (1/2) \times 0 = 1/2$
Expected value

Intuition: what value does the random variable take, on average?

- e.g., consider a fair coin where heads = 1, tails = 0
- Equal probability (1/2 both)
- Expected value is then $(1/2) \times 1 + (1/2) \times 0 = 1/2$

Discrete

$E[X] = \sum_{i=1}^{n} p_i x_i$

- expected value of random variable $X$
- number of possible outcomes
- probability of i-th outcome
- value of i-th outcome
Expected value

Intuition: what value does the random variable take, on average?

- e.g., consider a fair coin where heads = 1, tails = 0
- Equal probability (1/2 both)
- Expected value is then \(\frac{1}{2} \times 1 + \frac{1}{2} \times 0 = 1/2\)

**Discrete**

\[
E[X] = \sum_{i=1}^{n} p_i x_i
\]

**Continuous**

\[
E[X] = \int_{\mathbb{R}} p(x) x \, dx
\]
**Expected value**

**Intuition:** what value does the random variable take, on average?

- e.g., consider a fair coin where heads = 1, tails = 0
- Equal probability (1/2 both)
- Expected value is then \( (1/2) \times 1 + (1/2) \times 0 = 1/2 \)

**Discrete**

\[
E[X] = \sum_{i=1}^{n} p_i x_i
\]

- expected value of random variable \( X \)
- number of possible outcomes
- probability of i-th outcome
- value of i-th outcome

**Continuous**

\[
E[X] = \int_{\mathbb{R}} p(x) x \, dx
\]

**Properties**

- \( E[X_1 + X_2] = \)
- \( E[aX] = \)
Expected value

Intuition: what value does the random variable take, on average?
- e.g., consider a fair coin where heads = 1, tails = 0
- Equal probability (1/2 both)
- Expected value is then \((1/2) \times 1 + (1/2) \times 0 = 1/2\)

**Discrete**

\[ E[X] = \sum_{i=1}^{n} p_i x_i \]

- expected value of random variable \(X\)
- number of possible outcomes
- probability of i-th outcome
- value of i-th outcome

**Continuous**

\[ E[X] = \int_{\mathbb{R}} p(x) x \, dx \]

- Properties
  - \(E[X_1 + X_2] = E[X_1] + E[X_2]\)
  - \(E[aX] = aE[X]\)
Monte Carlo Integration

Motivation: want to compute the integral

\[ F = \int_{D} f(x) \, dx \]

Could we approximate \( F \) by averaging a number of realizations \( x_i \) of a random process?

\[ \frac{1}{N} \sum_{i=1}^{N} f(x_i) \]
Monte Carlo Integration

\[
E \left[ \frac{1}{N} \sum_{i=1}^{N} f(X_i) \right] = \frac{1}{N} \sum_{i=1}^{N} E[f(X_i)]
\]

\[
= E[f(X_i)]
\]

\[
= \int_{D} f(x) p_{X_i}(x) \, dx
\]

(oops, that’s not what we wanted!)
Monte Carlo Integration

Motivation: want to compute the integral

\[ F = \int_D f(x) \, dx \]

Solution: Approximate \( F \) by averaging realizations of a random variable \( X \), and explicitly accounting for its PDF:

\[ F \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \]
Monte Carlo Integration

\[
E \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \right] = \frac{1}{N} \sum_{i=1}^{N} E \left[ \frac{f(X_i)}{p(X_i)} \right]
\]

Monte Carlo integration is correct on average.

- This assumes that \( p(X_i) \neq 0 \) when \( f(X_i) \neq 0 \).
- This property is called unbiasedness.
Monte Carlo Integration

Requirement (why?)

\[ f(x) \neq 0 \Rightarrow p(x) > 0 \]

Domain \( D \) might be: plane, sphere, hemisphere, surface of an object

Reasonable default for \( p(x) \): uniform distribution
Monte Carlo Integration

\[ f(x) = e^{\sin(3x^2)} \]
Monte Carlo Integration

\[ F = \int_{0}^{1} e^{\sin(3x^2)} \, dx \]
Monte Carlo Integration

\[ F = \int_{0}^{1} e^{\sin(3x^2)} \, dx \approx F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \Rightarrow \frac{1}{N} \sum_{i=1}^{N} f(x_i) \]

double integrate(int N)
{
    double x, sum=0.0;
    for (int i = 0; i < N; ++i) {
        x = randf();
        p(x_i) = 1
        sum += exp(sin(3*x*x));
    }
    return sum / double(N);
}
Monte Carlo Integration

\[ F = \int_{a}^{b} e^{\sin(3x^2)} \, dx \approx F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \]

double integrate(int N, double a, double b)
{
    double x, sum=0.0;
    for (int i = 0; i < N; ++i) {
        x = randf();
        sum += exp(sin(3*x*x));
    }
    return sum / double(N);
}
Monte Carlo Integration

\[ F = \int_a^b e^{\sin(3x^2)} \, dx \approx F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \]

double integrate(int N, double a, double b)
{
    double x, sum=0.0;
    for (int i = 0; i < N; ++i) {
        x = a + randf()*(b-a);
        sum += exp(sin(3*x*x));
    }
    return sum / double(N);
}
Monte Carlo Integration

\[ F = \int_a^b e^{\sin(3x^2)} \, dx \approx F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)} \]

double integrate(int N, double a, double b)
{
    double x, sum=0.0;
    for (int i = 0; i < N; ++i) {
        x = a + randf()*(b-a);
        sum += exp(sin(3*x*x)) / (1/(b-a));
    }
    return sum / double(N);
}
Monte Carlo Integration

\[ f(x) = e^{\sin(3x^2)} \]

<table>
<thead>
<tr>
<th>( N )</th>
<th>( F_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.75039</td>
</tr>
<tr>
<td>10</td>
<td>1.9893</td>
</tr>
<tr>
<td>100</td>
<td>1.79139</td>
</tr>
<tr>
<td>1000</td>
<td>1.75146</td>
</tr>
<tr>
<td>10000</td>
<td>1.77313</td>
</tr>
<tr>
<td>100000</td>
<td>1.77862</td>
</tr>
</tbody>
</table>

**True value:** 1.760977217585905...
Monte Carlo Integration

\[ f(x) = e^{\sin(3x^2)} \]

<table>
<thead>
<tr>
<th>( N )</th>
<th>( F_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.75039</td>
</tr>
<tr>
<td>10</td>
<td>1.9893</td>
</tr>
<tr>
<td>100</td>
<td>1.79139</td>
</tr>
<tr>
<td>1000</td>
<td>1.75146</td>
</tr>
<tr>
<td>10000</td>
<td>1.77313</td>
</tr>
<tr>
<td>100000</td>
<td>1.77862</td>
</tr>
</tbody>
</table>

Remarkable thing about this: Dimension doesn’t matter

True value: 1.760977217585905...
Variance

Intuition: how far are the samples from the average, on average?
Variance

Intuition: how far are the samples from the average, on average?

Definition:  
\[ V[X] = E \left[ (X - E[X])^2 \right] \]
Variance

Intuition: how far are the samples from the average, on average?

Definition: \[ V[X] = E \left[ (X - E[X])^2 \right] \]

**Q: Which of these has higher variance?**
Variance

Intuition: how far are the samples from the average, on average?

Definition: \[ V[X] = E \left[ (X - E[X])^2 \right] \]

Q: Which of these has higher variance?

Properties

\[
\begin{align*}
V[X] &= \\
V[X_1 + X_2] &= \\
V[aX] &=
\end{align*}
\]

only if uncorrelated!
Monte Carlo Error

\[ E[\|F_N - F\|^2] = E[F_N^2 - 2F_N F + F^2] \]


\[ = E[F_N^2] - 2E[F_N]F + F^2 \]

\[ = E[F_N^2] - 2FF + F^2 \]

\[ = E[F_N^2] - F^2 \]

\[ = E[F_N^2] - E[F_N]^2 = V[F_N] \]

For an *unbiased* estimator, its average error is equal to its variance!
Monte Carlo error

Variance:

\[
V \left[ \langle F^N \rangle \right] = V \left[ \frac{1}{N} \sum_{i=0}^{N-1} \frac{f(X_i)}{\text{pdf}(X_i)} \right]
\]

\[
= \frac{1}{N^2} \sum_{i=0}^{N-1} V \left[ \frac{f(X_i)}{\text{pdf}(X_i)} \right]
\]

\[
= \frac{1}{N^2} \sum_{i=0}^{N-1} V [Y_i]
\]

\[
= \frac{1}{N} V [Y]
\]
Monte Carlo error

Variance:
\[ V \left[ \langle F^N \rangle \right] = V \left[ \frac{1}{N} \sum_{i=0}^{N-1} \frac{f(X_i)}{\text{pdf}(X_i)} \right] \sim \text{assume uncorrelated samples} \]

\[ = \frac{1}{N^2} \sum_{i=0}^{N-1} V \left[ \frac{f(X_i)}{\text{pdf}(X_i)} \right] \]

\[ = \frac{1}{N^2} \sum_{i=0}^{N-1} V [Y_i] \]

\[ = \frac{1}{N} V [Y] \]

Std. deviation:
\[ \sigma \left[ \langle F^N \rangle \right] = \frac{1}{\sqrt{N}} \sigma [Y] \]
Monte Carlo Methods

Pros
- Flexible
- Easy to implement
- Easily handles complex integrands
- Efficient for high dimensional integrands
- *Unbiased* estimator

Cons
- Variance (noise)
- Slow convergence* \( O(1/\sqrt{N}) \)
Monte Carlo Integration Summary

Goal: evaluate integral \( \int_{a}^{b} f(x) \, dx \)

Random variable \( X_i \sim p(x) \)

Monte Carlo Estimator \( F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \)

Expectation \( E[F_N] = \int_{a}^{b} f(x) \, dx \)
Remaining Agenda

\[ F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \]

Main practical issues:
- How to choose \( p(x) \)
- How to generate \( x_i \) according to \( p(x) \)

Ambient Occlusion

\[ L_r(x, \bar{\omega}_r) = \int_{H^2} f_r(x, \bar{\omega}_i, \bar{\omega}_r) L_i(x, \bar{\omega}_i) \cos \theta_i \, d\bar{\omega}_i \]
Sampling Random Variables

Sampling the function domain:

- Uniform unit interval (0,1)
- Uniform interval (a,b)
- Circle?
- Sphere?
- Hemisphere?
- More complex domains?
Example: uniformly sampling a disk

Uniform probability density on a unit disk

\[ p(x, y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 < 1 \\ 0 & \text{otherwise} \end{cases} \]

Goal: draw samples \( X_i, Y_i \) that are distributed as:

\( (X_i, Y_i) \sim p(x, y) \)

Problem: pseudo-random number generator only allows us to draw samples from a canonical uniform distribution
Rejection Sampling in a Disk

Vector2 v;
do{
  v.x = 1-2*randf();
  v.y = 1-2*randf();
} while (dot(v,v) > 1)

- Similar technique for sampling a sphere
Rejection Sampling in a Sphere

Vector3 v;

do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(dot(v,v) > 1)
Rejection Sampling on a Sphere

Vector3 v;
do{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(dot(v,v) > 1)

// Project onto sphere
v = v/length(v);
Rejection Sampling a Hemisphere

Vector3 v;

do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(dot(v,v) > 1)
Rejection Sampling a Hemisphere

Vector3 v;

do
{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(dot(v,v) > 1 ||
    v.z < 0)
Rejection Sampling a Hemisphere

Vector3 v;
do {
v.x = 1-2*randf();
v.y = 1-2*randf();
v.z = 1-2*randf();
} while(dot(v,v) > 1 || v.z < 0)

• Arbitrary orientation?
Rejection Sampling a Hemisphere

Vector3 v;
do{
    v.x = 1-2*randf();
    v.y = 1-2*randf();
    v.z = 1-2*randf();
} while(dot(v,v) > 1 ||
    dot(v,n) < 0)

• Arbitrary orientation?
Rejection Sampling a Hemisphere

- Or, just generate in canonical orientation, and then rotate
Rejection Sampling

More complex shapes

Pros:
- Flexible

Cons:
- Inefficient
- Difficult/impossible to combine with stratification or quasi-Monte Carlo
Directly sampling a disk?

Idea: transform samples to polar coordinates:

- pick two uniform random variables $\xi_1, \xi_2$
- select point at $(r, \phi)$ with $r = \xi_1$ and $\phi = 2\pi \xi_2$
- This algorithm **does not** produce the desired uniform sampling of the disk. Why?

not equi-areal
Wrong!
Samples are uniform in \((\theta, r)\), but non-uniform in \((x, y)\)!

Right!
Samples are non-uniform in \((\theta, r)\), but uniform in \((x, y)\)!

\[
\begin{align*}
\theta &= 2\pi \xi_1 \\
r &= \xi_2
\end{align*}
\]

This can be corrected by choosing \(r\) non-uniformly!

\[
\begin{align*}
\theta &= 2\pi \xi_1 \\
r &= \sqrt{\xi_2}
\end{align*}
\]
Transforming Between Distributions

Given a random variable $X_i \sim p(x)$

$Y_i = T(X_i)$ is also a random variable

- but what is its probability density?

$$p_y(y) = p_y(T(x)) = \frac{p_x(x)}{|J_T(x)|}$$

- where $|J_T(x)|$ is the absolute value of the determinant of the Jacobian of $T$
Polar coordinate parameterization

\[ T(r, \phi) \mapsto \begin{bmatrix} r \cos \phi \\ r \sin \phi \end{bmatrix} \]

\[ J_T(r, \phi) = \begin{bmatrix} \frac{\partial T_x}{\partial r} & \frac{\partial T_x}{\partial \phi} \\ \frac{\partial T_y}{\partial r} & \frac{\partial T_y}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \cos \phi & -r \sin \phi \\ \sin \phi & r \cos \phi \end{bmatrix} \]

\[ |\det J_T(r, \phi)| = r \]
Account for parameterization

Desired distribution on target domain

\[ p(x, y) = \begin{cases} \frac{1}{\pi}, & x^2 + y^2 < 1 \\ 0, & \text{otherwise} \end{cases} \]

If we sample in spherical coordinates:

\[
\begin{align*}
\text{target domain} & \quad p(x, y) = p(T(r, \phi)) = p(r, \phi) \\ 
\text{sampling domain} & \quad \frac{p(r, \phi)}{|\det J_T(r, \phi)|}
\end{align*}
\]

Thus, need this distribution on source domain:

\[
\begin{align*}
p(r, \phi) & = p(T(r, \phi)) \cdot |\det J_T(r, \phi)| = \frac{r}{\pi} \\
& = 1/\pi \quad \Rightarrow \quad r
\end{align*}
\]
Sampling 2D Distributions

Draw samples \((X, Y)\) from a 2D distribution \(p(x, y)\)

If \(p(x, y)\) is separable, i.e., \(p(x, y) = p(x) \ p(y)\), we can independently sample \(p(x)\), and \(p(y)\)

Otherwise, compute the marginal density function:

\[
p(x) = \int p(x, y) \ dy
\]

and, the conditional density:

\[
p(y \mid x) = \frac{p(x, y)}{p(x)}
\]

Procedure: first sample \(X_i \sim p(x)\), then \(Y_i \sim p(y \mid X_i)\)
Account for parameterization

Thus: need this distribution on source domain

\[ p(r, \phi) = p(T(r, \phi)) \cdot |\text{det} J_T(r, \phi)| = \frac{r}{\pi} \]

\[ = \frac{1}{\pi} \cdot r \]

Step 1: generate \( \phi \) proportional to

\[ p_1(\phi) = \frac{1}{2\pi} \quad (\phi \in [0, 2\pi]) \]

Step 2: generate \( r \) proportional to

\[ p_2(r) \propto r = 2r \quad (r \in [0, 1]) \]

Constant PDF in \( \phi \), linearly increasing PDF in \( r \)
Sampling arbitrary distributions

The inversion method:

1. Compute the CDF
   \[ P(x) = \int_0^x p(x') \, dx' \]
2. Compute its inverse
   \[ P^{-1}(y) \]
3. Obtain a uniformly distributed random number \( \xi \)
4. Compute
   \[ X_i = P^{-1}(\xi) \]
Sampling a linear ramp

Goal: sample with PDF: \[ p(r) = 2r \]

Step 1: \[ P(r) = r^2 \]

Step 2: \[ P^{-1}(y) = \sqrt{y} \]

Step 3: \[ r_i = \sqrt{\xi} \]
Uniformly Sampling a Disk

Pick two uniform random variables $\xi_1, \xi_2$

Sample in polar coordinates with:

$$ (r, \phi) = (\xi_1, 2\pi \xi_2) $$

$$ (r, \phi) = \left( \sqrt{\xi_1}, 2\pi \xi_2 \right) $$

not equi-area

equi-area
Recipe

1. Express the desired distribution in a convenient coordinate system

2. Account for distortion by coordinate system
   - Requires computing the determinant of the Jacobian

3. Compute marginal and conditional 1D PDFs

4. Sample 1D PDFs using the inversion method
Directly Sampling on a Sphere

Can we use this?

Given a random variable $X_i \sim p(x)$

$Y_i = T(X_i)$ is also a random variable

- but what is its probability density?

$$p_y(y) = p_y(T(x)) = \frac{p_x(x)}{|J_T(x)|}$$

- where $|J_T(x)|$ is the absolute value of the determinant of the Jacobian of $T$
Directly Sampling on a Sphere

Different transformation rule:

\[ p_x(x(u, v)) = \frac{p(u,v)(u, v)}{\|x_u(u,v) \times x_v(u,v)\|} \]

Where does this come from?

- Expression for differential area (e.g., as in area integral):

\[ dA(x) = \|x_u(u,v) \times x_v(u,v)\| dudv \]
Directly Sampling on a Sphere

Pick two uniform random variables $\xi_1, \xi_2$

Idea: select point at $(\theta, \varphi)$ with $\theta = \pi \xi_1$ and $\varphi = 2\pi \xi_2$

- Problem: not uniform with respect to surface area!

Correct solution: $\theta = \cos^{-1}(2\xi_1 - 1)$ and $\varphi = 2\pi \xi_2$

Algorithm

\[
\begin{align*}
\theta & = \cos^{-1}(2\xi_1 - 1) \\
\phi & = 2\pi \xi_2 \\
\vec{\omega}_x & = \sin \theta \cos \phi \\
\vec{\omega}_y & = \sin \theta \sin \phi \\
\vec{\omega}_z & = \cos \theta
\end{align*}
\]

Better

\[
\begin{align*}
\vec{\omega}_z & = 2\xi_1 - 1 \\
r & = \sqrt{1 - \vec{\omega}_z^2} \\
\phi & = 2\pi \xi_2 \\
\vec{\omega}_x & = r \cos \phi \\
\vec{\omega}_y & = r \sin \phi
\end{align*}
\]
Archimedes' Hat-Box Theorem

The surface area of a sphere between any two horizontal planes is equal to the corresponding area on the circumscribing cylinder.

- i.e.: uniform areas on a cylinder map to uniform areas on a sphere
- What is $|J_T|$ for cylindrical mapping?
Archimedes' Hat-Box Theorem

The surface area of a sphere between any two horizontal planes is equal to the corresponding area on the circumscribing cylinder.

- i.e.: uniform areas on a cylinder map to uniform areas on a sphere

- What is $|J_T|$ for cylindrical mapping?

\[
\begin{align*}
\tilde{\omega}_z &= \frac{2\xi_1 - 1}{1 - \tilde{\omega}_z^2} \\
r &= \sqrt{1 - \tilde{\omega}_z^2} \\
\phi &= 2\pi\tilde{\xi}_2 \\
\tilde{\omega}_x &= r\cos\phi \\
\tilde{\omega}_y &= r\sin\phi
\end{align*}
\]

- point on unit cylinder
- projection onto sphere

Directly Sampling a Hemisphere

Just like a sphere

Use Hat-Box theorem with shorter cylinder
More Random Sampling

Other useful sampling domains:

- triangles
- 1- or 2-D discrete PDFs (e.g. environment maps)

Much more!
### Sampling Various Distributions

<table>
<thead>
<tr>
<th>Target Space</th>
<th>Density</th>
<th>Domain</th>
<th>Transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius $R$ disk</td>
<td>$p(r, \theta) = \frac{1}{\pi R^2}$</td>
<td>$\theta \in [0, 2\pi]$</td>
<td>$\theta = 2\pi u$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$r \in [0, R]$</td>
<td>$r = R u$</td>
</tr>
<tr>
<td>Sector of radius $R$ disk</td>
<td>$p(r, \theta) = \frac{2}{(\theta_2 - \theta_1)(r_2^2 - r_1^2)}$</td>
<td>$\theta \in [\theta_1, \theta_2]$</td>
<td>$\theta = \theta_1 + u(\theta_2 - \theta_1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$r \in [r_1, r_2]$</td>
<td>$r = \sqrt{r_1^2 + u(r_2^2 - r_1^2)}$</td>
</tr>
<tr>
<td>Phong density</td>
<td>$p(\theta, \phi) = \frac{n + 1}{2\pi} \cos^n \theta$</td>
<td>$\theta \in \left[0, \frac{\pi}{2}\right]$</td>
<td>$\theta = \arccos((1-u)^{1/(n+1)})$</td>
</tr>
<tr>
<td>exponent $n$</td>
<td></td>
<td>$\phi \in [0, 2\pi]$</td>
<td>$\phi = 2\pi v$</td>
</tr>
<tr>
<td>Separated triangle filter</td>
<td>$p(x, y; 1 -</td>
<td>x</td>
<td>)(1 -</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y \in [-1, 1]$</td>
<td>$y = \begin{cases} 1 - \sqrt{2(1-v)} &amp; \text{if } v \geq 0.5 \ -1 + \sqrt{2v} &amp; \text{if } v &lt; 0.5 \end{cases}$</td>
</tr>
<tr>
<td>Triangle with vertices $a_0, a_1, a_2$</td>
<td>$p(\alpha) = \frac{1}{\text{area}}$</td>
<td>$s \in [0, 1]$</td>
<td>$s = 1 - \frac{1-u}{\sqrt{2}}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t \in [0, 1-s]$</td>
<td>$t = (1-s)u$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\alpha = a_0 + s(a_1 - a_0) + t(a_2 - a_0)$</td>
<td>$\alpha = \arccos(1 - 2u)$</td>
</tr>
<tr>
<td>Surface of unit sphere</td>
<td>$p(\theta, \phi) = \frac{1}{4\pi}$</td>
<td>$\theta \in [0, \pi]$</td>
<td>$\theta = 2\pi u$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi \in [0, 2\pi]$</td>
<td>$\phi = 2\pi v$</td>
</tr>
<tr>
<td>Sector on surface</td>
<td>$p(\theta, \phi)$</td>
<td>$\theta \in [\theta_1, \theta_2]$</td>
<td>$\theta = \arccos(\cos \theta_1$</td>
</tr>
<tr>
<td>of unit sphere</td>
<td></td>
<td>$\phi \in [\phi_1, \phi_2]$</td>
<td>$\phi = \phi_1 + v(\phi_2 - \phi_1)$</td>
</tr>
<tr>
<td>Interior of radius $R$ sphere</td>
<td>$p = \frac{3}{4\pi R^3}$</td>
<td>$\theta \in [0, \pi]$</td>
<td>$\theta = \arccos(1 - 2u)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\phi \in [0, 2\pi]$</td>
<td>$\phi = 2\pi v$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R \in [0, R]$</td>
<td>$R = \sqrt{1-2u}$</td>
</tr>
</tbody>
</table>

*The symbols $u, v,$ and $\alpha$ represent instances of uniformly distributed random variables ranging over $[0, 1].$*
Ambient Occlusion

Consider diffuse objects illuminated by an ambient overcast sky

$$L_r(x, \omega_x) \equiv \int_{H^2} \int_{H} f_r(x, \omega_i, \omega_r) L_i(x, \omega_i) \cos \theta_i \, d\omega_i$$

integral over hemisphere
Ambient Occlusion

Consider diffuse objects illuminated by an ambient overcast sky

\[ L_r(x) = \frac{\rho}{\pi} \int_{H^2} V(x, \omega_i) \cos \theta_i \, d\omega_i \]

\( V = 0 \), zero contribution
Ambient Occlusion

Consider diffuse objects illuminated by an ambient overcast sky

\[ L_r(x) = \frac{\rho}{\pi} \int_{H^2} V(x, \tilde{\omega}_i) \cos \theta_i \, d\tilde{\omega}_i \approx \frac{\rho}{\pi} \frac{V(x, \tilde{\omega}_i) \cos \theta_i}{p(\tilde{\omega}_i)} \]

Uniform sampling: \[ p(\tilde{\omega}_i) = \frac{1}{2\pi} \]

\[ \approx 2\rho V(x, \tilde{\omega}_i) \cos \theta_i \]

\[ V = 0, \text{ zero contribution} \]
Hemispherical Sampling (1 Sample)
Hemispherical Sampling (4 Samples)
Hemispherical Sampling (9 Samples)
Hemispherical Sampling (16 Samples)
Hemispherical Sampling (256 Samples)
Hemispherical Sampling (1024 Samples)
Ambient Occlusion

http://www.3dluvr.com/marcoss/
Strategies for reducing variance

The standard MC estimator:

\[ F = \int_{\mu(x)} f(x) \, d\mu(x) \]

\[ \langle F^N \rangle = \frac{1}{N} \sum_{i=0}^{N-1} \frac{f(X_i)}{\text{pdf}(X_i)} \]

\[ \sigma \left[ \langle F^N \rangle \right] = \frac{1}{\sqrt{N}} \sigma [Y] \]

How do we reduce the variance of \( Y \)?

- Importance sampling
Importance sampling

\[ \int f(x) \, dx \quad \quad F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \]

Assume:

\[ p(x) = cf(x) \]

\[ \int p(x) \, dx = 1 \quad \rightarrow \quad c = \frac{1}{\int f(x) \, dx} \]

Estimator:

\[ \frac{f(X_i)}{p(X_i)} = \frac{1}{c} = \int f(x) \, dx \quad \text{zero variance!} \]
Importance sampling

$p(x) = cf(x)$ requires knowledge of the integral we are trying to compute in the first place!

But: If PDF is similar to integrand, variance can be significantly reduced

Common strategy: sample according to part of the integrand
Ambient occlusion

\[ L_r(x) = \frac{\rho}{\pi} \int_{H^2} V(x, \vec{\omega}_i) \cos \theta_i \, d\vec{\omega}_i \]

What terms can we importance sample?

- incident radiance
- cosine term
Ambient occlusion

\[ L_r(x) = \frac{\rho}{\pi} \int_{H^2} V(x, \vec{\omega}_i) \cos \theta_i d\vec{\omega}_i \]

What terms can we importance sample?

- incident radiance
- cosine term
Ambient Occlusion

\[ L_r(x) \approx \frac{\rho}{\pi N} \sum_{k=1}^{N} \frac{V(x, \bar{\omega}_{i,k}) \cos \theta_{i,k}}{p(\bar{\omega}_{i,k})} \]

**Uniform hemispherical sampling**

\[ p(\bar{\omega}_{i,k}) = \frac{1}{2\pi} \]

\[ L_r(x) \approx \frac{2\rho}{N} \sum_{k=1}^{N} V(x, \bar{\omega}_{i,k}) \cos \theta_{i,k} \]

**Cosine-weighted importance sampling**

\[ p(\bar{\omega}_{i,k}) = \frac{\cos \theta_{i,k}}{\pi} \]

\[ L_r(x) \approx \frac{\rho}{N} \sum_{k=1}^{N} V(x, \bar{\omega}_{i,k}) \]
Cosine-weighted Hemispherical Sampling

Could proceed as before: compute marginal and conditional densities, then use inversion method.

It turns out that:

- Generating points uniformly on the disc, and then project these points vertically onto the hemisphere produces the desired distribution.
Cosine-weighted Hemispherical Sampling

Generate points on sphere
(unit directions)
Cosine-weighted Hemispherical Sampling

Generate points on sphere (unit directions)

Add unit normal
Cosine-weighted Hemispherical Sampling

Generate points on sphere (unit directions)

Add unit normal
normalize

unit normal
Uniform hemispherical sampling

1 sample/pixel

Cosine-weighted importance sampling
Uniform hemispherical sampling

4 sample/pixel

Cosine-weighted importance sampling
Uniform hemispherical sampling  16 sample/pixel  Cosine-weighted importance sampling
Uniform hemispherical sampling

1024 sample/pixel

Cosine-weighted importance sampling
More Integration Dimensions

Anti-aliasing (image space)
Light visibility (surface of area lights)
Depth-of-field (camera aperture)
Motion blur (time)
Many lights
Multiple bounces of light
Participating media (volume)