Participating media

http://graphics.cs.cmu.edu/courses/15-468

15-468, 15-668, 15-868
Physics-based Rendering
Spring 2024, Lecture 12

Course announcements

- Take-home quiz 6 posted, due Tuesday 3/14 at 23:59 (after spring break).

- Programming assignment 3 posted, due Friday 3/17 at 23:59 (after spring break).
  - How many of you have looked at/started/finished it?
  - Any questions?

- Suggest topics for Friday's reading group on Piazza.

Overview of today’s lecture

- Participating media.
- Scattering material characterization.
- Volume rendering equation.
- Ray marching.
- Volumetric path tracing.
- Delta tracking.

Slide credits

Most of these slides were directly adapted from:

- Wojciech Jarosz (Dartmouth).

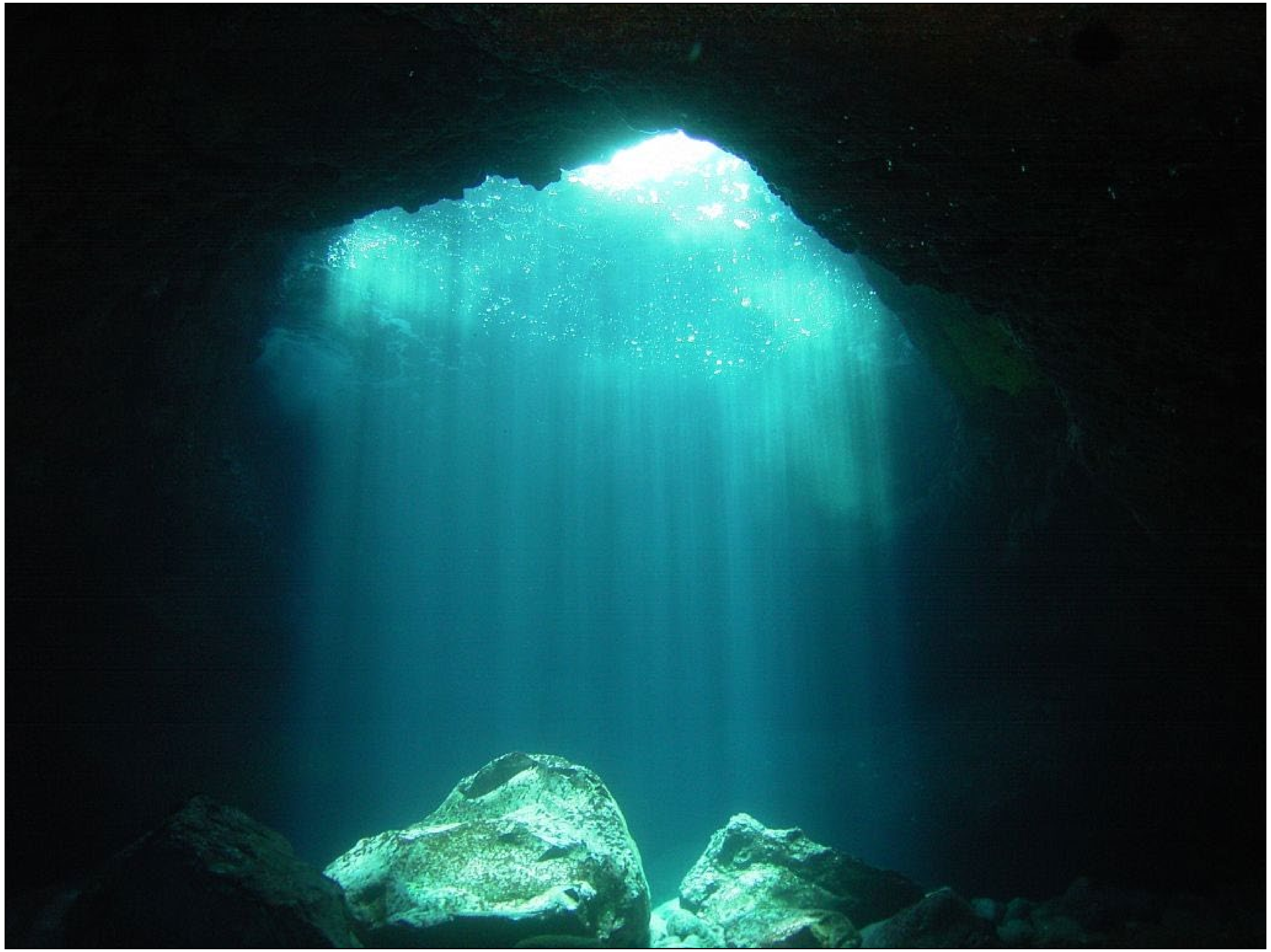
Fog


Steve Lacey

Clouds & Crepuscular rays

Fire

Underwater

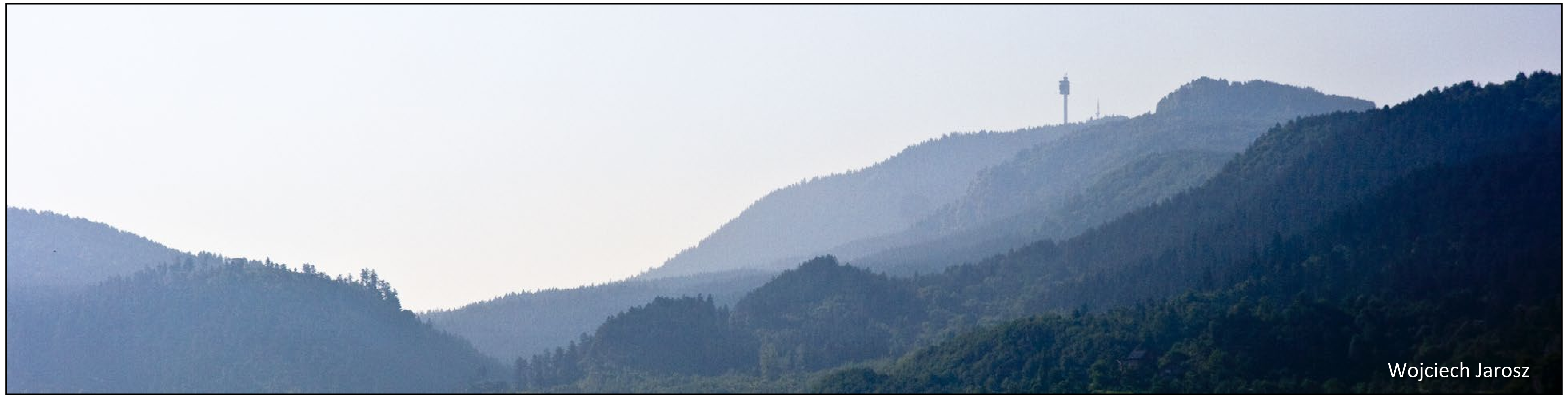


Surface or Volume?

source: Studio Lernert & Sander

Antelope Canyon, Az.

Aerial (Atmospheric) Perspective

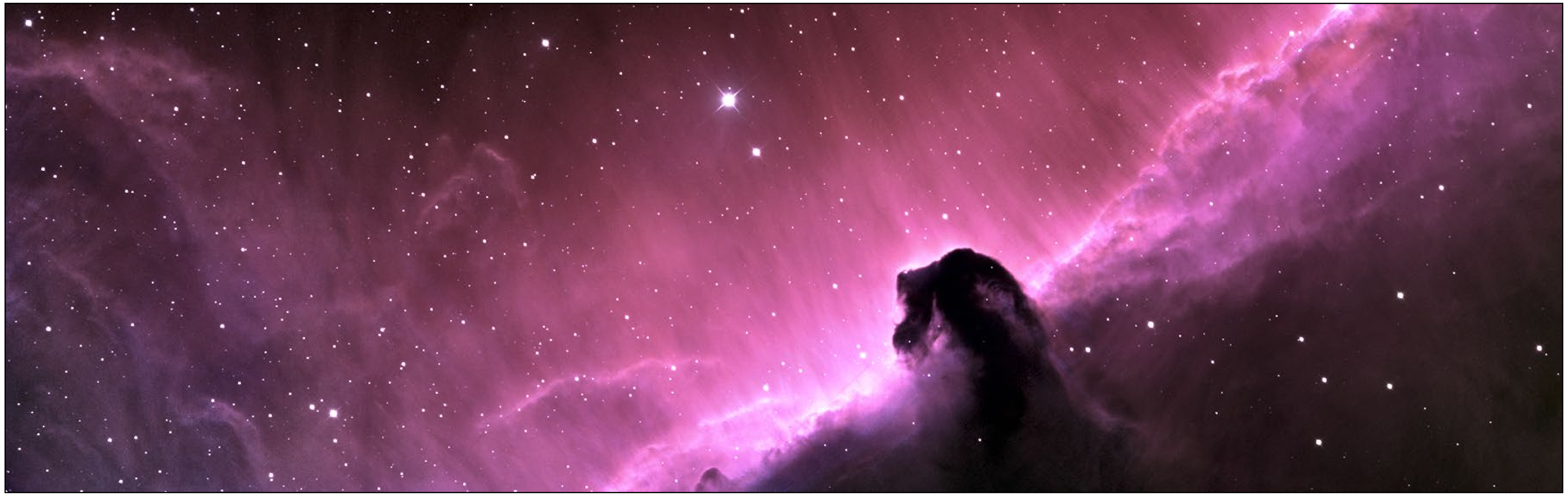


Leonardo da Vinci (1480)

Thus, if one is to be five times as distant, make it five times bluer.

—Treatise on Painting, Leonardo Da Vinci, pp 295, circa 1480.

Nebula



T.A.Rector (NOAO/AURA/NSF) and the Hubble Heritage Team (STScI/AURA/NASA)

Emission

Absorption

Scattering

Defining Participating Media

Typically, we do not model particles of a medium explicitly (wouldn’t fit in memory, completely impractical to ray trace)

The properties are described statistically using various coefficients and densities

- Conceptually similar idea as microfacet models

Defining Participating Media

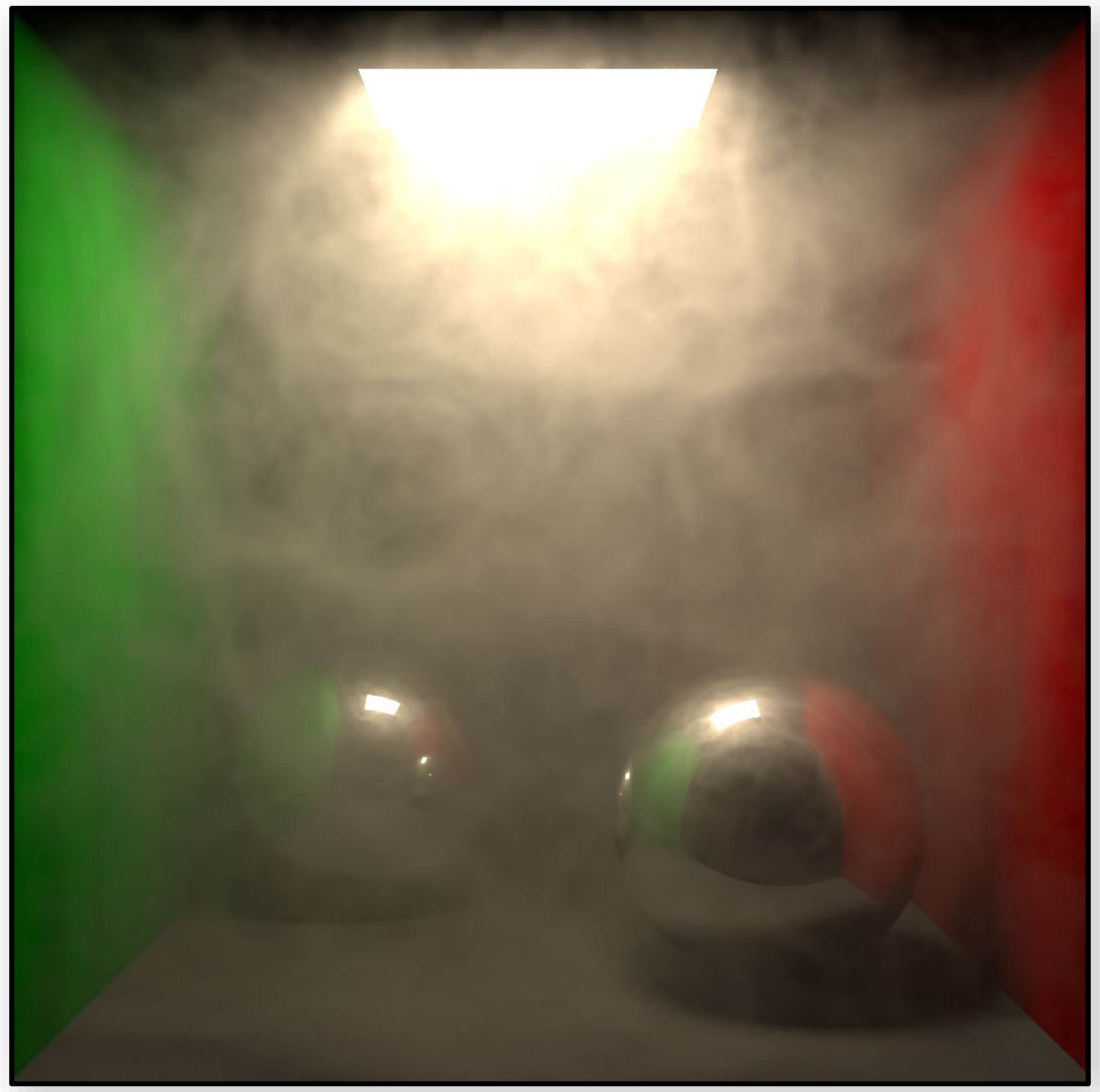
Homogeneous:

- Infinite or bounded by a surface or simple shape

Defining Participating Media

Heterogeneous (spatially varying coefficients):

- Procedurally, e.g., using a noise function
- Simulation + volume discretization, e.g., a voxel grid

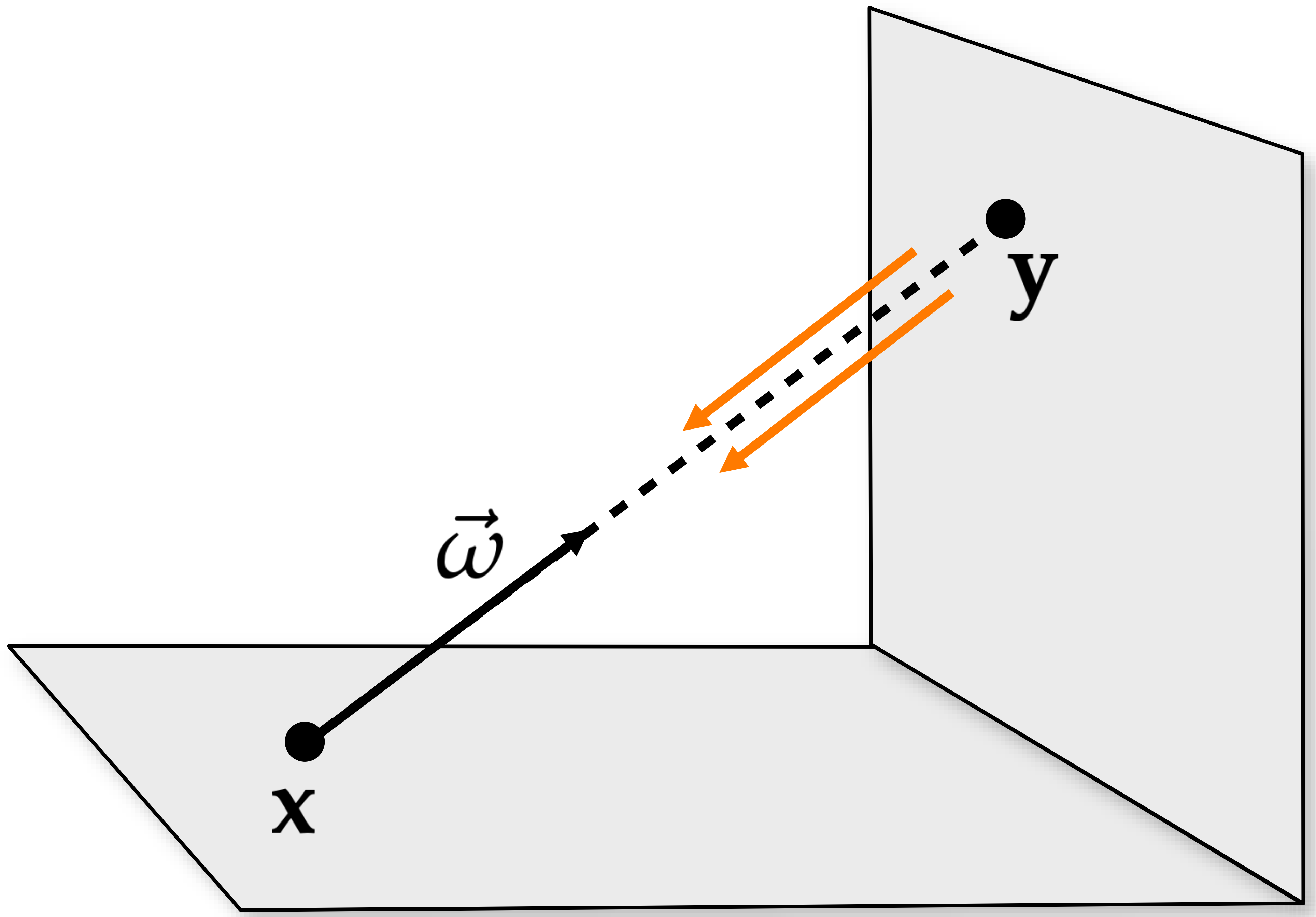


Radiance

The main quantity we are interested in for rendering is radiance

Previously: radiance *remains constant* along rays between surfaces

$$L_i(\mathbf{x}, \vec{\omega}) = L_o(\mathbf{y}, -\vec{\omega})$$
$$\mathbf{y} = r(\mathbf{x}, \vec{\omega})$$



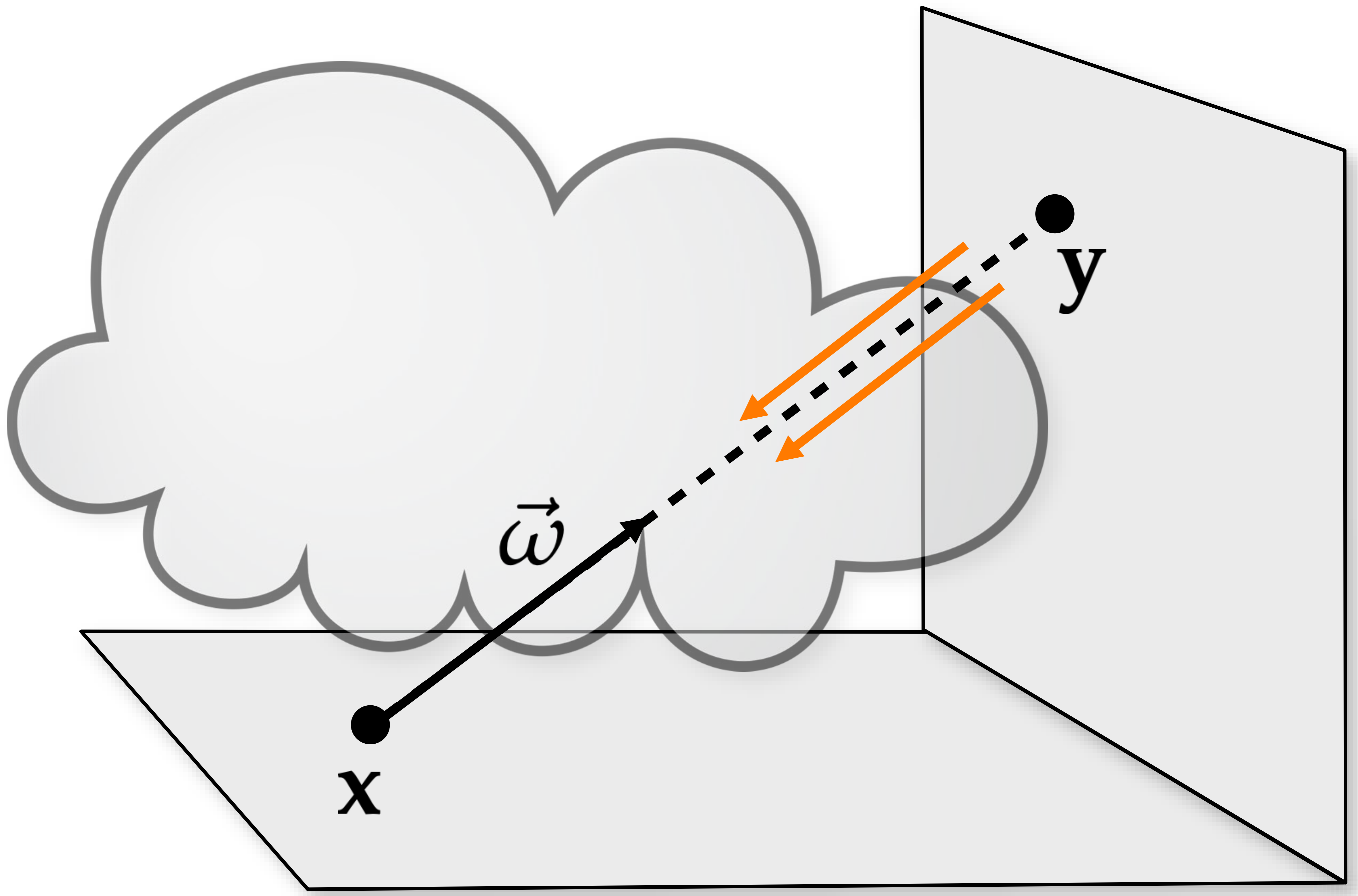
Radiance

The main quantity we are interested in for rendering is radiance

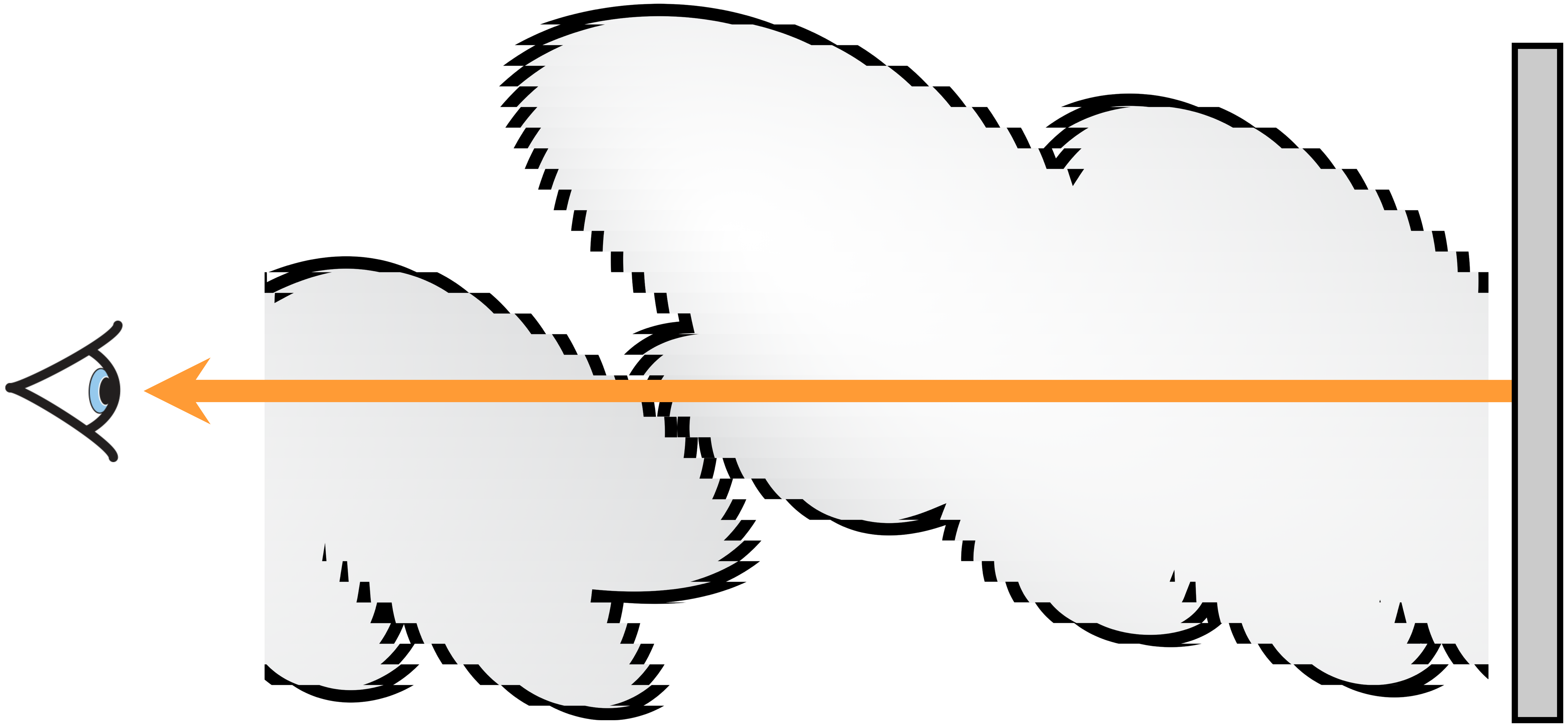
Now: radiance may *change* along rays between surfaces

$$L_i(\mathbf{x}, \vec{\omega}) \neq L_o(\mathbf{y}, -\vec{\omega})$$

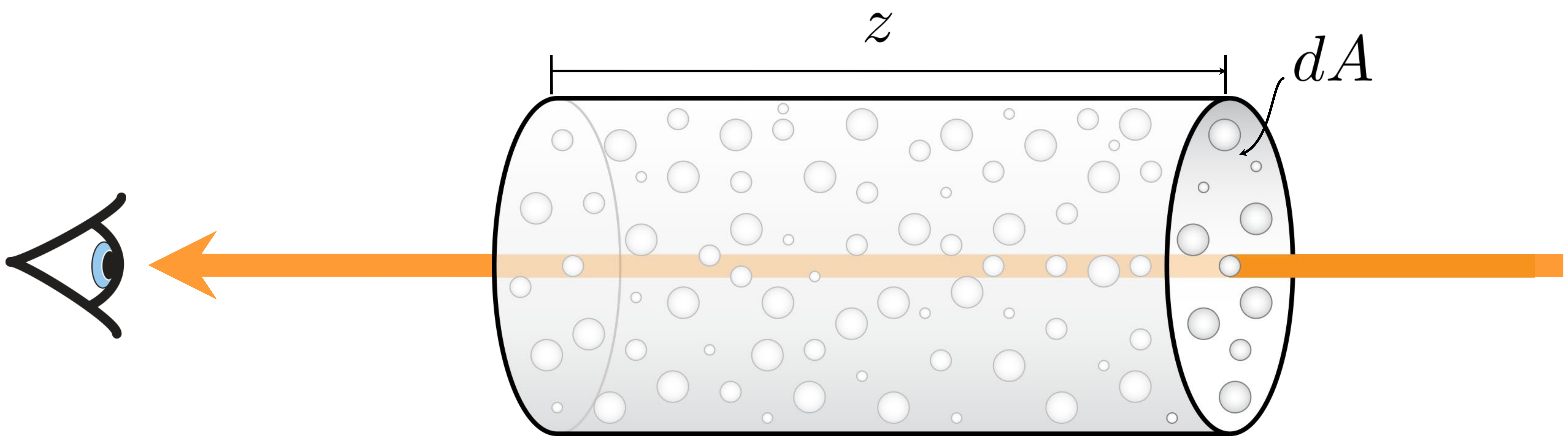
$$\mathbf{y} = r(\mathbf{x}, \vec{\omega})$$



Participating Media

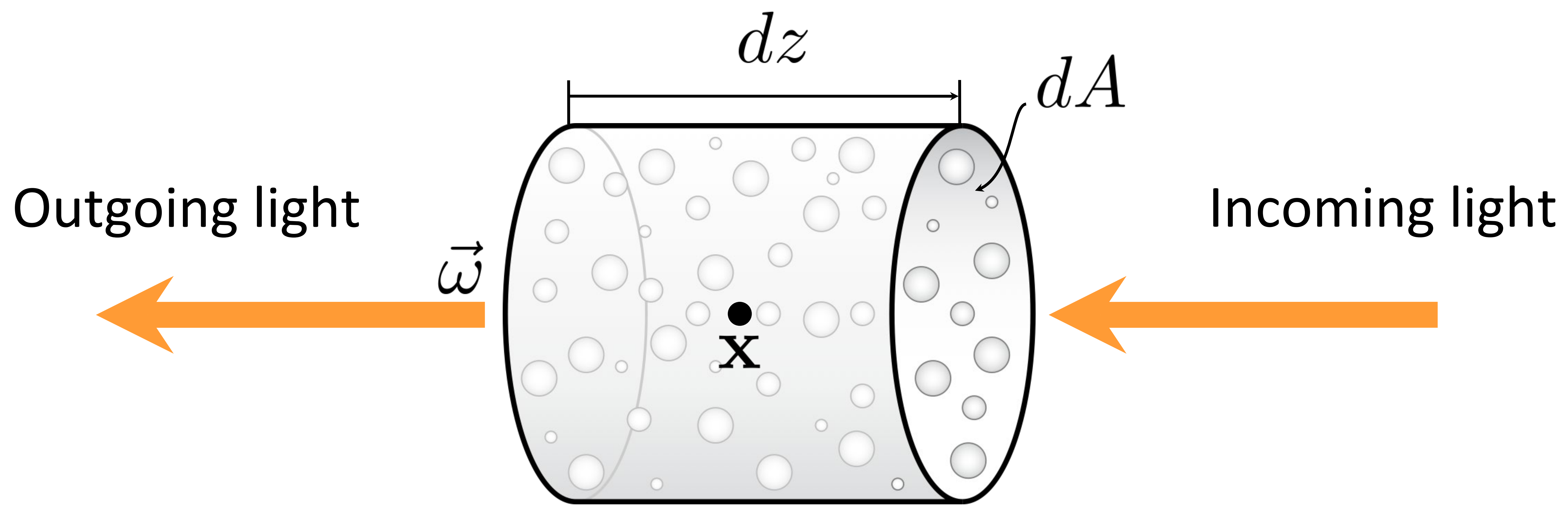


Differential Beam

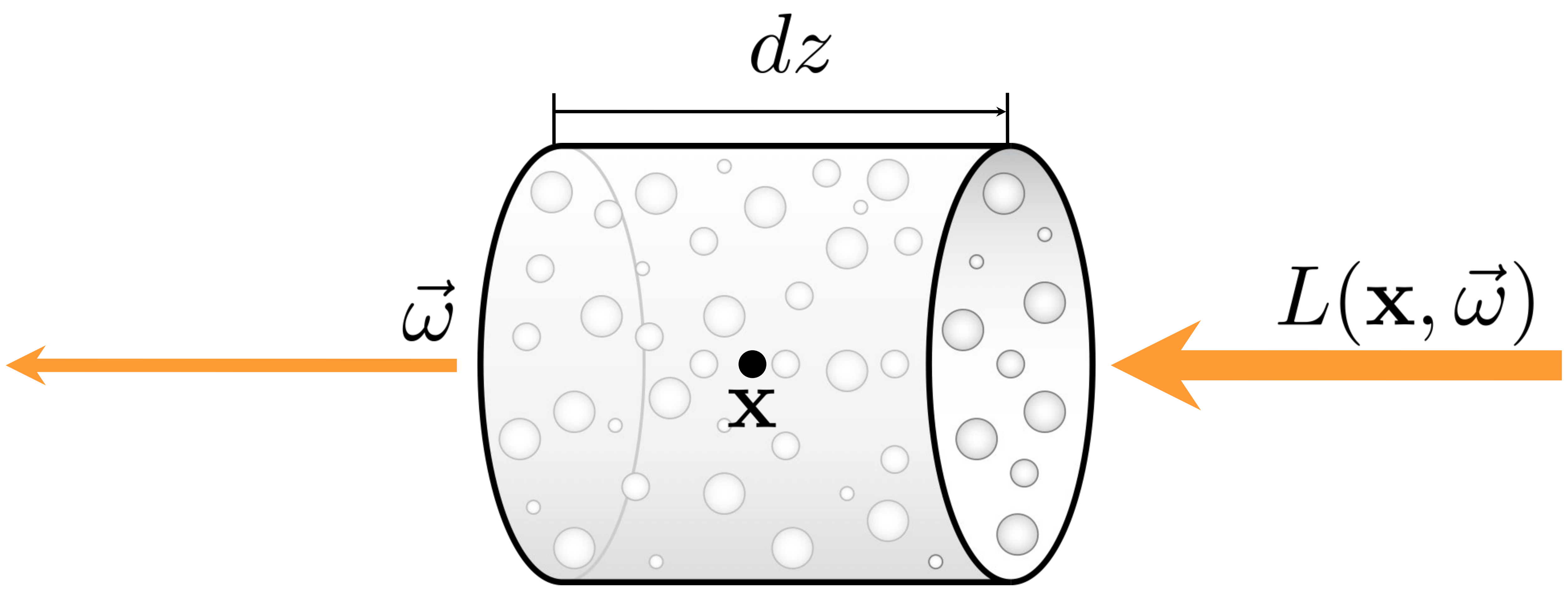


How much light is *lost/gained* along the differential beam due to interactions of light with the medium?

Differential Beam Segment



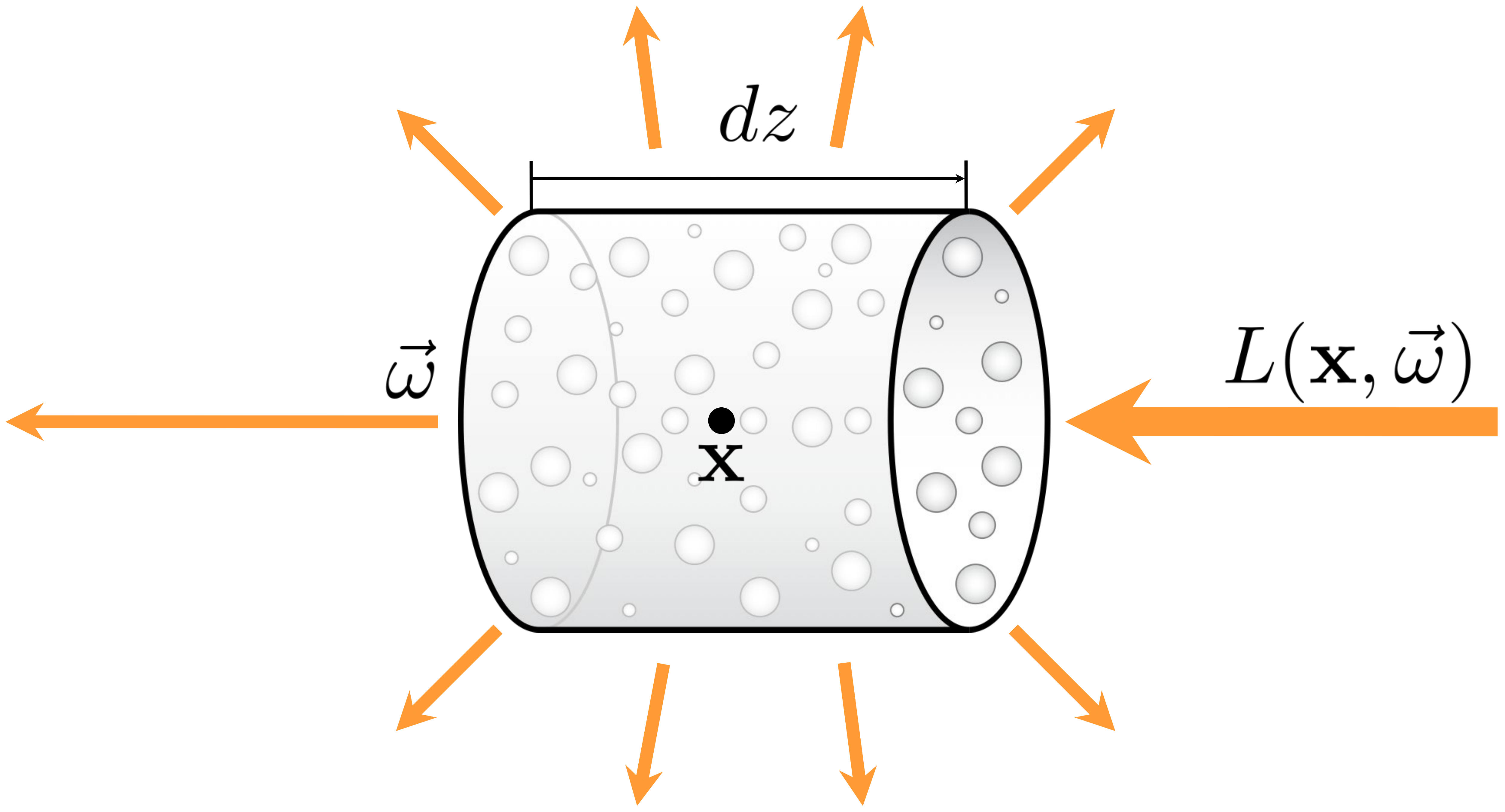
Absorption



$$dL(\mathbf{x}, \vec{\omega}) = -\sigma_a(\mathbf{x}) L(\mathbf{x}, \vec{\omega}) dz$$

$\sigma_a(\mathbf{x})$: absorption coefficient $[m^{-1}]$

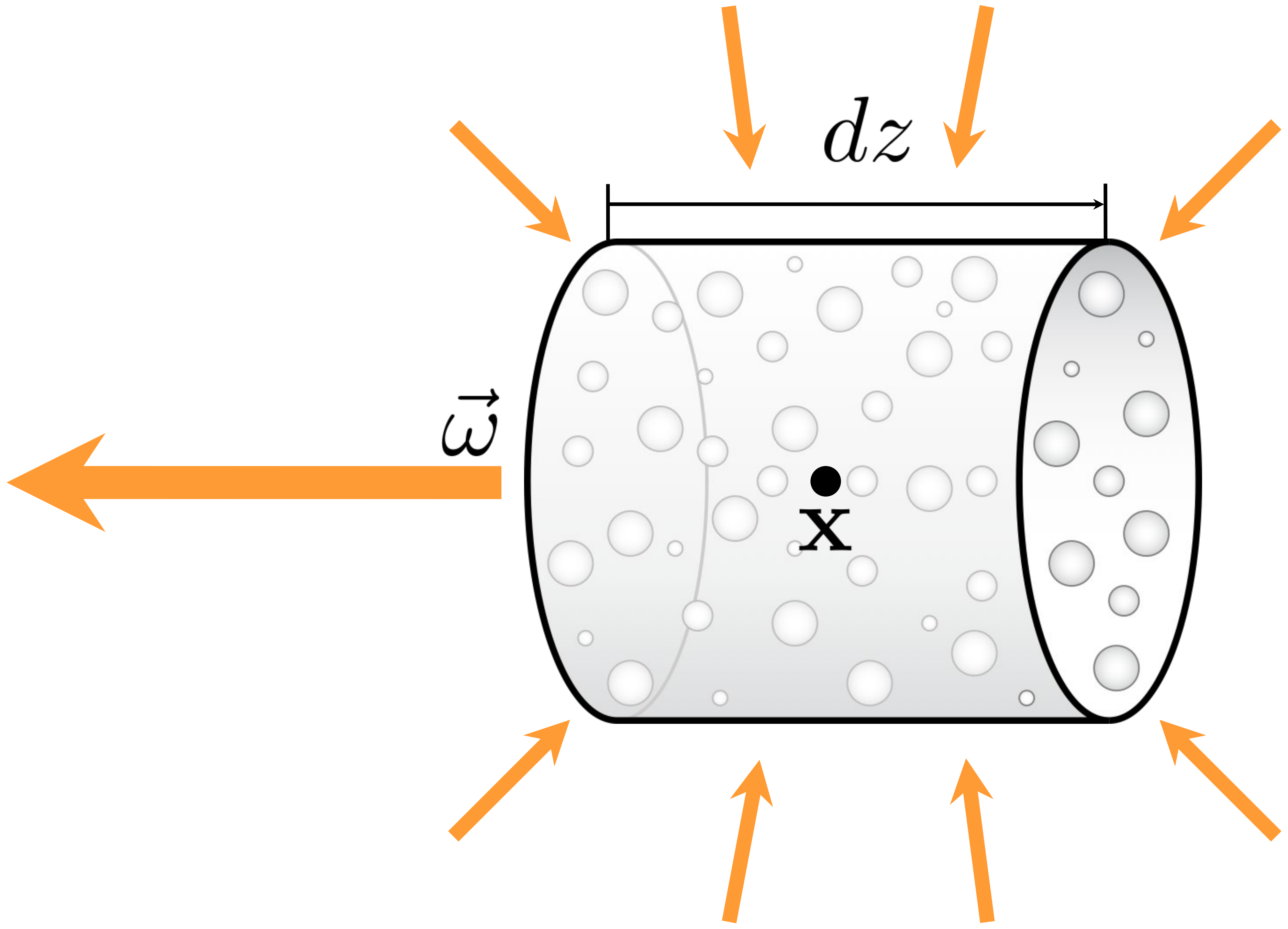
Out-scattering



$$dL(\mathbf{x}, \vec{\omega}) = -\sigma_s(\mathbf{x}) L(\mathbf{x}, \vec{\omega}) dz$$

$\sigma_s(\mathbf{x})$: scattering coefficient $[m^{-1}]$

In-scattering

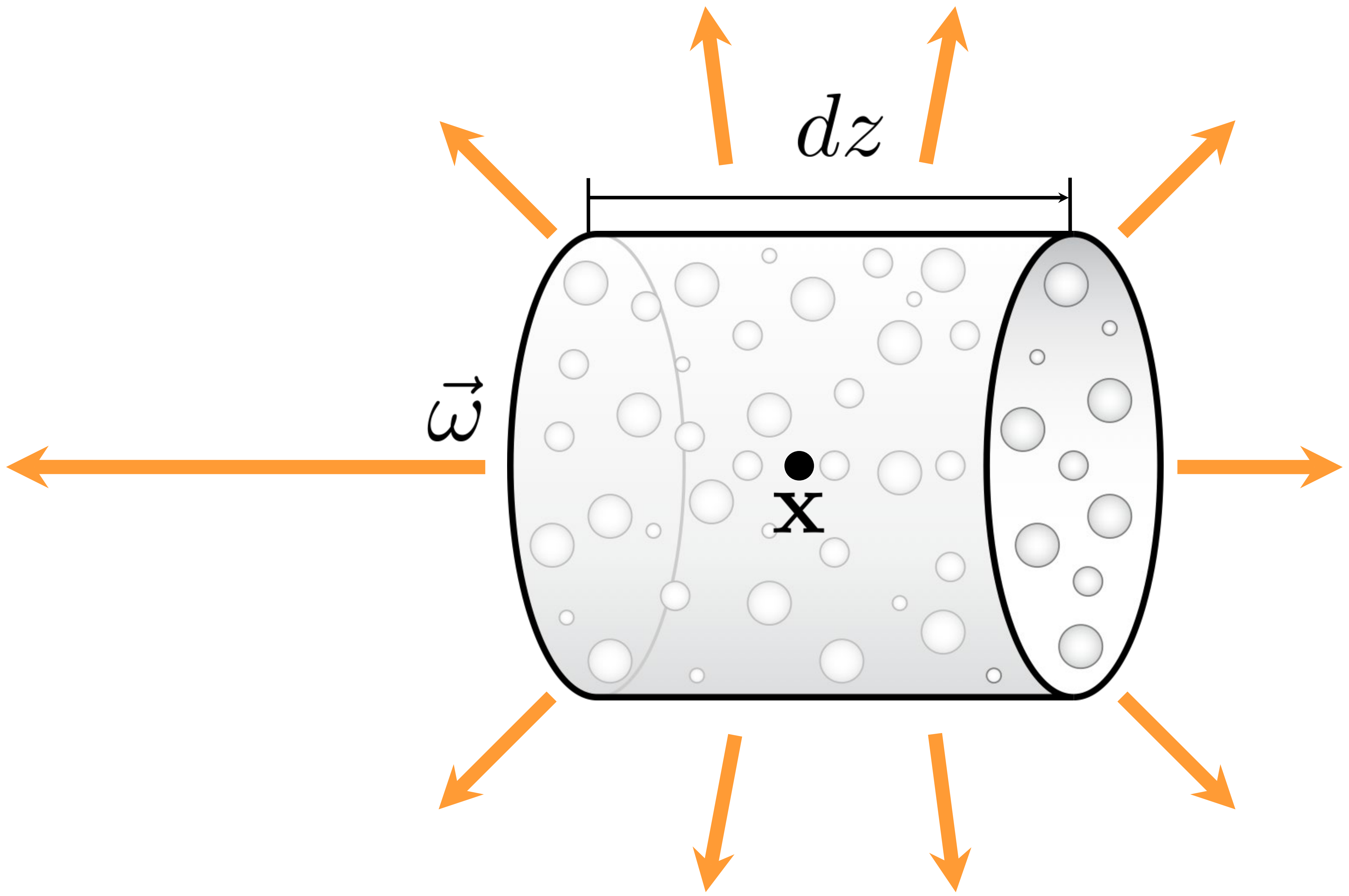


$$dL(\mathbf{x}, \vec{\omega}) = \sigma_s(\mathbf{x}) L_s(\mathbf{x}, \vec{\omega}) dz$$

$\sigma_s(\mathbf{x})$: scattering coefficient $[m^{-1}]$

$L_s(\mathbf{x}, \vec{\omega})$: in-scattered radiance

Emission



$$dL(\mathbf{x}, \vec{\omega}) = \sigma_a(\mathbf{x}) L_e(\mathbf{x}, \vec{\omega}) dz$$

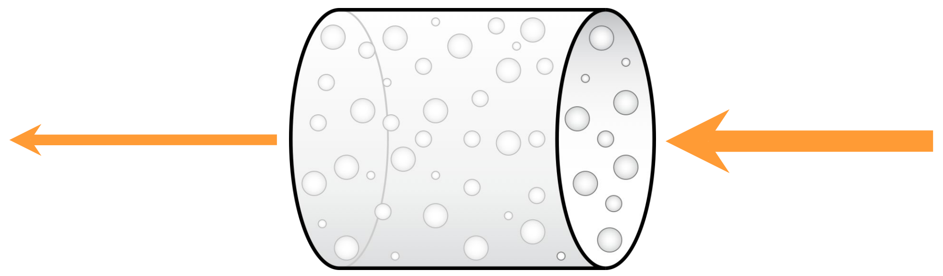
*Sometimes modeled without the absorption coefficient just by specifying a “source” term

$\sigma_a(\mathbf{x})$: absorption coefficient $[m^{-1}]$

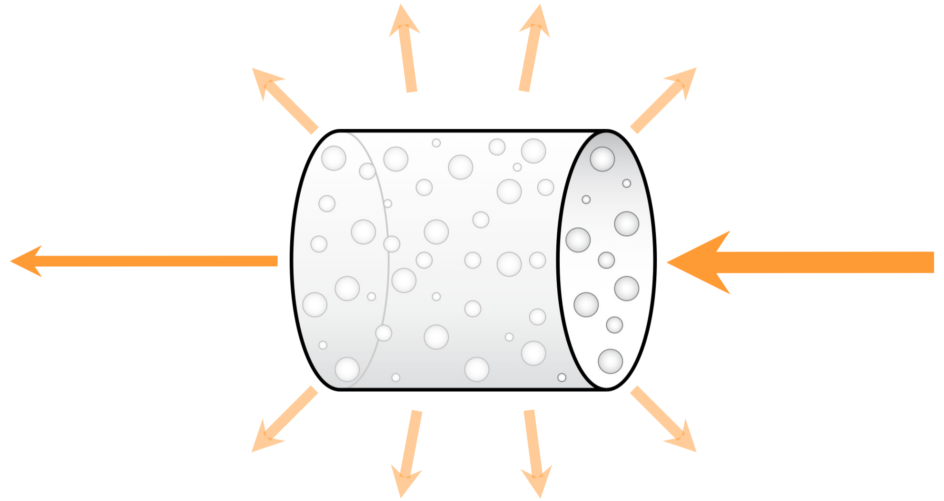
$L_e(\mathbf{x}, \vec{\omega})$: emitted radiance

Radiative Transfer Equation (RTE)

Absorption



Out-scattering

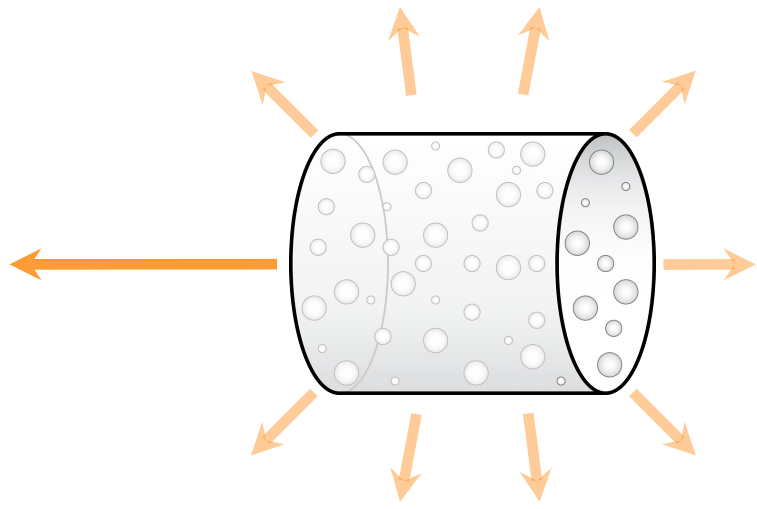


$$dL(\mathbf{x},\vec{\omega}) = -\sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega})dz \quad -\sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega})dz$$

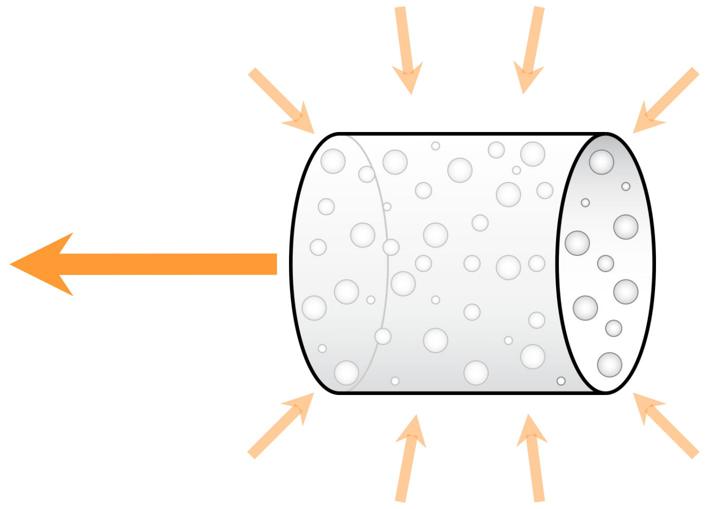
Losses

$$+\sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})dz + \sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega})dz$$

Gains



Emission



In-scattering

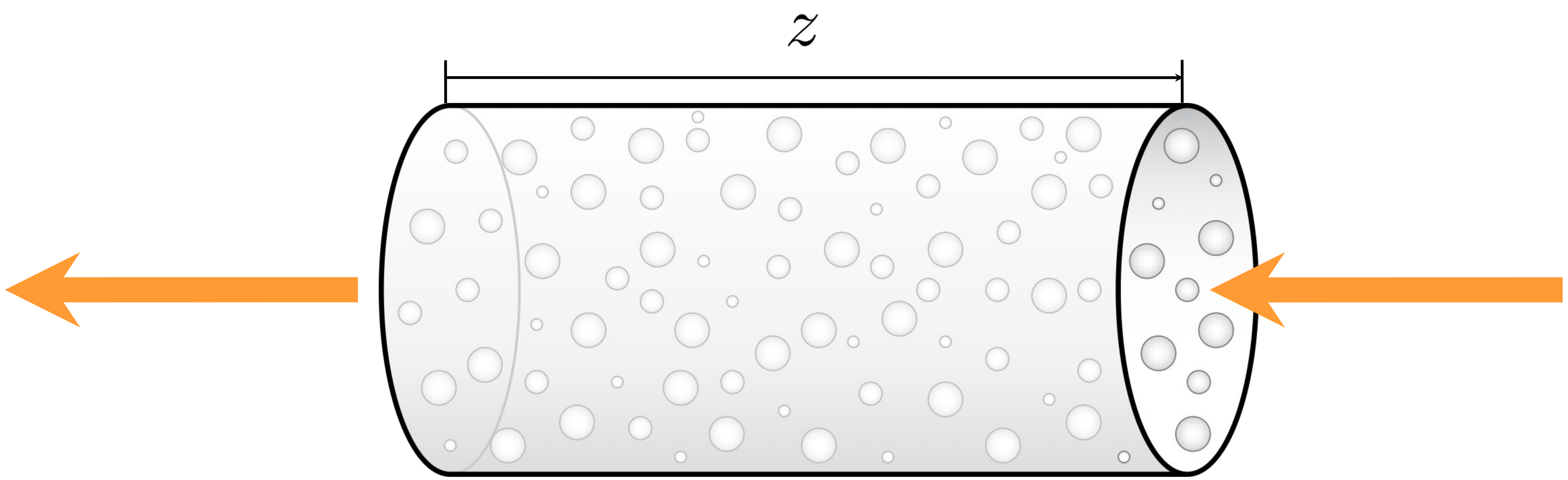
Losses (Extinction)

Absorption   Out-scattering

$$dL(\mathbf{x},\vec{\omega}) = -\sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega})dz - \sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega})dz$$
$$= -\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega})dz$$

$\sigma_t(\mathbf{x})$: extinction coefficient $[m^{-1}]$

: total loss of light per unit distance



What about a beam with a finite length?

Extinction Along a Finite Beam

$$dL(\mathbf{x},\vec{\omega}) = -\sigma_t(\mathbf{x})L(\mathbf{x},\vec{\omega})dz$$ // Assume constant $\sigma_t(\mathbf{x})$, reorganize

$$\frac{dL(\mathbf{x},\vec{\omega})}{L(\mathbf{x},\vec{\omega})} = -\sigma_t dz$$ // Integrate along beam from 0 to z

$$\ln(L_z) - \ln(L_0) = -\sigma_t z$$

$$\ln\left(\frac{L_z}{L_0}\right) = -\sigma_t z$$ // Exponentiate

$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

Beer-Lambert Law

Expresses the remaining radiance after traveling a finite distance through a medium with constant extinction coefficient

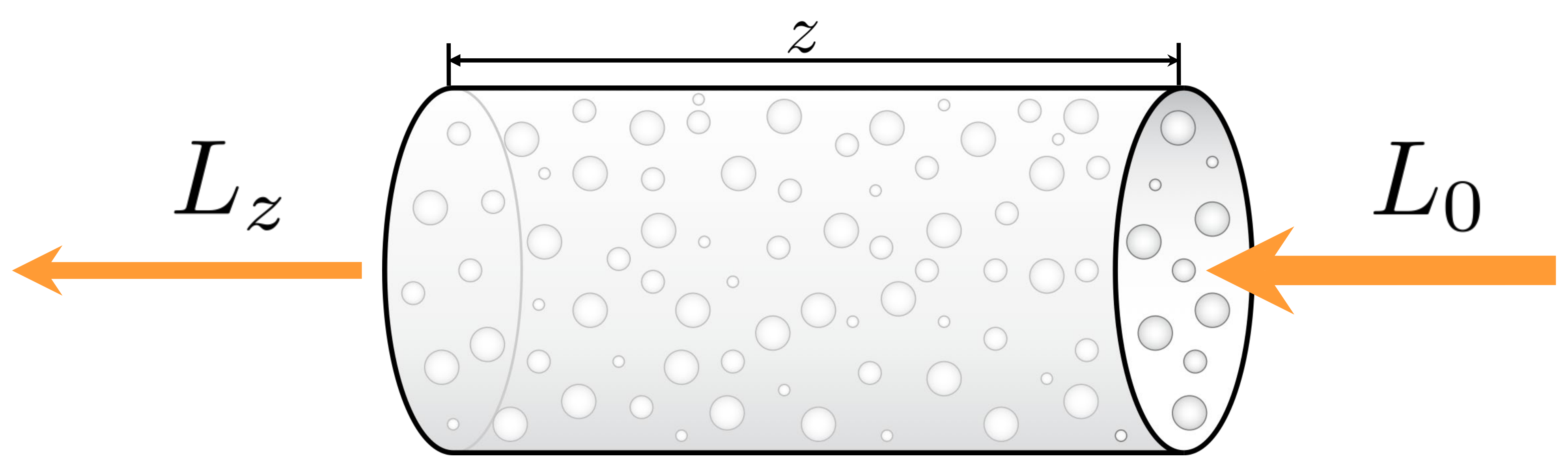
The fraction is referred to as the *transmittance*

Think of this as fractional visibility between points

Radiance at distance z

$$\frac{L_z}{L_0} = e^{-\sigma_t z}$$

Radiance at the beginning of the beam



Transmittance

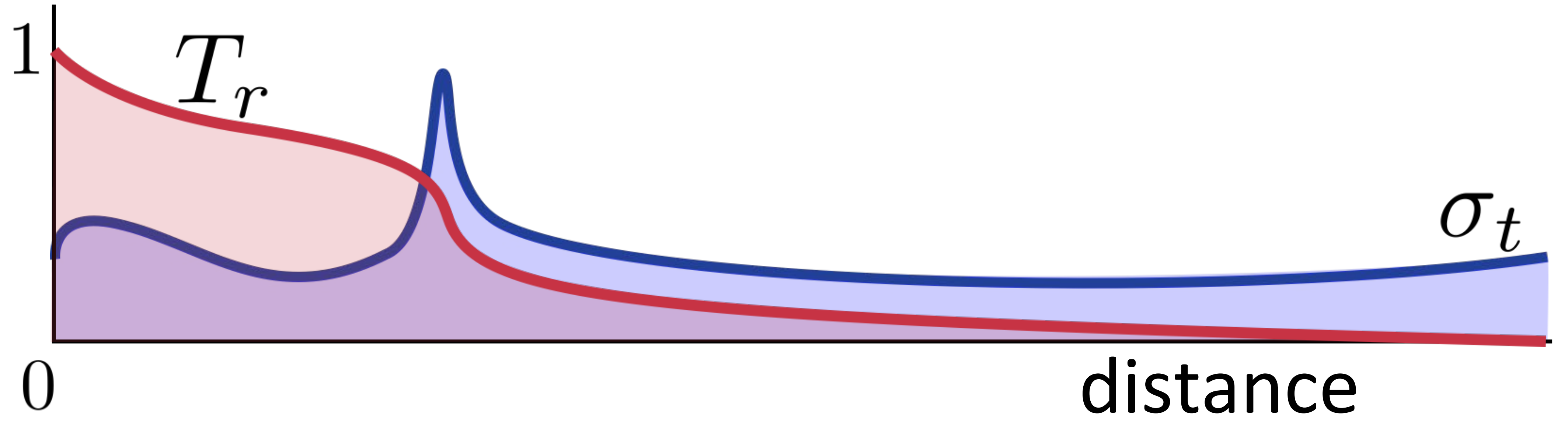
Homogeneous volume:

$$T_r(\mathbf{x}, \mathbf{y}) = e^{-\boxed{\sigma_t \|\mathbf{x}-\mathbf{y}\|}}$$

Heterogeneous volume (spatially varying σ_t):

$$T_r(\mathbf{x}, \mathbf{y}) = e^{-\boxed{\int_0^{\|\mathbf{x}-\mathbf{y}\|} \sigma_t(t)dt}}$$

Optical thickness



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Transmittance

Homogeneous volume:

$$T_r(\mathbf{x},\mathbf{y}) = e^{-\boxed{\sigma_t \|\mathbf{x}-\mathbf{y}\|}}$$

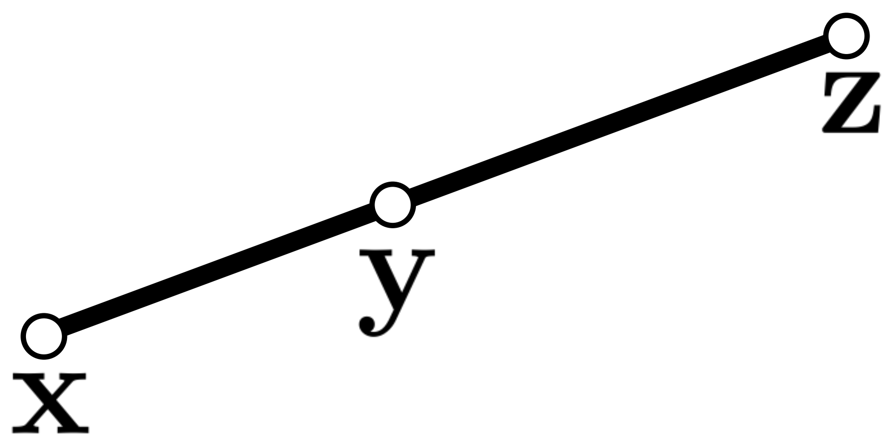
Heterogeneous volume (spatially varying σ_t):

$$T_r(\mathbf{x},\mathbf{y}) = e^{-\boxed{\int_0^{\|\mathbf{x}-\mathbf{y}\|} \sigma_t(t)dt}}$$

Optical thickness

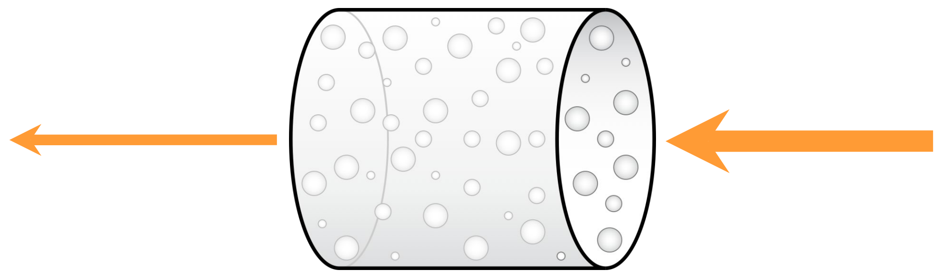
Transmittance is multiplicative:

$$T_r(\mathbf{x},\mathbf{z}) = T_r(\mathbf{x},\mathbf{y})T_r(\mathbf{y},\mathbf{z})$$

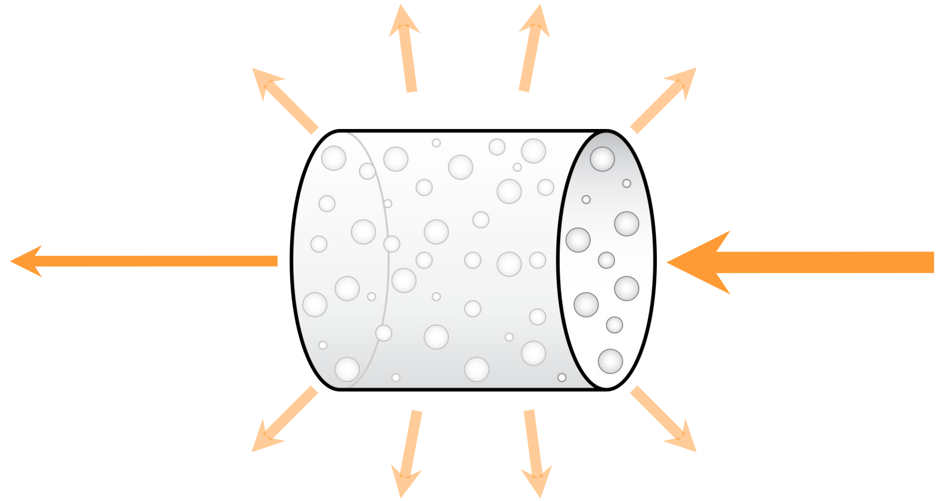


Radiative Transfer Equation (RTE)

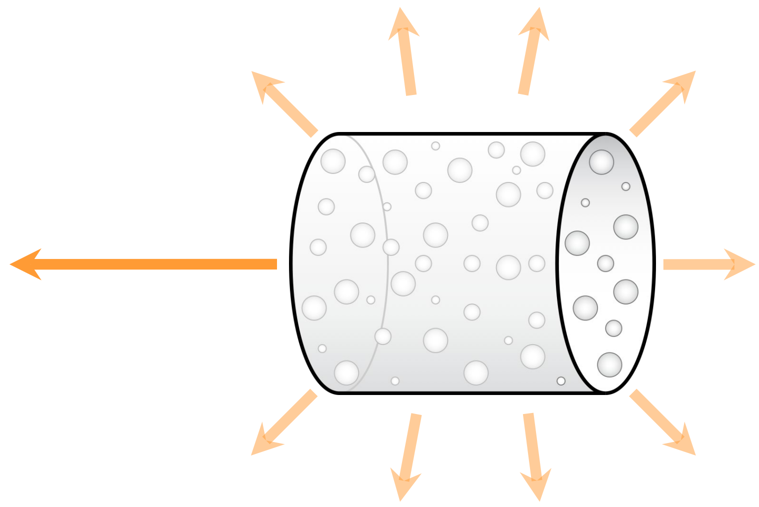
Absorption



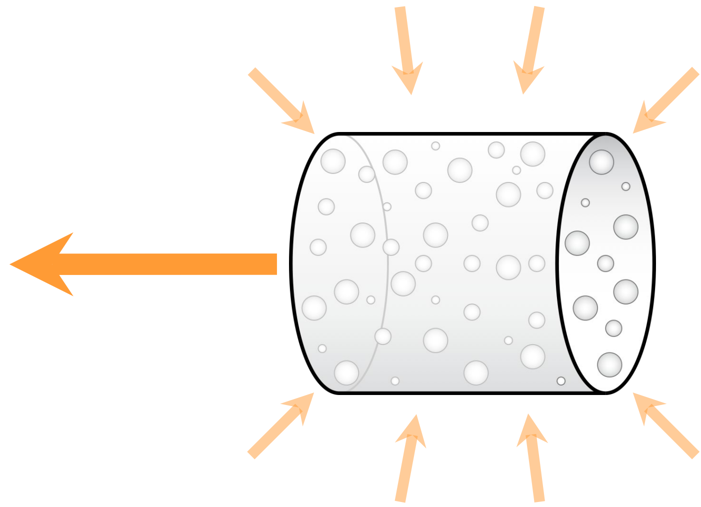
Out-scattering



$$dL(\mathbf{x},\vec{\omega}) = -\,\sigma_a(\mathbf{x})L(\mathbf{x},\vec{\omega})dz \quad -\,\sigma_s(\mathbf{x})L(\mathbf{x},\vec{\omega})dz$$
$$+\,\sigma_a(\mathbf{x})L_e(\mathbf{x},\vec{\omega})dz \,+\,\sigma_s(\mathbf{x})L_s(\mathbf{x},\vec{\omega})dz$$



Emission

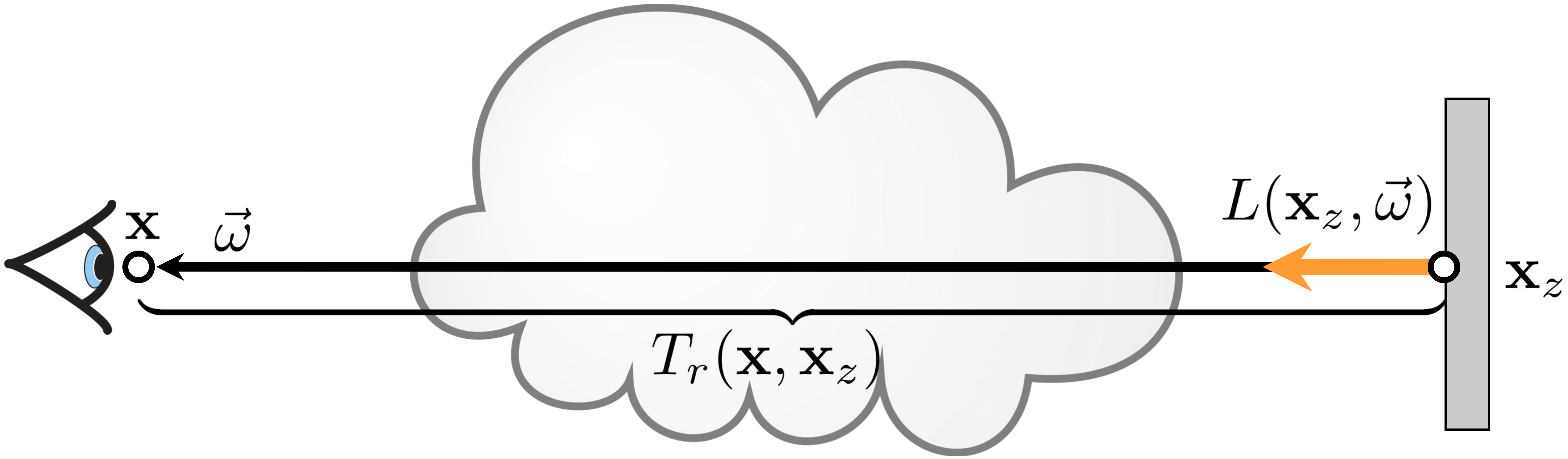


In-scattering

Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$

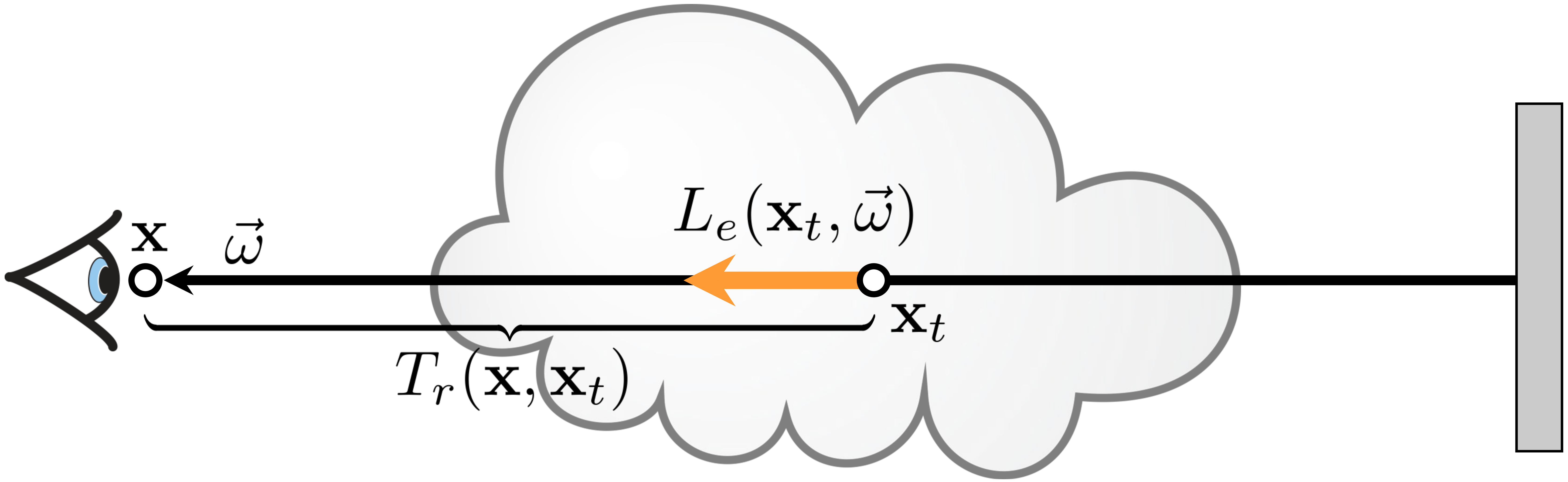
Reduced (background) surface radiance



Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt$$

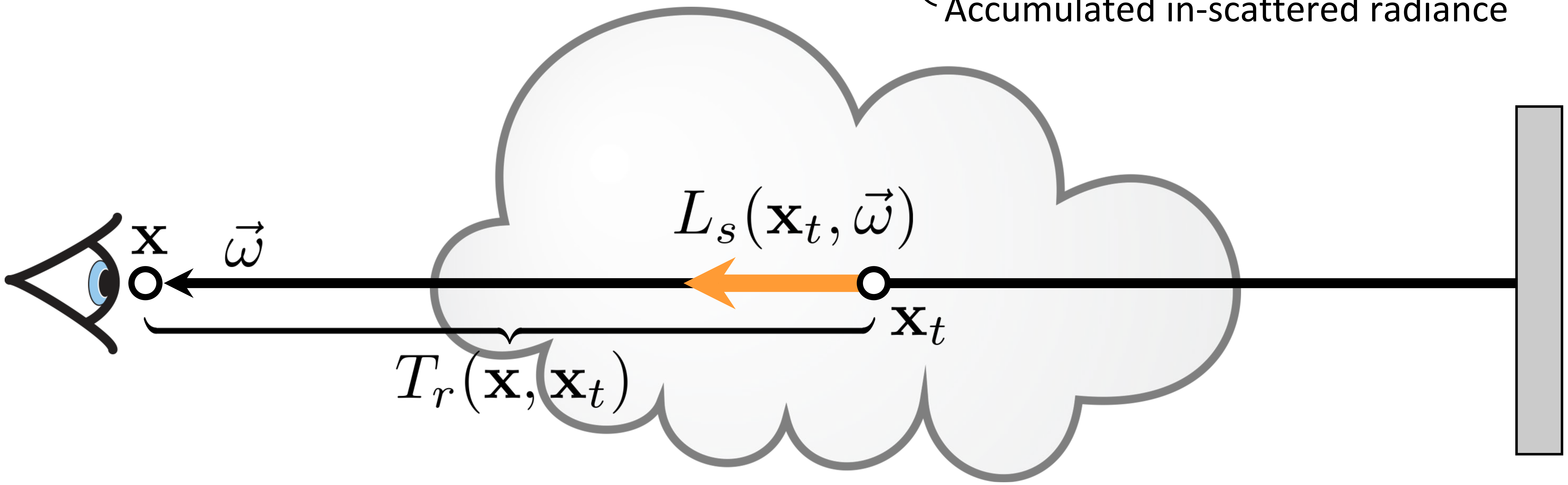
Accumulated emitted radiance



Volume Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega}) + \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega})dt + \boxed{\int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})dt}$$

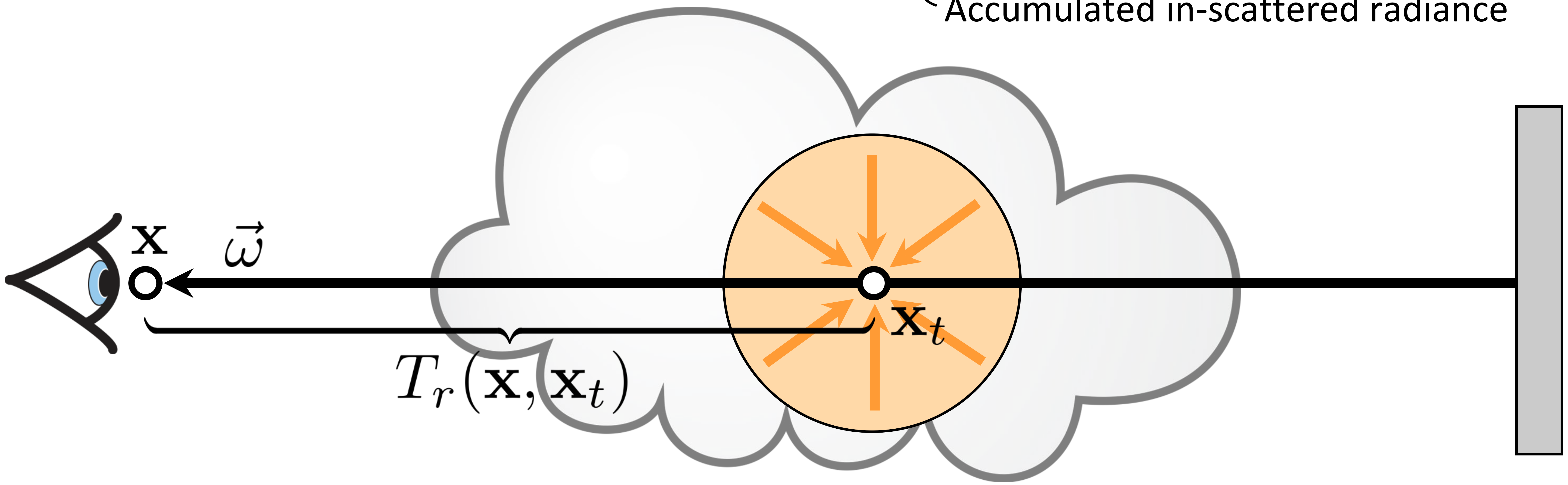
Accumulated in-scattered radiance



Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$
$$+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt$$
$$+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}' dt$$

Accumulated in-scattered radiance



Volume Rendering Equation

$$
\begin{aligned}
L(\mathbf{x}, \vec{\omega}) &= T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) \\
&+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt \\
&+ \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}' dt
\end{aligned}
$$

Scattering in Media

Phase Function f_p

Describes distribution of scattered light

Analog of BRDF but for scattering in media

Integrates to unity (unlike BRDF)

$$\int_{S^2} f_p(\mathbf{x}, \vec{\omega}', \vec{\omega}) d\vec{\omega}' = 1$$

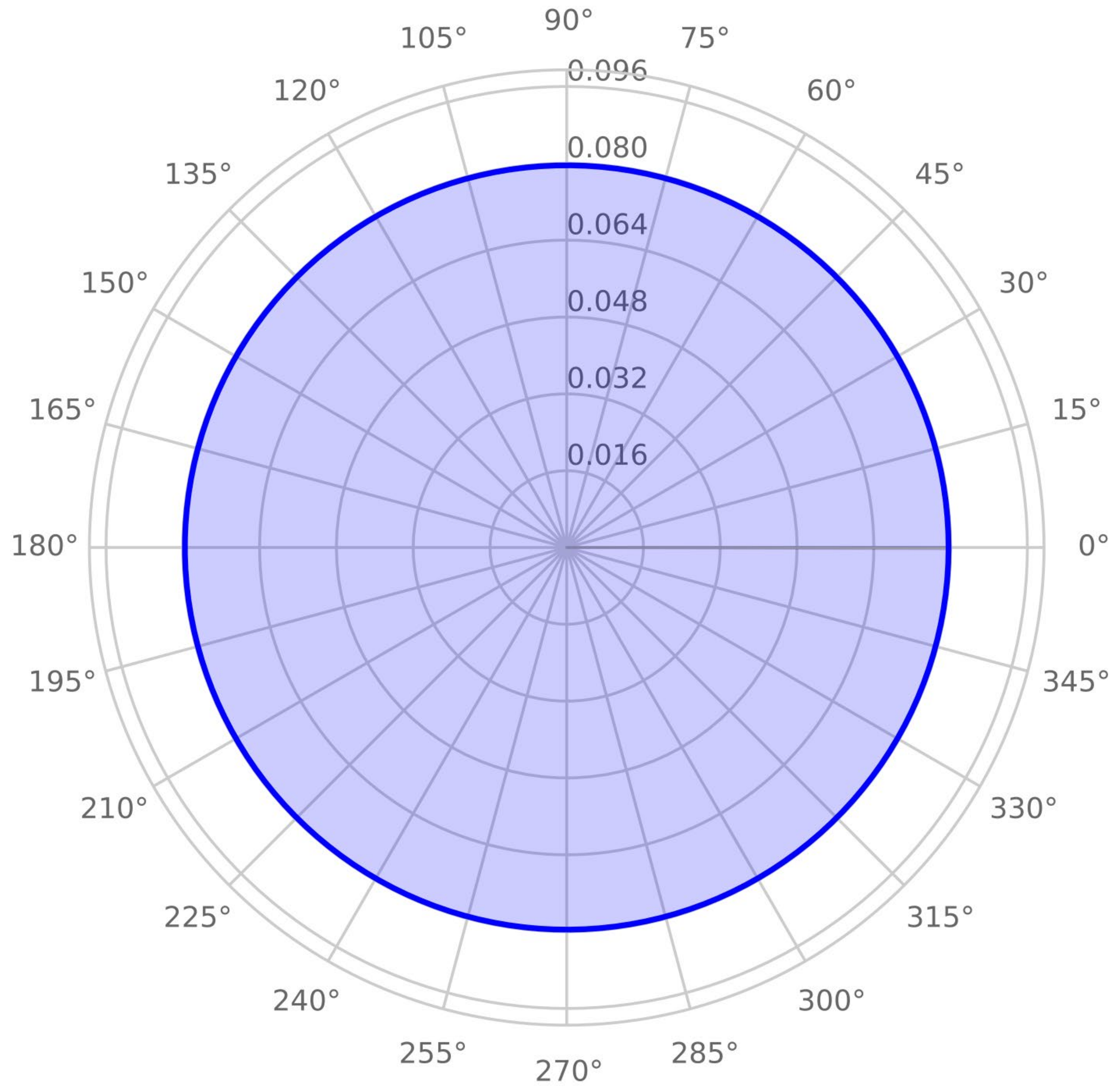
Why do we have this property?

*We will use the same convention that phase function direction vectors always *point away* from the shading point **x**. Many publications, however, use a different convention for phase functions, in which direction vectors “follow” the light, i.e. one direction points *towards* **x** and the other *away* from **x**. When reading papers, be sure to clarify the meaning of the vectors to avoid misinterpretation.

Isotropic Scattering

Uniform scattering, analogous to Lambertian BRDF

$$f_p(\vec{\omega}', \vec{\omega}) = \frac{1}{4\pi}$$



Where does this value come from?

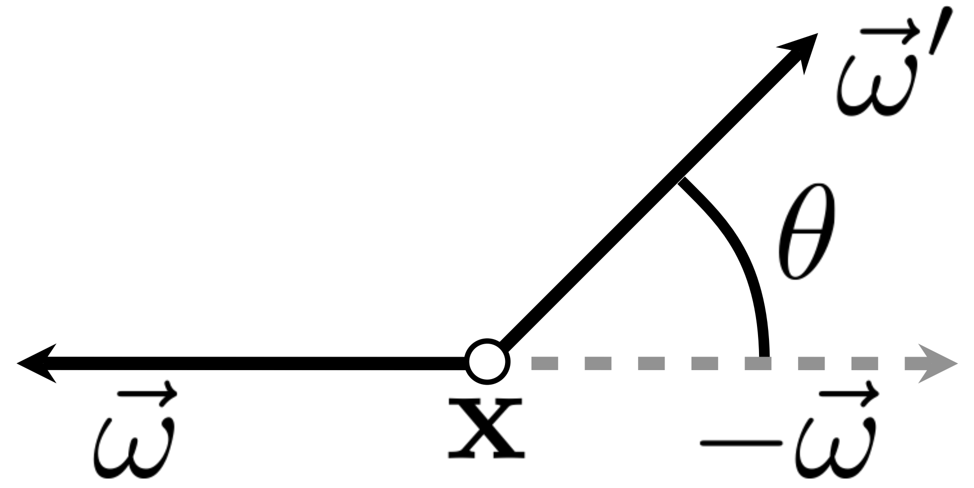
Anisotropic Scattering

Quantifying anisotropy (g, "average cosine"):

$$g = \int_{S^2} f_p(\mathbf{x}, \vec{\omega}', \vec{\omega}) \cos\theta \, d\vec{\omega}'$$

where:

$$\cos\theta = -\vec{\omega} \cdot \vec{\omega}'$$

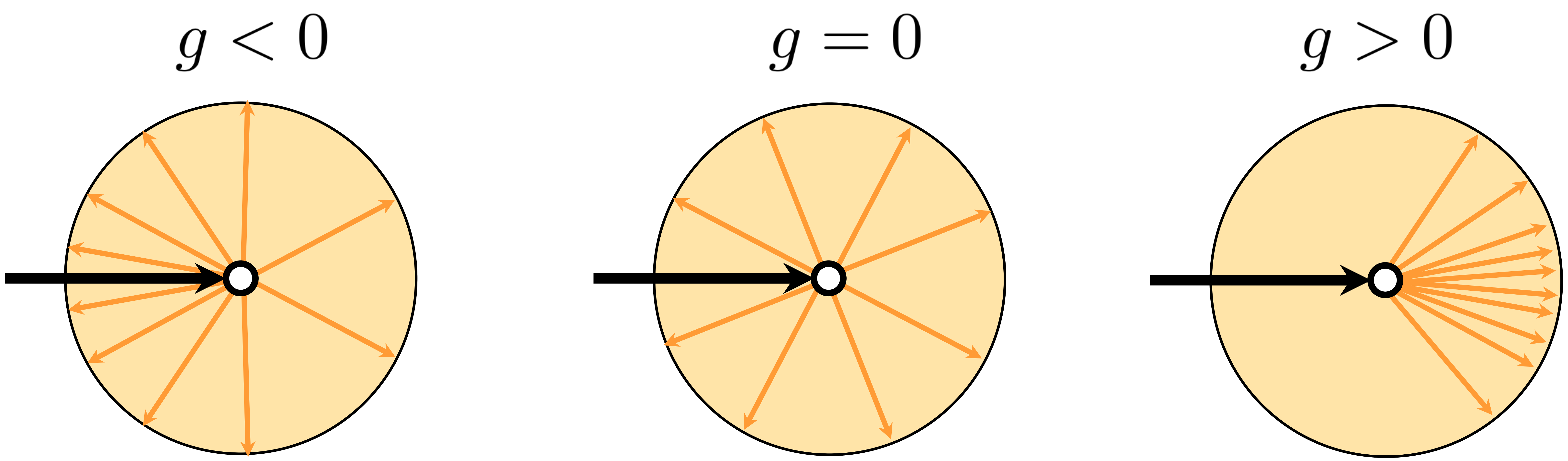


$g = 0$: isotropic scattering (on average)
$g > 0$: forward scattering
$g < 0$: backward scattering

Henyey-Greenstein Phase Function

Anisotropic scattering

$$f_{p\mathrm{HG}}(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g\cos\theta)^{3/2}}$$

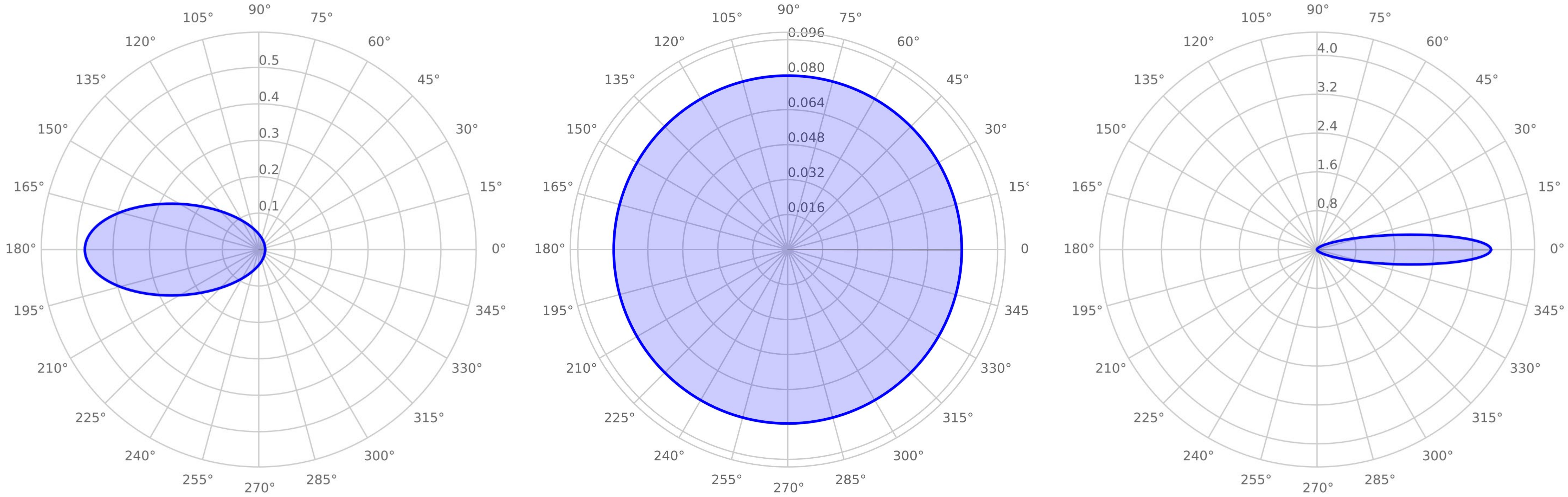


Henyey-Greenstein Phase Function

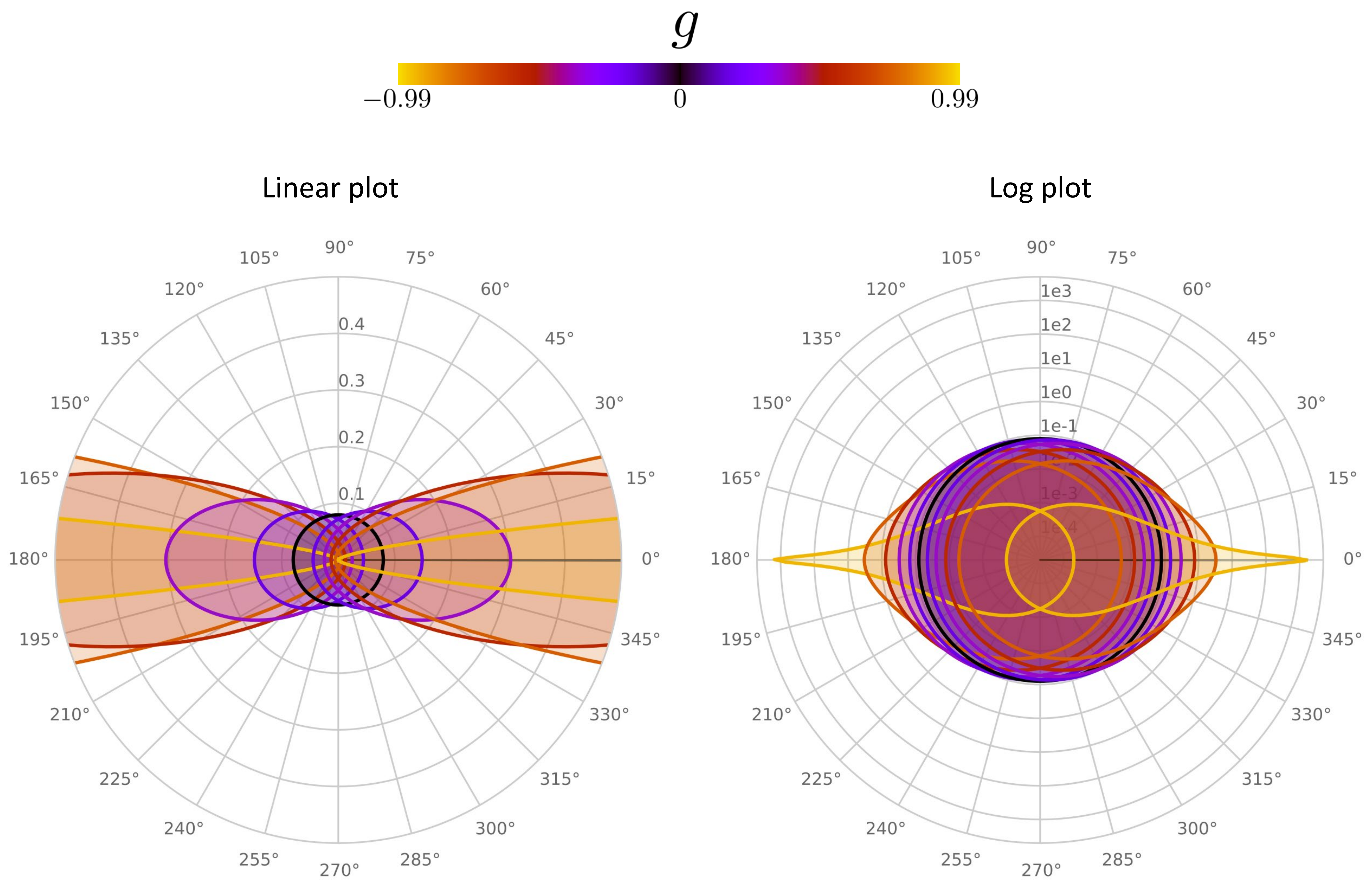
Anisotropic scattering

$$f_{p\mathrm{HG}}(\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g\cos\theta)^{3/2}}$$

$g = -0.5$ $\qquad$ $g = 0$ $\qquad$ $g = 0.8$



Henyey-Greenstein Phase Function



Henyey-Greenstein Phase Function

Empirical phase function

Introduced for intergalactic dust

Very popular in graphics and other fields

Schlick's Phase Function

Empirical phase function

Faster approximation of HG

$$f_{p\mathrm{Schlick}}(\theta) = \frac{1}{4\pi} \frac{1 - k^2}{(1 - k\cos\theta)^2}$$

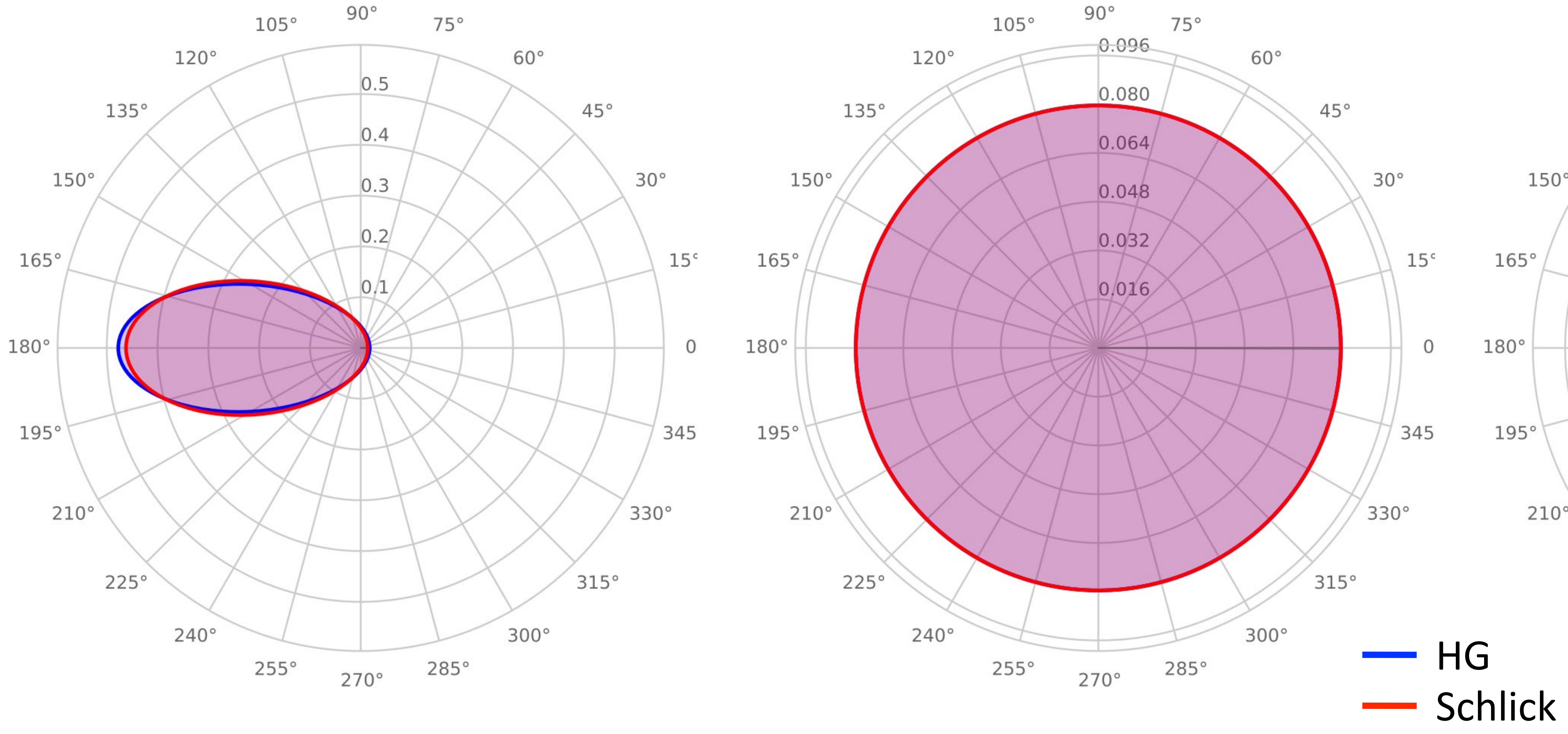
$$k = 1.55g - 0.55g^3$$

Schlick's Phase Function

Empirical phase function

Faster approximation of HG

$g = -0.5 \quad k = -0.706$ $\qquad$ $g = 0 \quad k = 0$ $\qquad$ $g = 0.8 \quad k = 0.96$



Lorenz-Mie Scattering

If the diameter of scatterers is on the order of light wavelength, we cannot neglect the wave nature of light

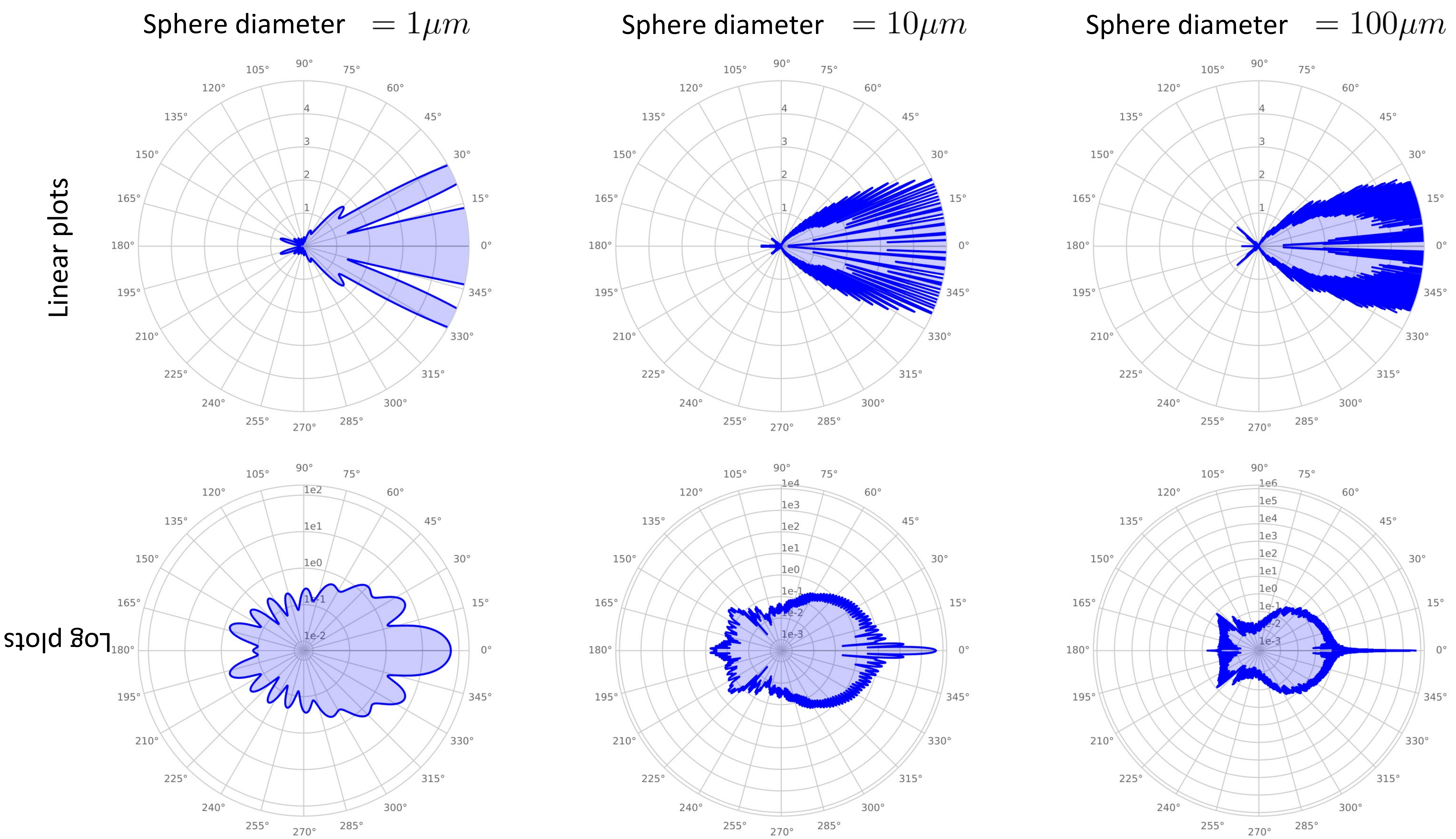
Solution to Maxwell's equations for scattering from any spherical dielectric particle

Explains many phenomena

Complicated:

- Solution is an infinite analytic series

Lorenz-Mie Phase Function



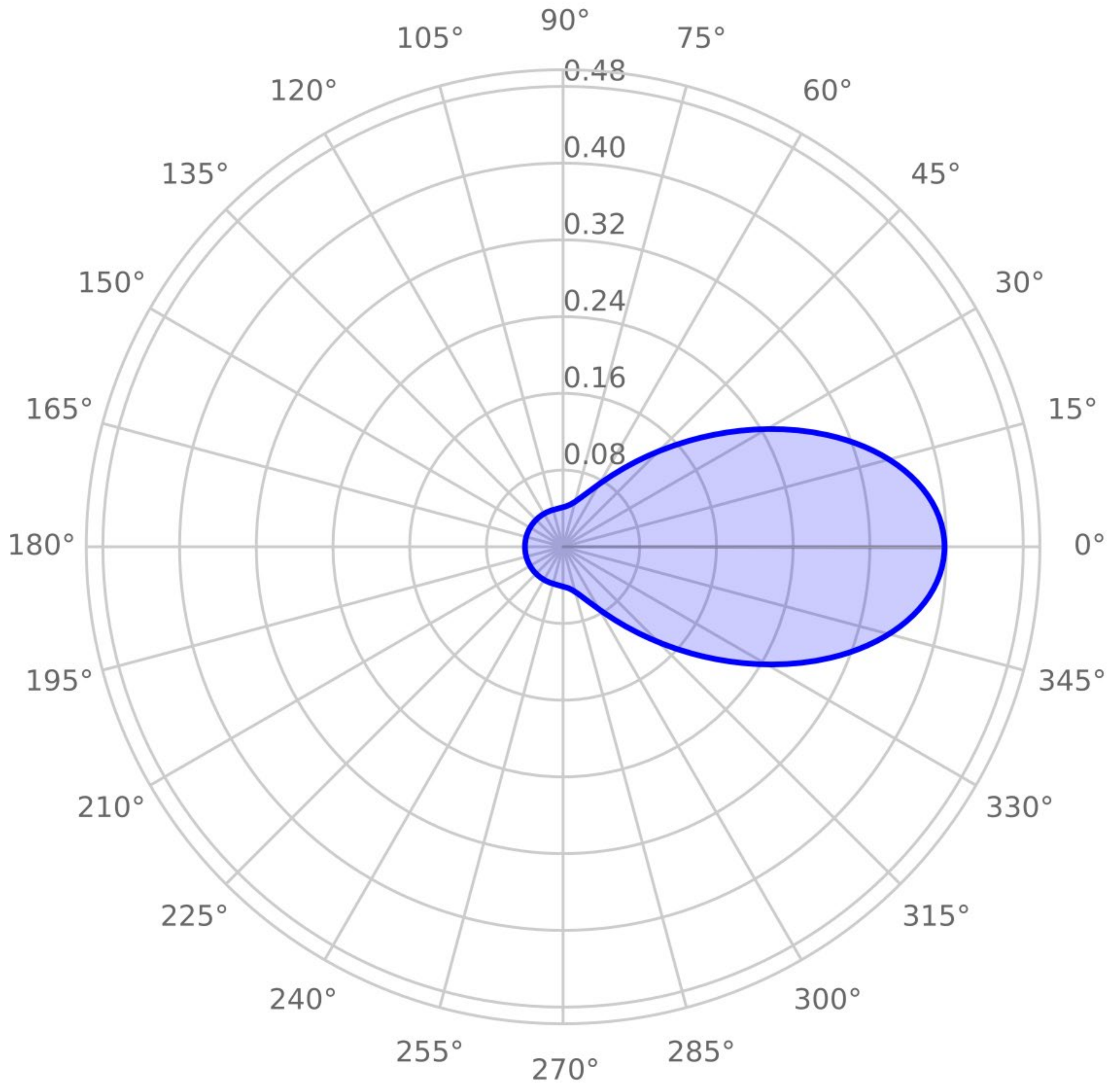
Data obtained from http://www.philiplaven.com/mieplot.htm

Rainbows

Lorenz-Mie Approximations

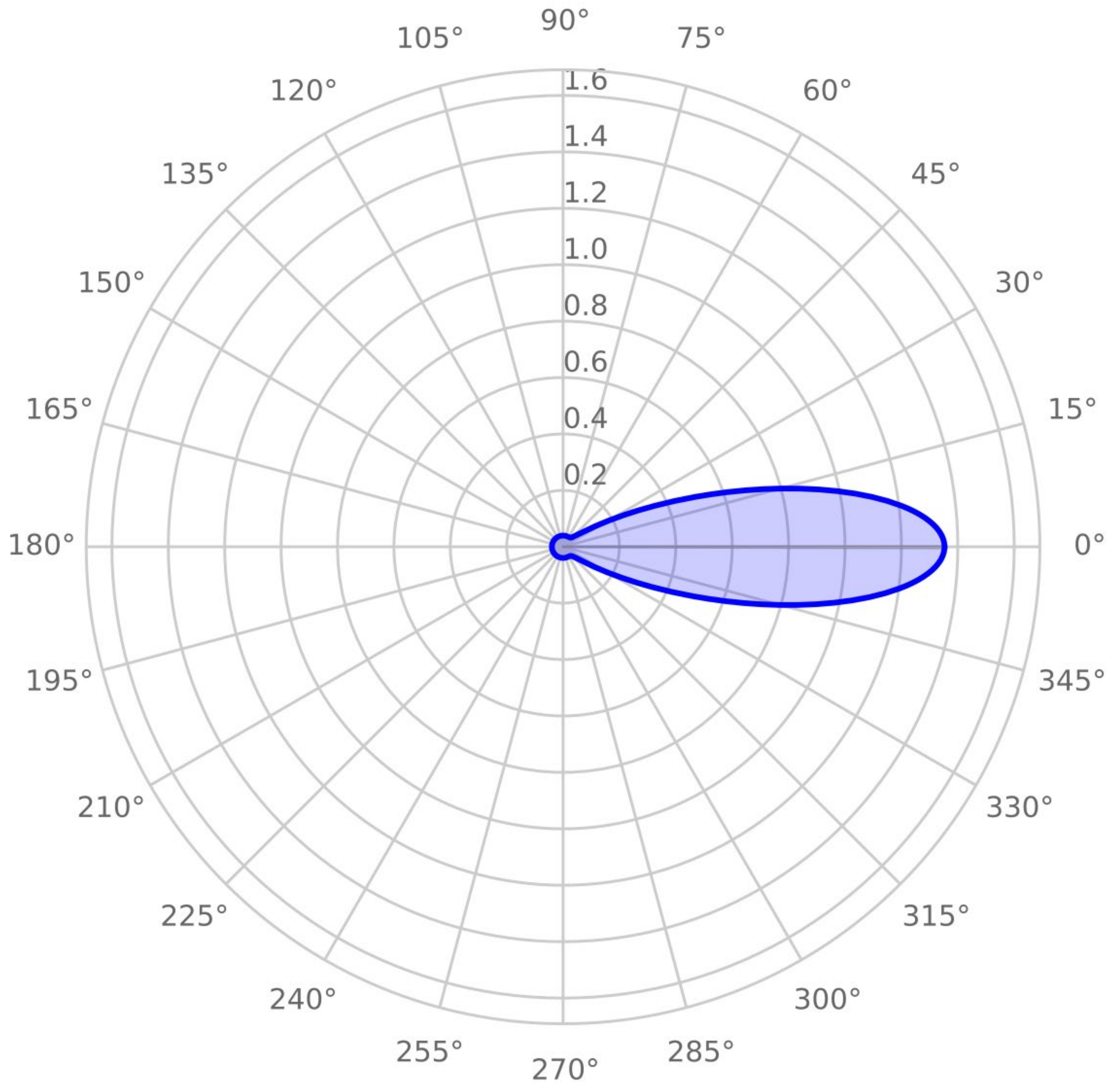
Hazy atmosphere

$$f_{p\,\text{hazy}}(\theta) = \frac{1}{4\pi}\left(5 + \left(\frac{1+\cos\theta}{2}\right)^{8}\right)$$



Murky atmosphere

$$f_{p\,\text{murky}}(\theta) = \frac{1}{4\pi}\left(17 + \left(\frac{1+\cos\theta}{2}\right)^{32}\right)$$



Lorenz-Mie Approximations

Hazy atmosphere

$$f_{p\,\mathrm{hazy}}(\theta) = \frac{1}{4\pi}\left(5 + \left(\frac{1+\cos\theta}{2}\right)^{8}\right)$$

Murky atmosphere

$$f_{p\,\mathrm{murky}}(\theta) = \frac{1}{4\pi}\left(17 + \left(\frac{1+\cos\theta}{2}\right)^{32}\right)$$

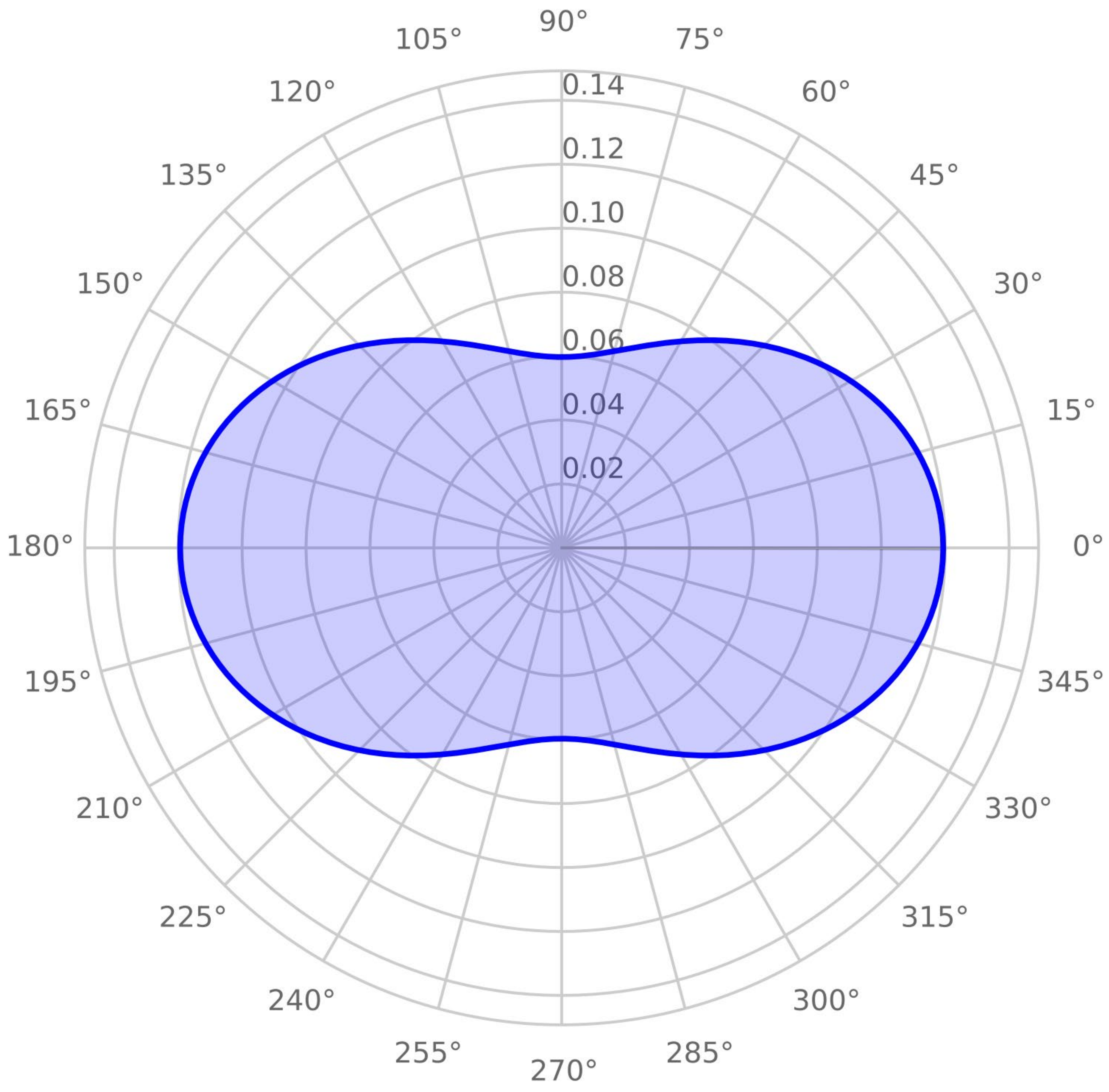
Rayleigh Scattering

Approximation of Lorenz-Mie for tiny scatterers that are typically smaller than 1/10th the wavelength of visible light

Used for atmospheric scattering, gasses, transparent solids

Highly wavelength dependent

Rayleigh Phase Function



$$f_{p\text{Rayleigh}}(\theta) = \frac{3}{16\pi}(1 + \cos^2\theta)$$

Scattering at right angles is half as likely as scattering forward or backward

Rayleigh Scattering

Wavelength

Index of refraction

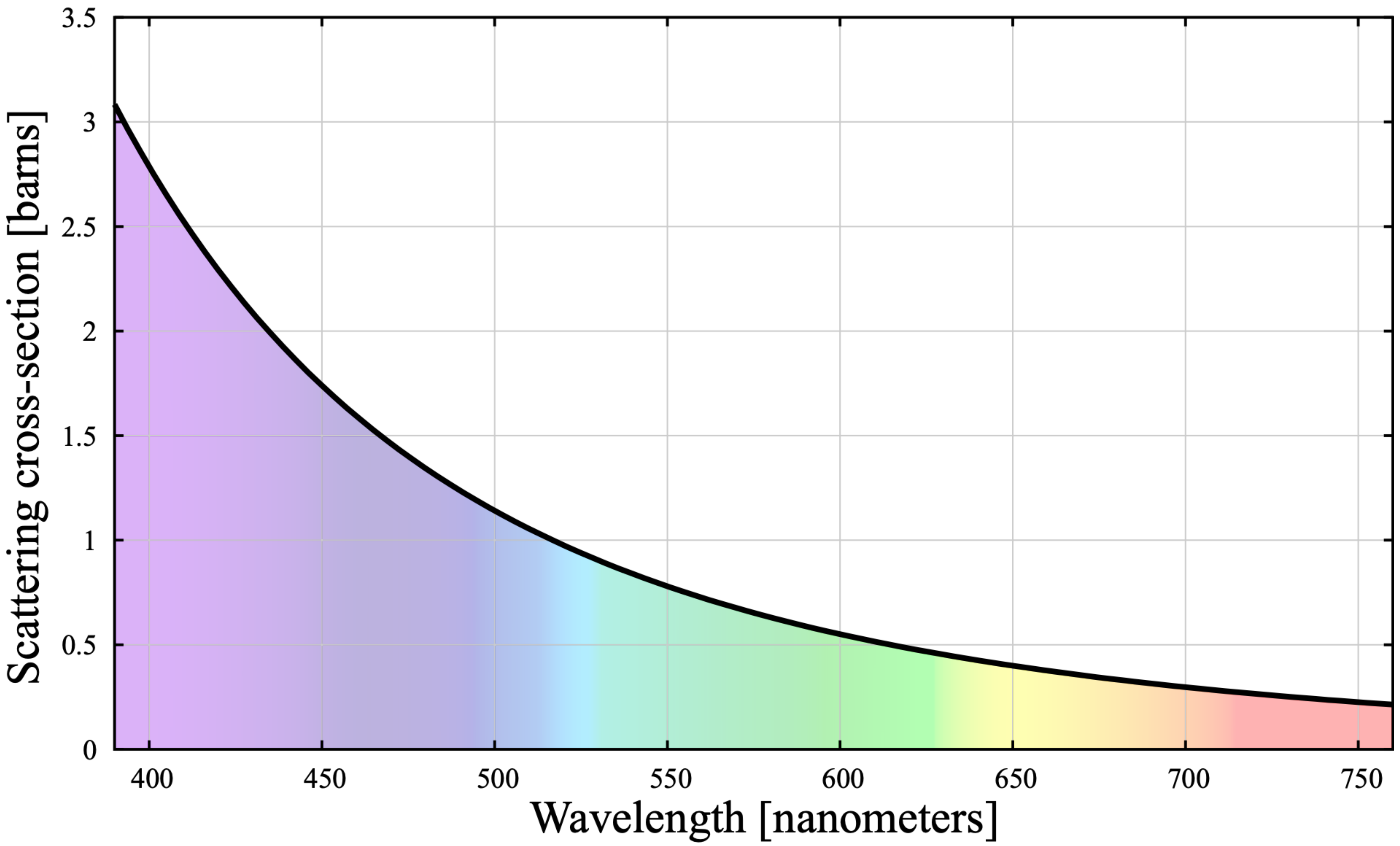
$$\sigma_{s\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \, \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$

Density of scatterers

Diameter of scatterers

Rayleigh Scattering

$$\sigma_{s\text{Rayleigh}}(\lambda, d, \eta, \rho) = \rho \, \frac{2\pi^5 d^6}{3\lambda^4} \left(\frac{\eta^2 - 1}{\eta^2 + 2} \right)^2$$



Examples

Examples

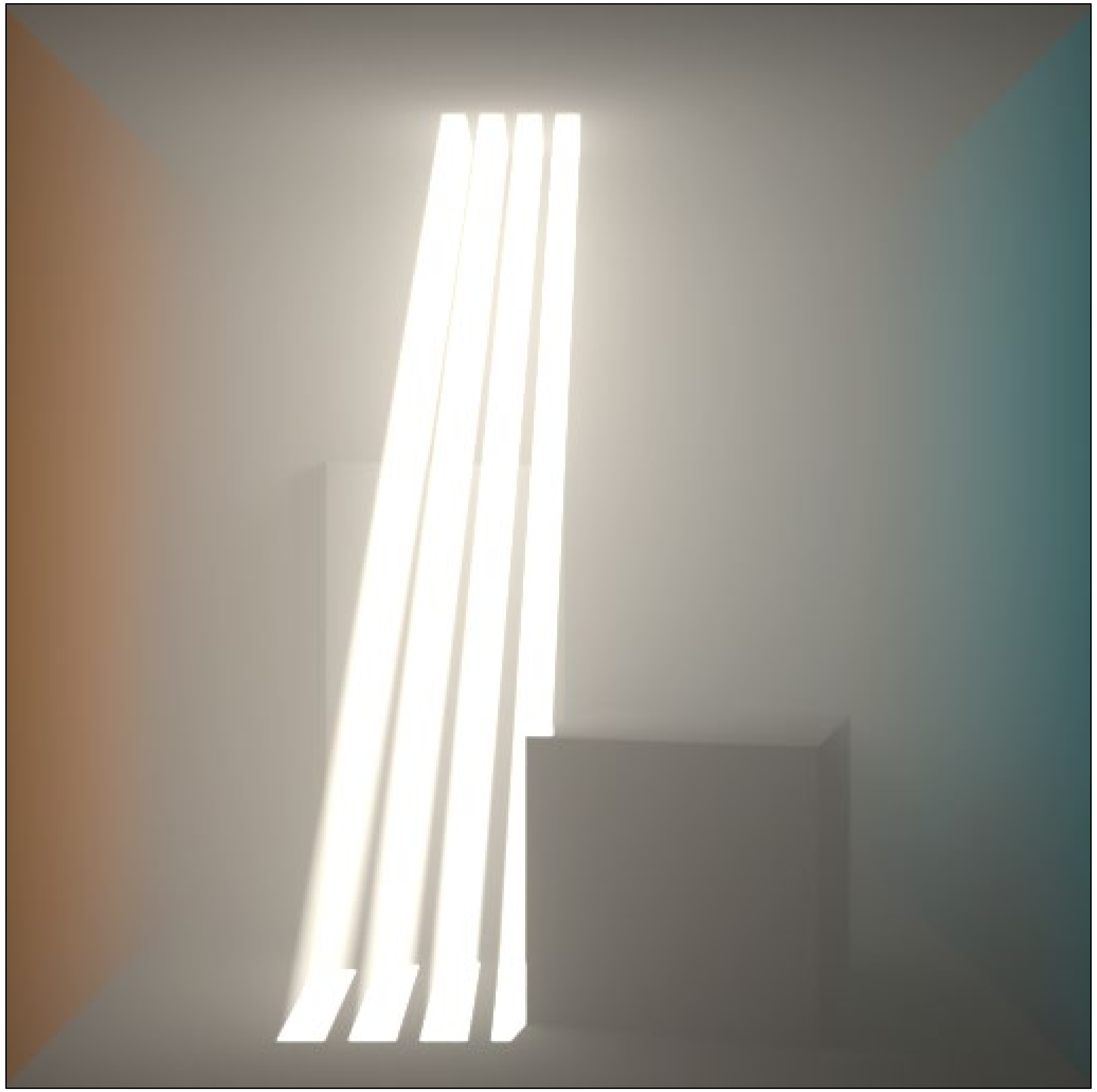
Steam

Forward scattering

Smoke

Backward scattering

Examples

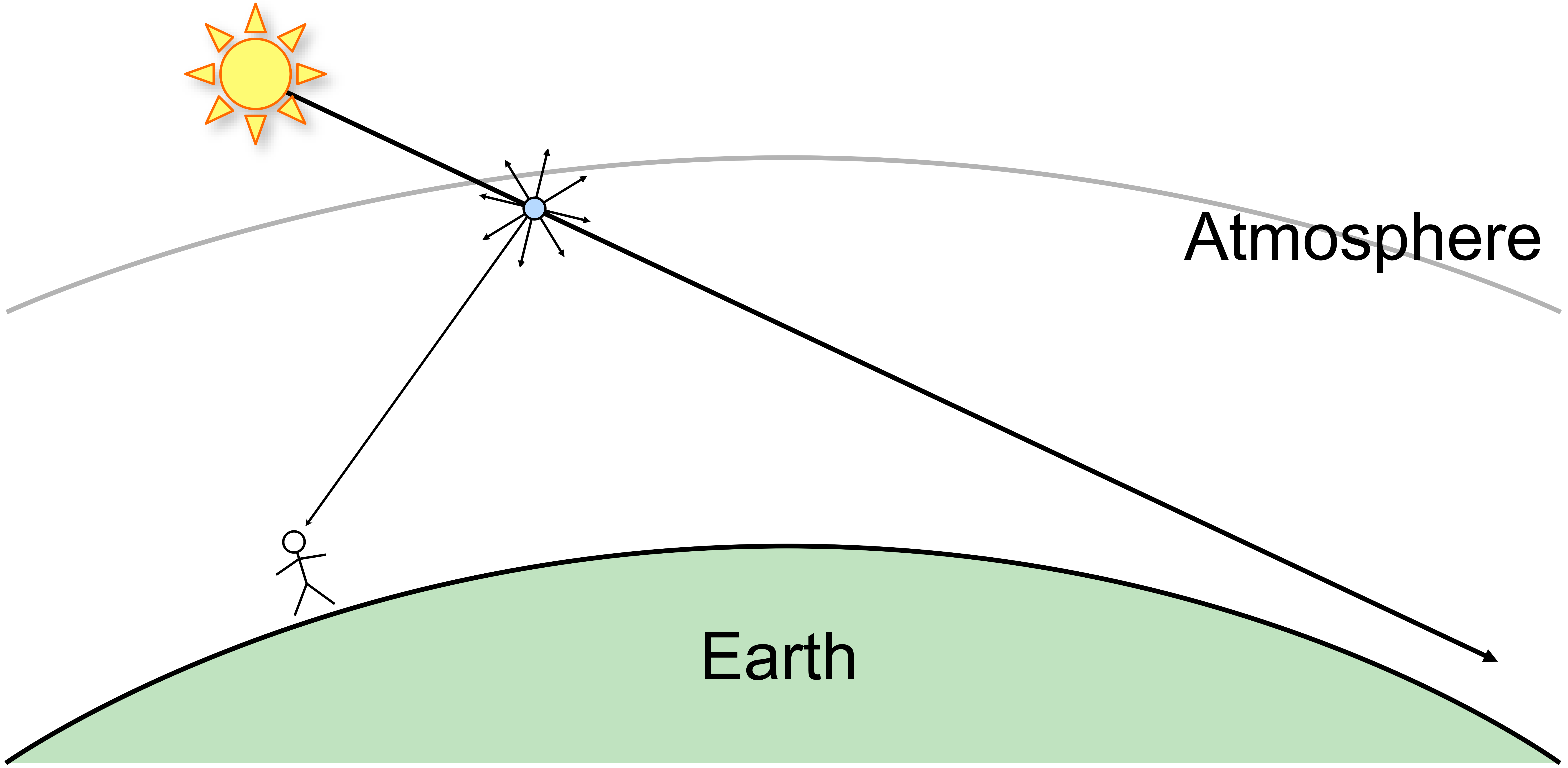


Isotropic scattering

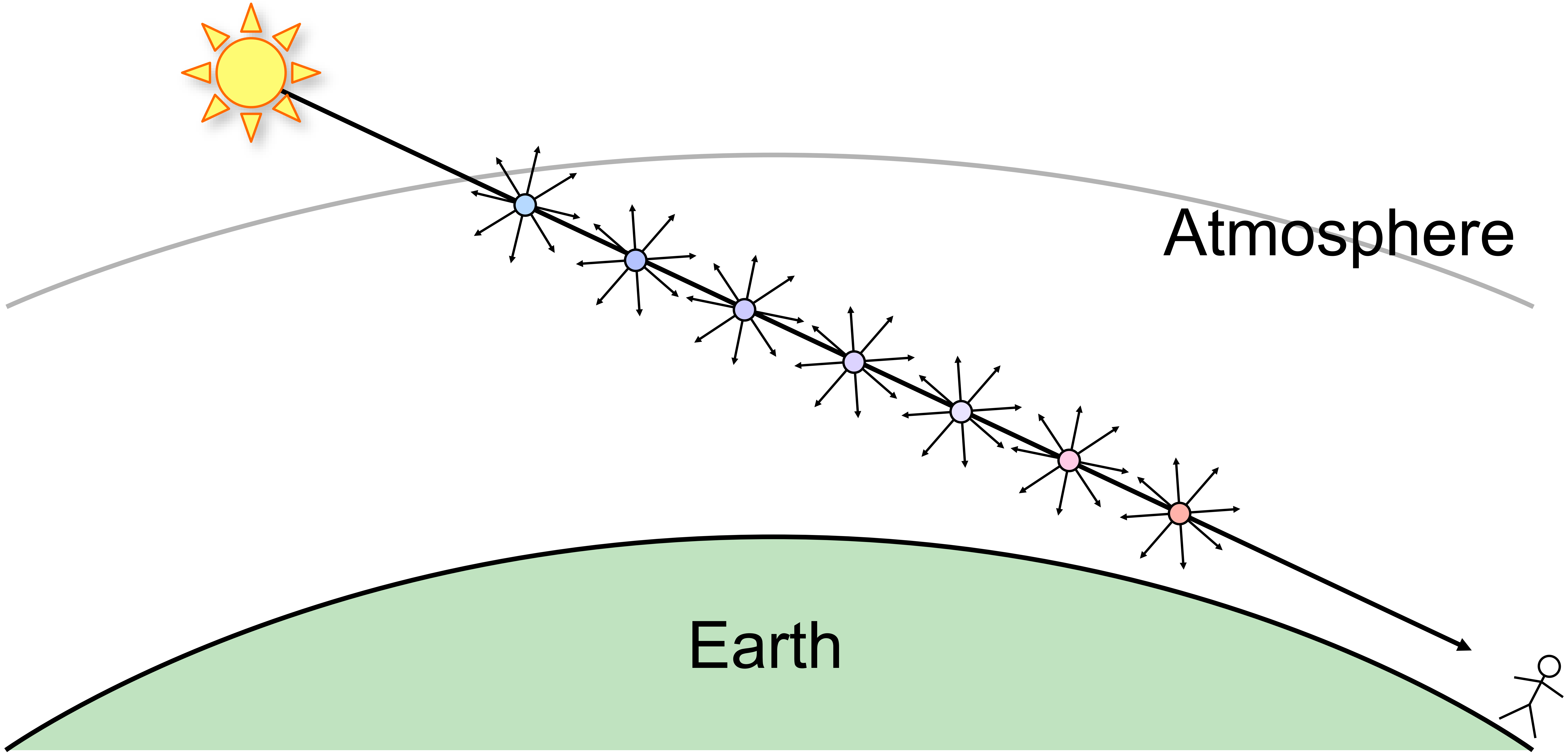
Examples

Forward scattering

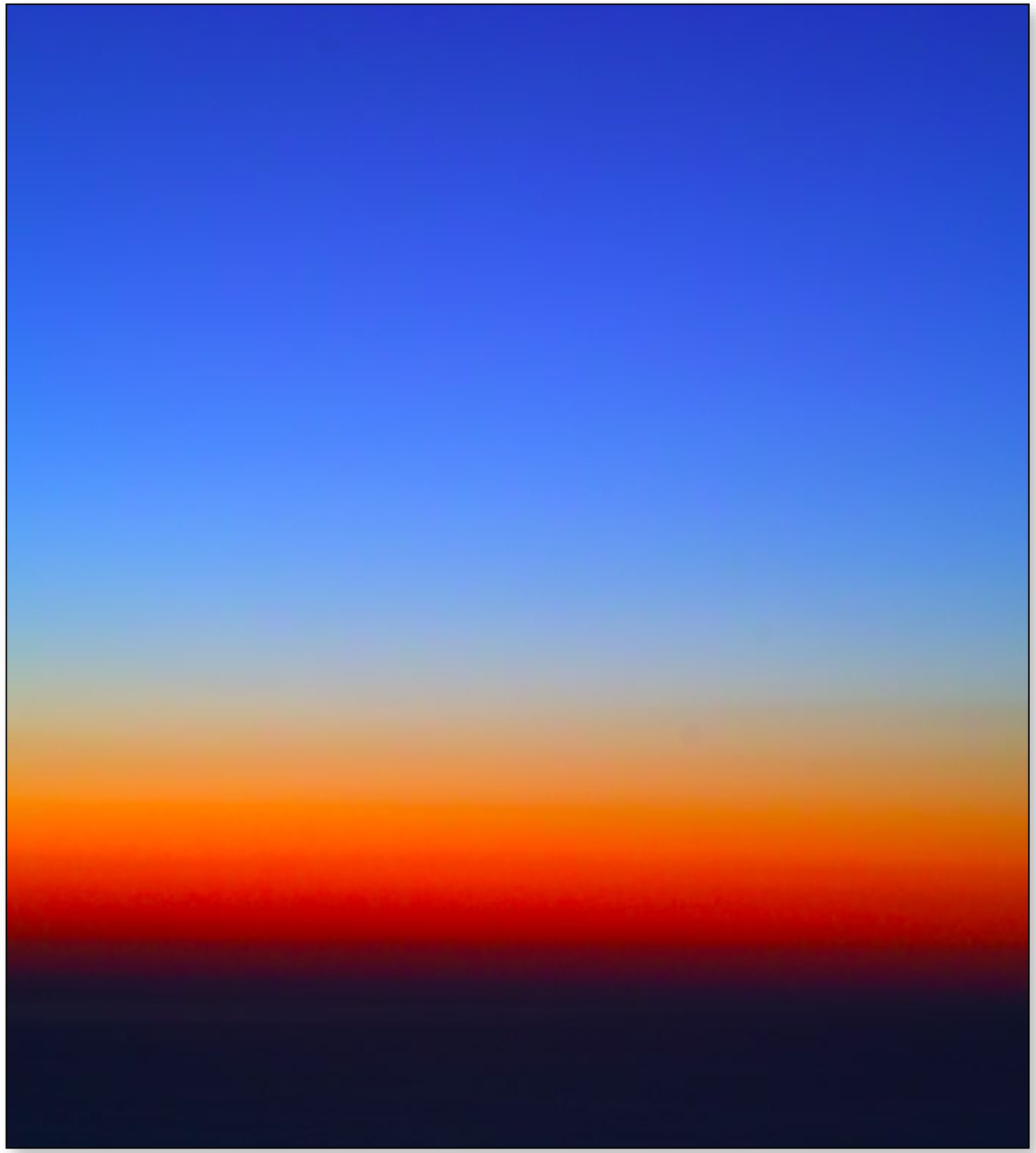
Why is the Sky Blue?

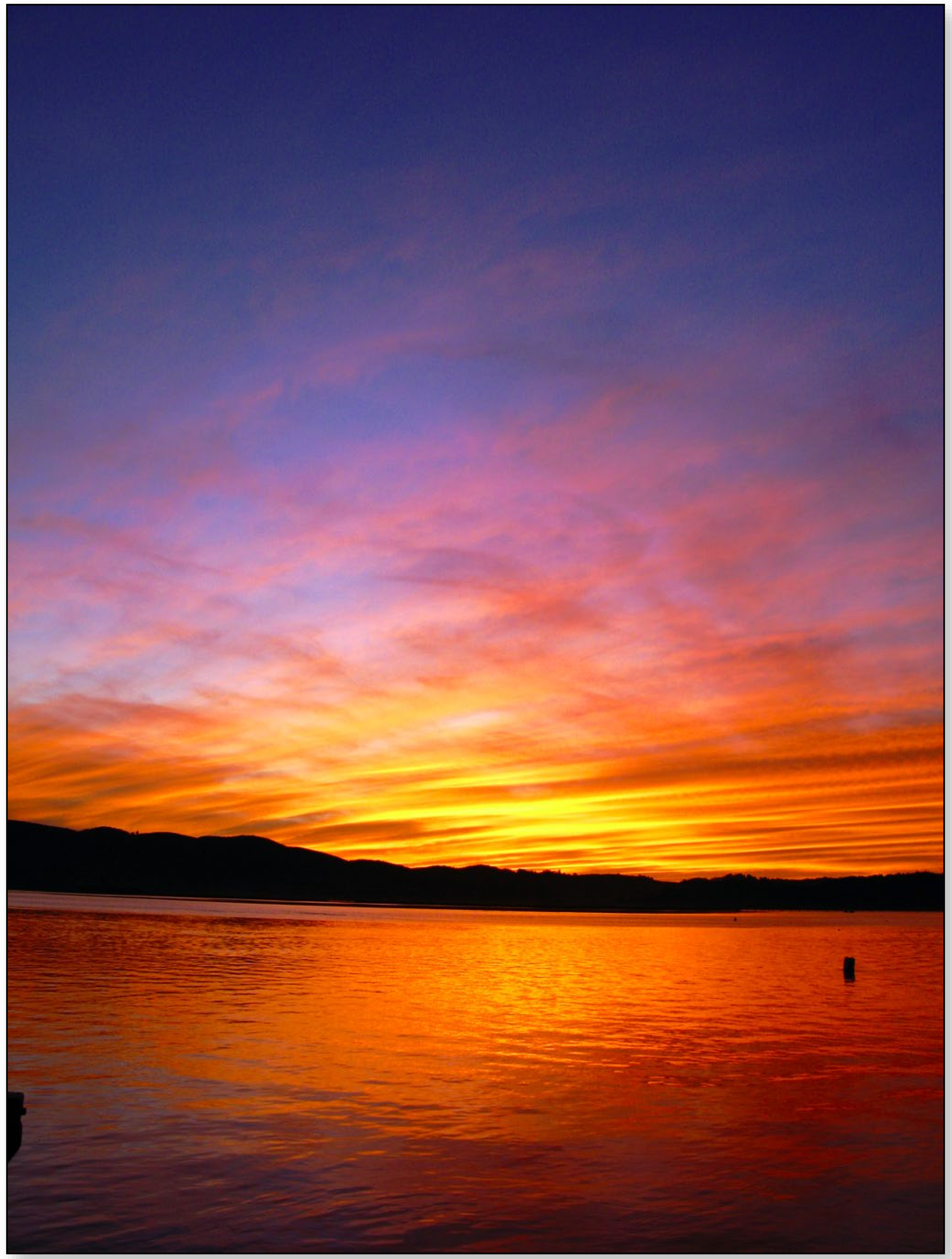


Why is the Sunset Red?



Rayleigh Scattering





Media Properties (Recap)

Given:

- Absorption coefficient $\sigma_a(\mathbf{x})$ $[\mathrm{m}^{-1}]$
- Scattering coefficient $\sigma_s(\mathbf{x})$ $[\mathrm{m}^{-1}]$
- Phase function $f_p(\mathbf{x}, \vec{\omega}', \vec{\omega})$ $[\mathrm{sr}^{-1}]$

Derived:

- Extinction coefficient $\sigma_t(\mathbf{x}) = \sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x})$ $[\mathrm{m}^{-1}]$
- Albedo $\alpha(\mathbf{x}) = \sigma_s(\mathbf{x}) / \sigma_t(\mathbf{x})$ $[\mathrm{none}]$
- Mean-free path $1/\sigma_t(\mathbf{x})$ $[\mathrm{m}]$
- Transmittance $T_r(\mathbf{x}, \mathbf{y}) = e^{-\int_0^{\|\mathbf{x}-\mathbf{y}\|} \sigma_t(t) dt}$ $[\mathrm{none}]$

Homogeneous Isotropic Medium

Given:

- Absorption coefficient σ_a $[\mathrm{m}^{-1}]$
- Scattering coefficient σ_s $[\mathrm{m}^{-1}]$
- Phase function $\frac{1}{4\pi}$ $[\mathrm{sr}^{-1}]$

Derived:

- Extinction coefficient $\sigma_t = \sigma_a + \sigma_s$ $[\mathrm{m}^{-1}]$
- Albedo $\alpha = \sigma_s / \sigma_t$ $[\mathrm{none}]$
- Mean-free path $1/\sigma_t$ $[\mathrm{m}]$
- Transmittance $T_r(\mathbf{x}, \mathbf{y}) = e^{-\sigma_t \|\mathbf{x} - \mathbf{y}\|}$ $[\mathrm{none}]$

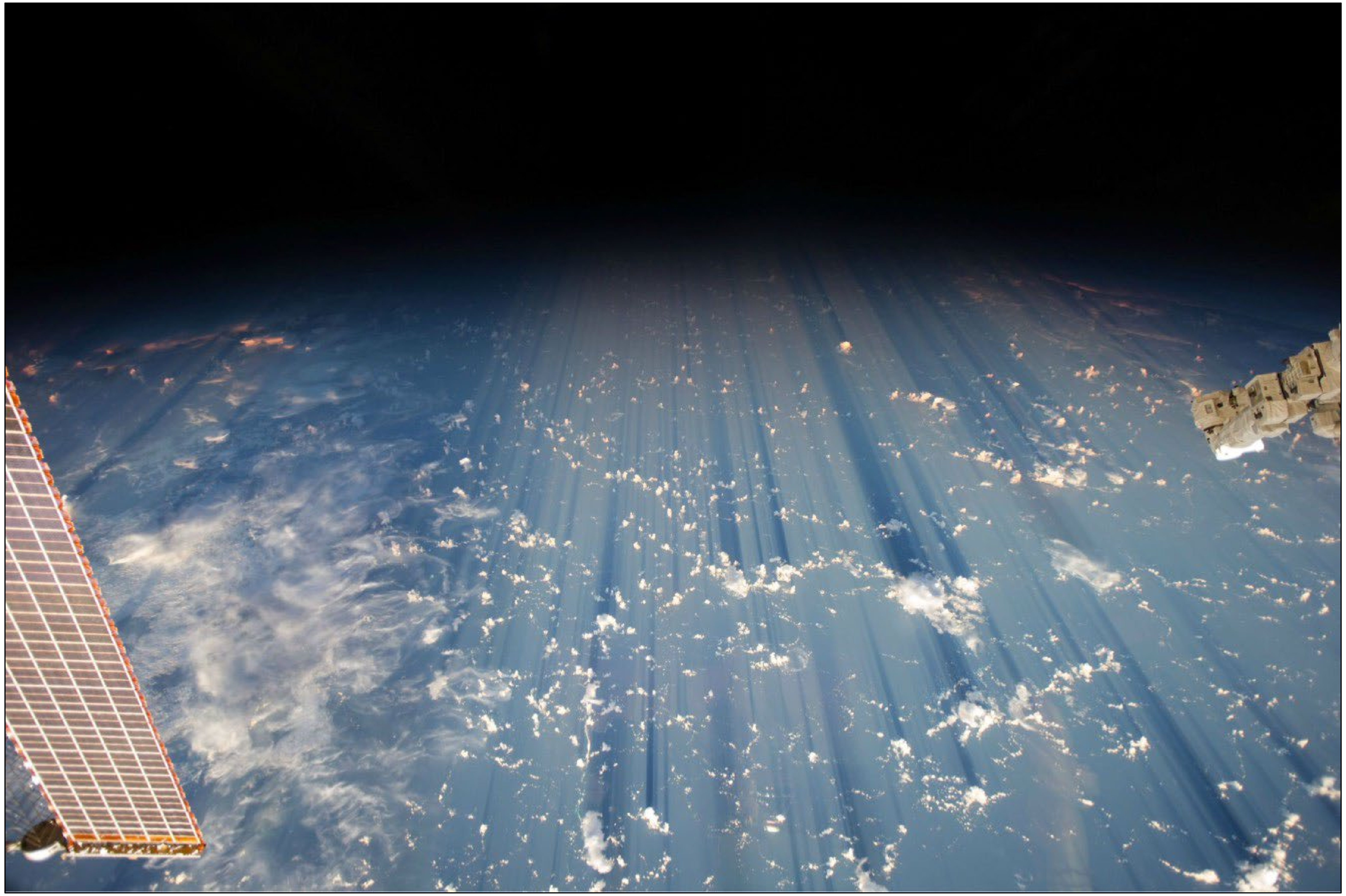
What is this?

Crepuscular Rays

Anti-Crepuscular Rays

source: wikipedia

Crepuscular rays from space



Solving the
Volume Rendering Equation

Complexity Progression

homogeneous vs. heterogeneous

scattering

- none
- fake ambient
- single
- multiple

Volume Rendering Equation

Attenuated background radiance

Accumulated emitted radiance

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt$$

Accumulated in-scattered radiance

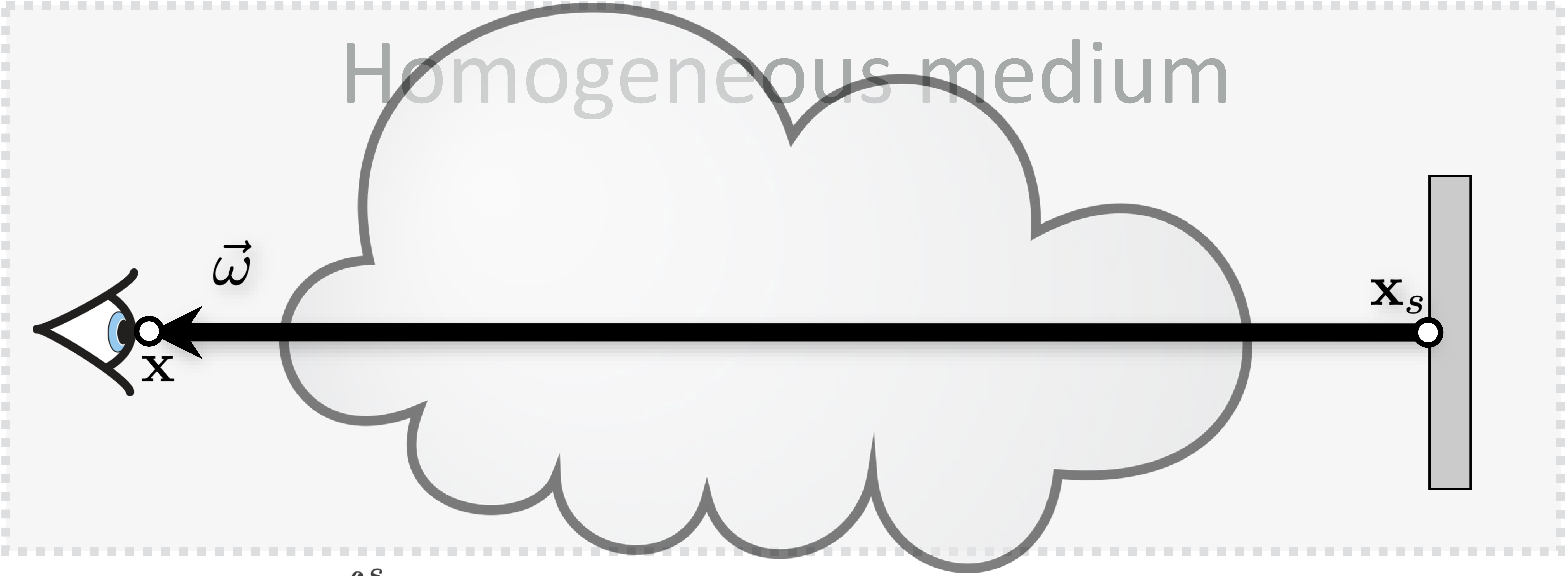
Purely absorbing media

Attenuated background radiance

$$L(\mathbf{x}, \vec{\omega}) = \boxed{T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})}$$

Fog

Participating Media



$$L(\mathbf{x},\vec{\omega}) = \int_0^s T_r(\mathbf{x}\leftrightarrow\mathbf{x}_t)\,\sigma_s(\mathbf{x}_t)\,L_i(\mathbf{x}_t,\vec{\omega})\,dt + T_r(\mathbf{x}\leftrightarrow\mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s T_r(\mathbf{x}\leftrightarrow\mathbf{x}_t)\,L_i(\mathbf{x}_t,\vec{\omega})\,dt + T_r(\mathbf{x}\leftrightarrow\mathbf{x}_s)L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t}\,L_i(\mathbf{x}_t,\vec{\omega})\,dt + e^{-s\sigma_t}L(\mathbf{x}_s,\vec{\omega})$$

Fog

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t}\, L_i(\mathbf{x}_t,\vec{\omega})\, dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant:

$$L(\mathbf{x}, \vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t} \boxed{L_i(\mathbf{x}_t, \vec{\omega})} dt + e^{-s\sigma_t} L(\mathbf{x}_s, \vec{\omega})$$

Homogeneous Ambient Media

Assume in-scattered radiance is an ambient constant:

$$L(\mathbf{x},\vec{\omega}) = \sigma_s \int_0^s e^{-t\sigma_t}\, L_i(\mathbf{x}_t,\vec{\omega})\,dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

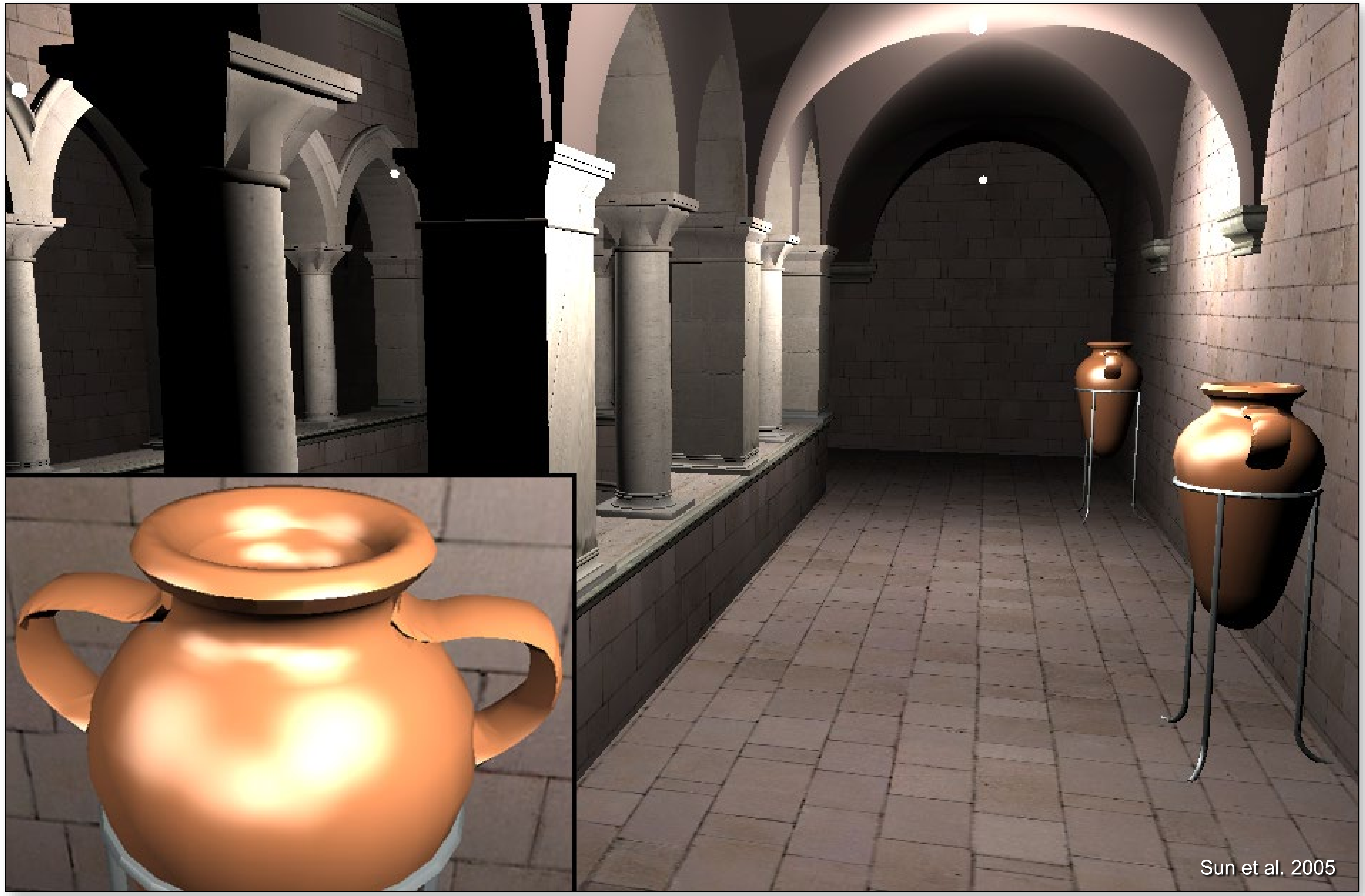
$$L(\mathbf{x},\vec{\omega}) = \sigma_s\, L_i \int_0^s e^{-t\sigma_t}\,dt + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \sigma_s\, L_i \frac{1-e^{-s\sigma_t}}{\sigma_t} + e^{-s\sigma_t} L(\mathbf{x}_s,\vec{\omega})$$

$$L(\mathbf{x},\vec{\omega}) = \mathrm{lerp}\left(\frac{\sigma_s}{\sigma_t} L_i,\ L(\mathbf{x}_s,\vec{\omega}),\ e^{-s\sigma_t}\right)$$

OpenGL Fog

OpenGL Clear Day



Fog


http://anordinarymom.wordpress.com

Andreas Levers

Andreas Levers

Volume Rendering Equation

$$L(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega}) + \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) dt + \boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt}$$

Accumulated in-scattered radiance

In-scattered Radiance

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) dt$$

$$L_s(\mathbf{x}_t, \vec{\omega}) = \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}'$$

Single scattering

- L_i arrives directly from a light source (direct illum.)
  i.e.:

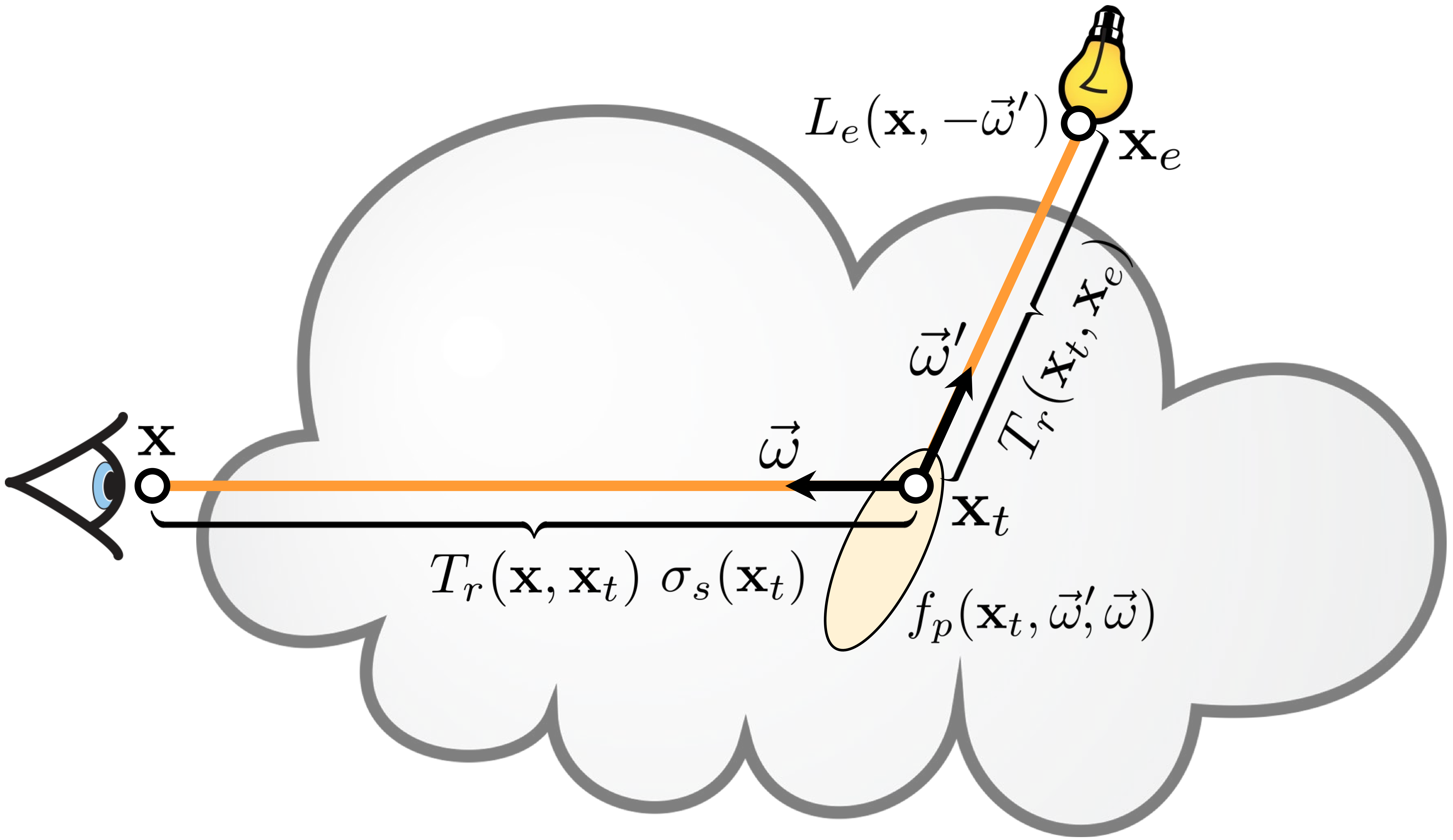
$$L_i(\mathbf{x}, \vec{\omega}) = T_r(\mathbf{x}, r(\mathbf{x}, \vec{\omega}))\, L_e(r(\mathbf{x}, \vec{\omega}), -\vec{\omega})$$

Multiple scattering

- L_i arrives through multiple bounces (indirect illum.)

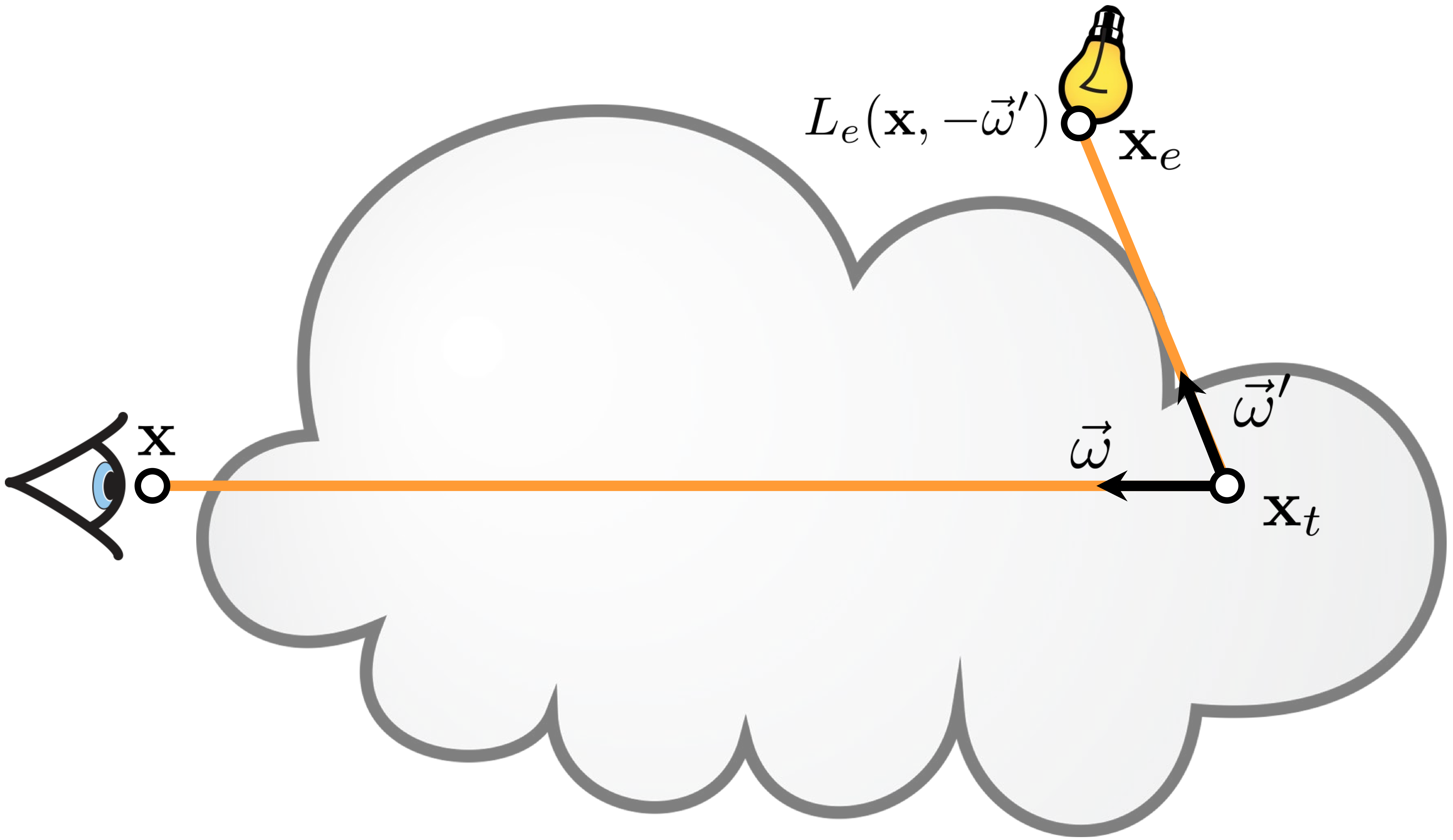
Single Scattering

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) T_r(\mathbf{x}_t, \mathbf{x}_e) L_e(\mathbf{x}_e, -\vec{\omega}') d\vec{\omega}' dt$$



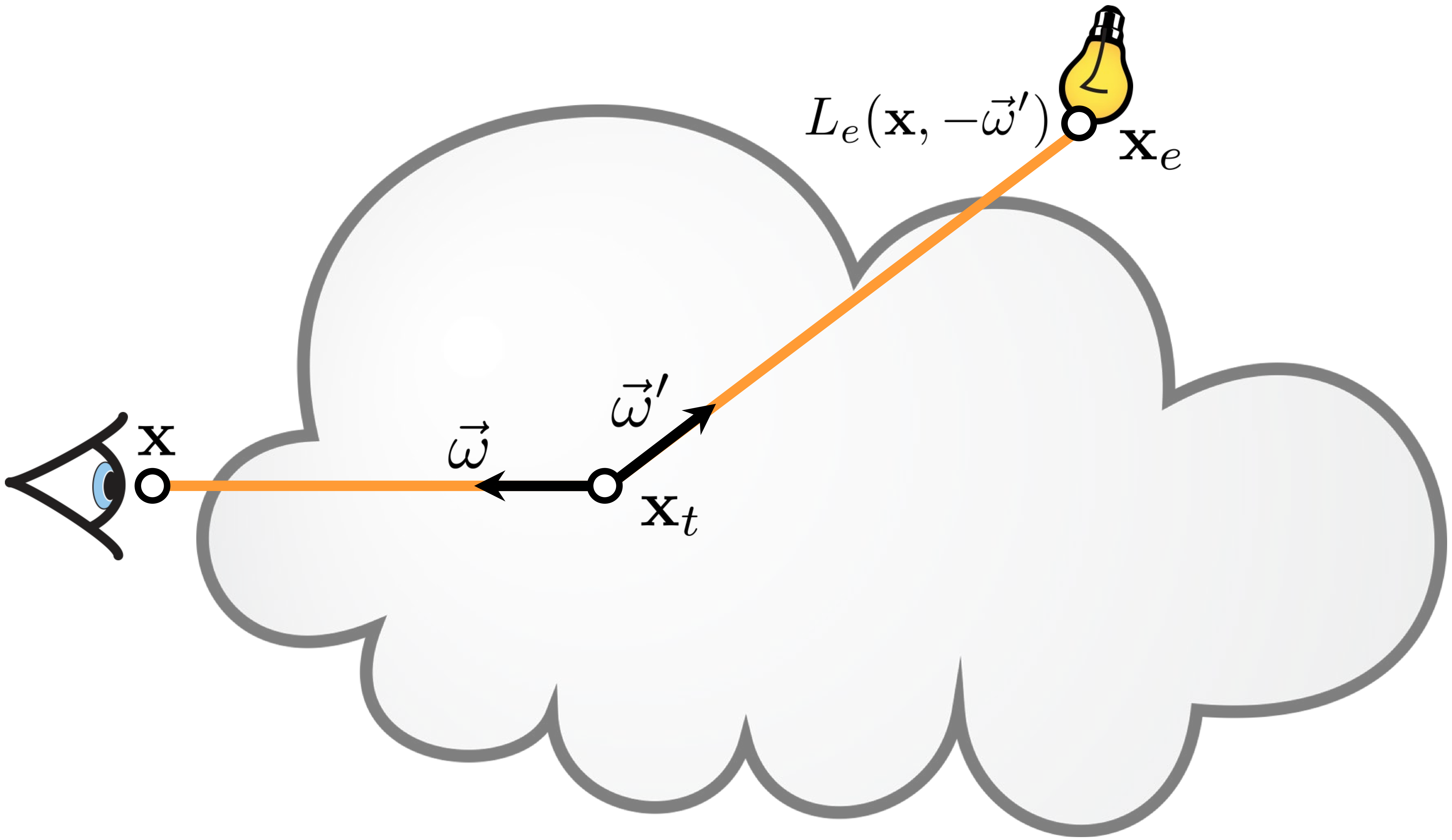
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$$



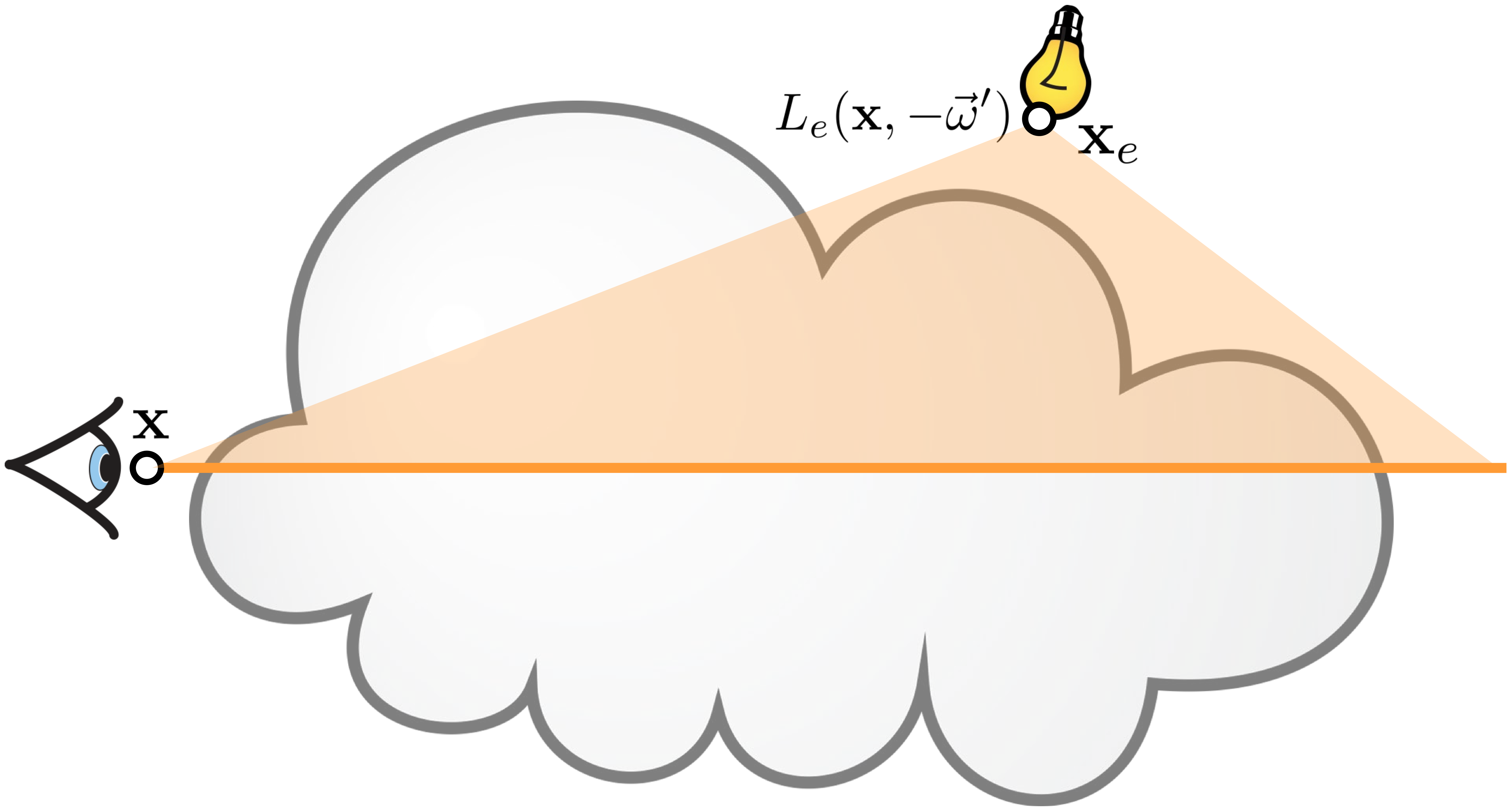
Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$$



Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$$



Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$$

(Semi-)analytic solutions:

- Sun et al. [2005]
- Pegoraro et al. [2009, 2010]

Numerical solutions:

- Ray-marching
- Equiangular sampling

Analytic Single Scattering

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\sigma_s(\mathbf{x}_t)\int_{S^2} f_p(\mathbf{x}_t,\vec{\omega}',\vec{\omega})T_r(\mathbf{x}_t,\mathbf{x}_e)L_e(\mathbf{x}_e,-\vec{\omega}')d\vec{\omega}'dt$$

Assumptions:

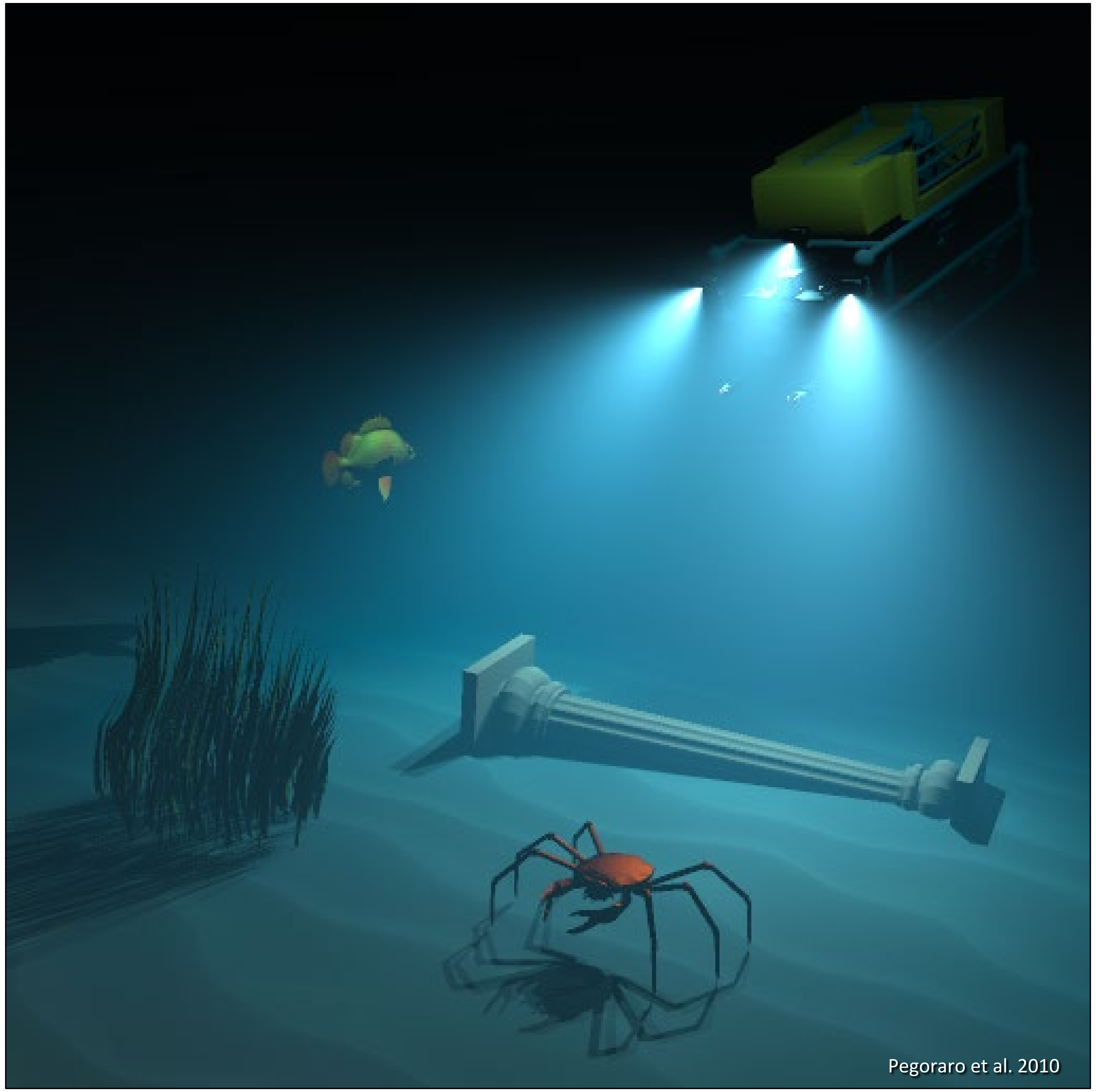
- Homogeneous medium
- Point or spot light
- Relatively simple phase function
- No occlusion

$$L(\mathbf{x},\vec{\omega}) = \frac{\Phi}{4\pi}\frac{1}{4\pi}\sigma_s\int_0^z e^{-\sigma_t\|\mathbf{x},\mathbf{x}_t\|}\frac{e^{-\sigma_t\|\mathbf{x}_t,\mathbf{x}_p\|}}{\|\mathbf{x}_t,\mathbf{x}_p\|^2}\mathrm{d}t$$

OpenGL Fog

Analytic Single Scattering

Analytic Single Scattering



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Analytic Single Scattering

$$L_m(x_a, x_b, \vec{\omega}) = \frac{\kappa_s}{h} e^{\kappa_t (x_a - x_h)} 2 \sum_{n=0}^{N-1} c(n) \sum_{k=0}^{2n} d(n,k) \int_{v_a}^{v_b} \frac{e^{-Hv}}{(v^2+1)^{n+1}} v^k \mathrm{d}v$$

$$\begin{aligned}
\int \frac{e^{av}}{(v^2+1)^m} v^n \mathrm{d}v = \frac{1}{2^{m-1}} \sum_{l=0}^{m-1} \frac{1}{2^l} \binom{m-1+l}{m-1} \Bigg(\sum_{k=0}^{\min\{m-1-l,n\}} \binom{n}{k} \Bigg(\frac{a^{m-1-l-k}}{(m-1-l-k)!} E(a, v, m-n-l+k) \\
- e^{av} \sum_{j=1}^{m-1-l-k} \frac{(j-1)!}{(m-1-l-k)!} \frac{a^{m-1-l-k-j}}{(v^2+1)^j} \sum_{\substack{i=(m-n-l+k-j) \bmod 2 \\ i+=2}}^{\leq j} (-1)^{\frac{m-n-l+k-j+i}{2}} \binom{j}{i} v^i \Bigg) \\
+ \frac{e^{av}}{a} \sum_{k=0}^{\leq n-m+l} \binom{n}{k} \sum_{j=0}^{n-m+l-k} \frac{(n-m+l-k)!}{j!} \frac{1}{(-a)^{n-m+l-k-j}} \sum_{\substack{i=(-m+l+k-j) \bmod 2 \\ i+=2}}^{\leq j} (-1)^{\frac{-m+l+k-j+i}{2}} \binom{j}{i} v^i \Bigg)
\end{aligned}$$

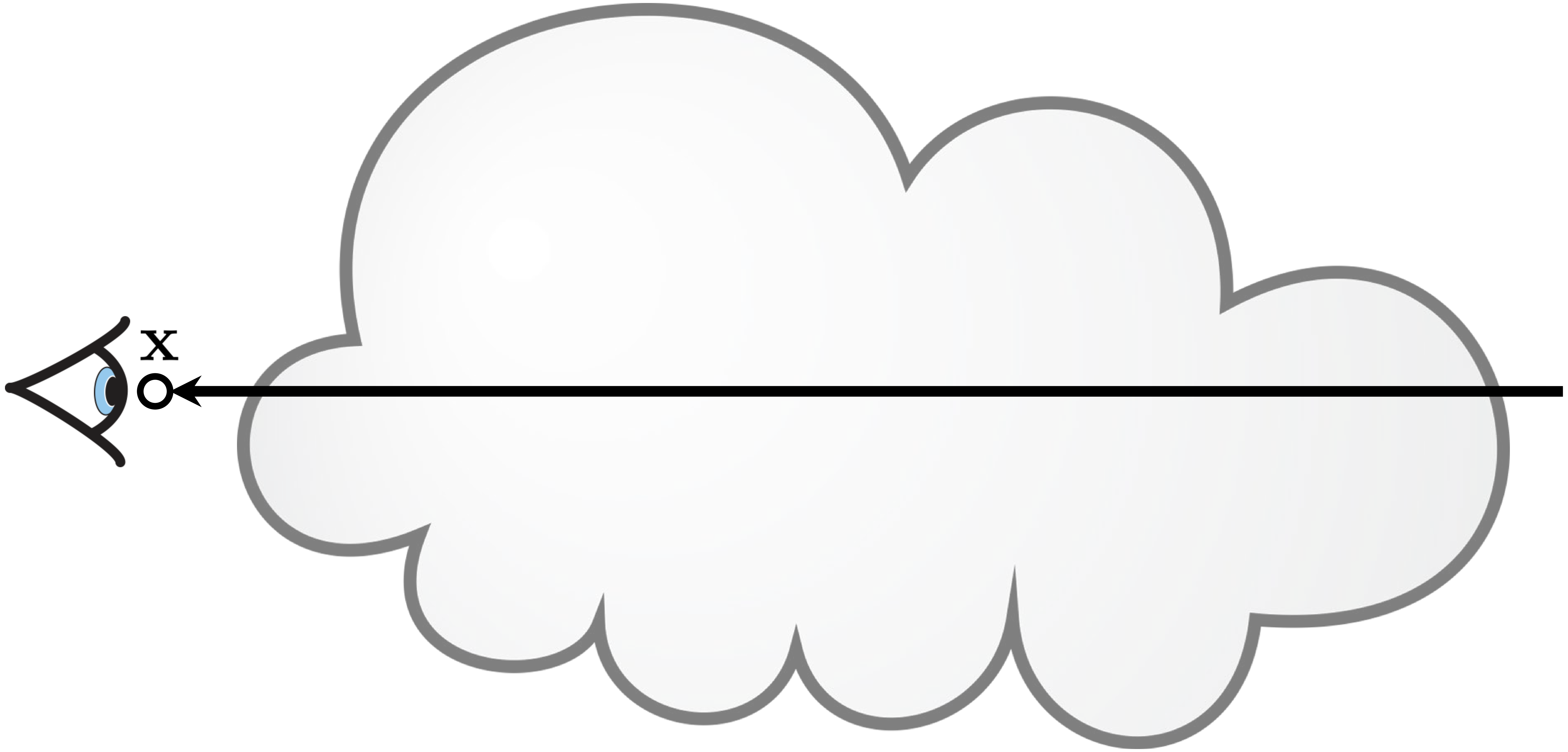
No shadows, implementation nightmare, computationally intensive...

Let’s try brute force!

Ray-Marching

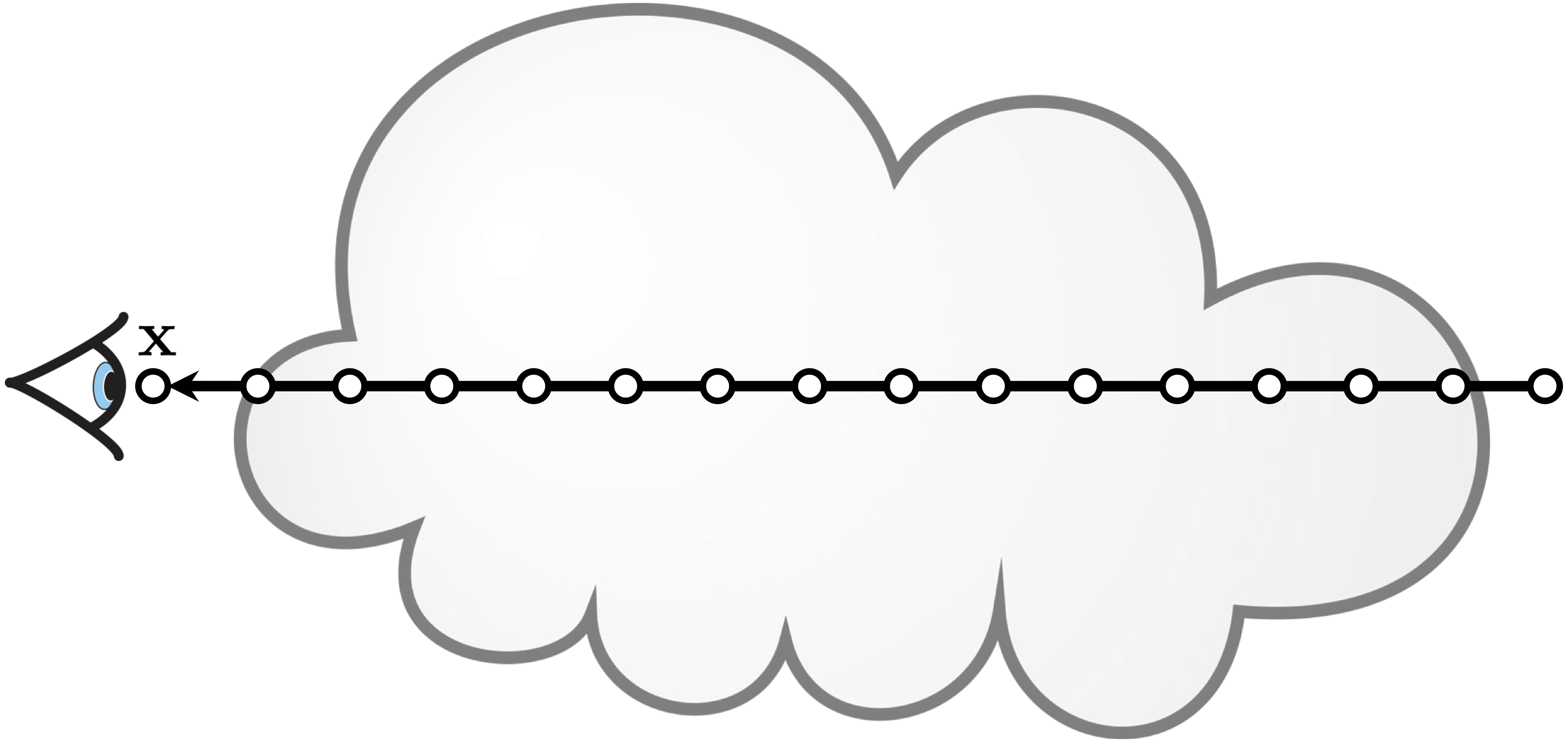
$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t, \vec{\omega})dt$$

Approximate with Riemann sum



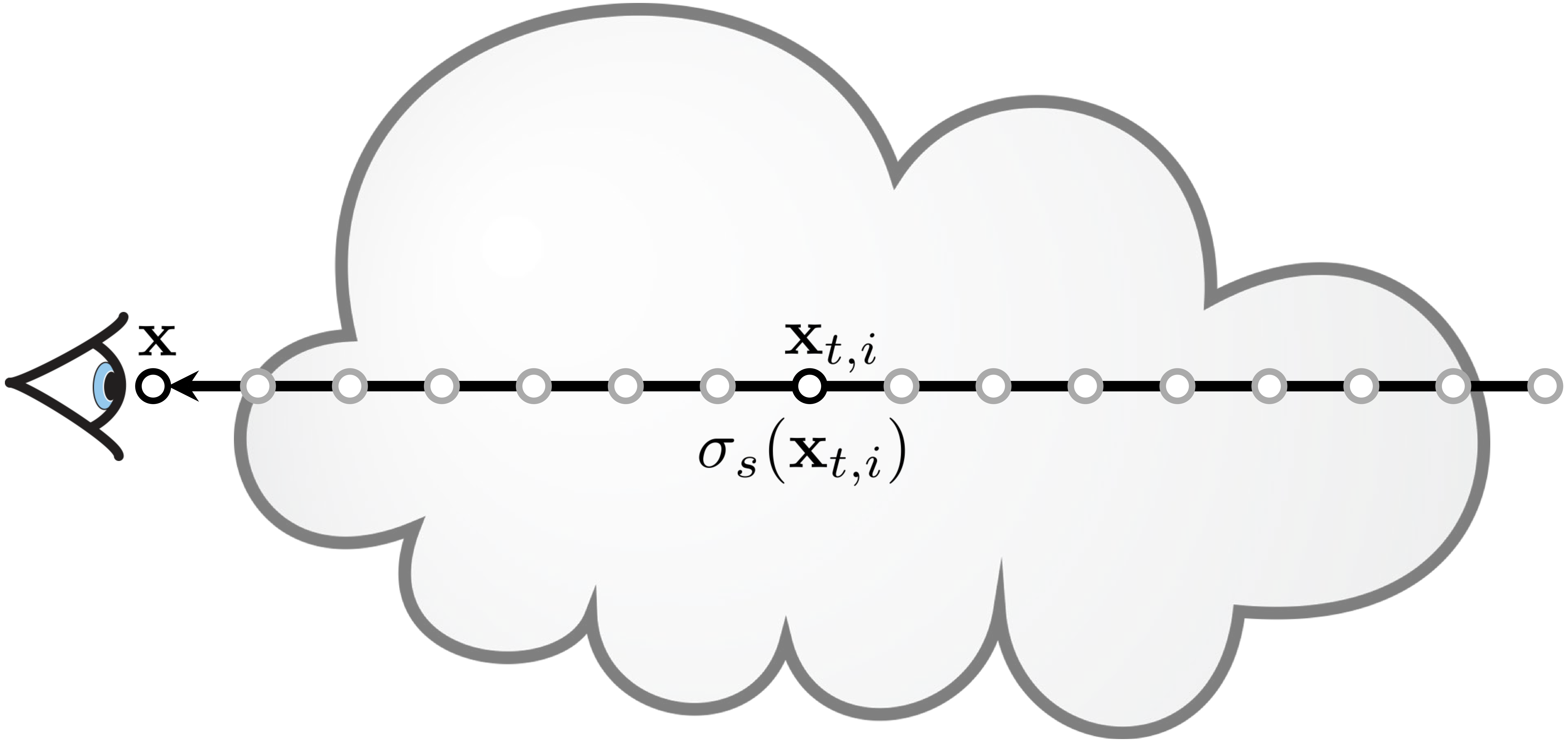
Ray-Marching

$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$



Ray-Marching

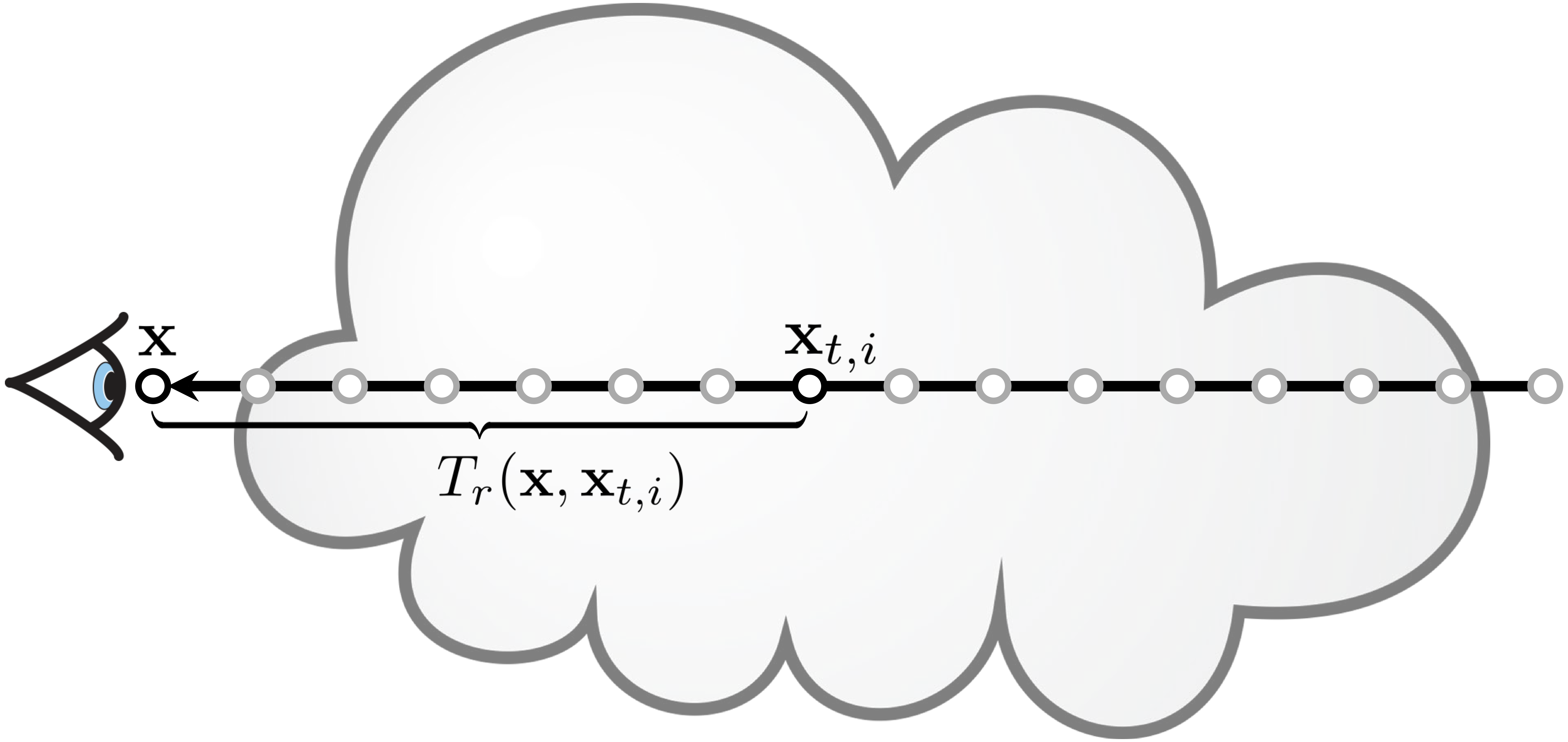
$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) L_s(\mathbf{x}_{t,i}, \vec{\omega}) \Delta t$$



Ray-Marching

$$L(\mathbf{x},\vec{\omega}) \approx \sum_{i=1}^{N} T_r(\mathbf{x},\mathbf{x}_{t,i})\sigma_s(\mathbf{x}_{t,i})L_s(\mathbf{x}_{t,i},\vec{\omega})\Delta t$$

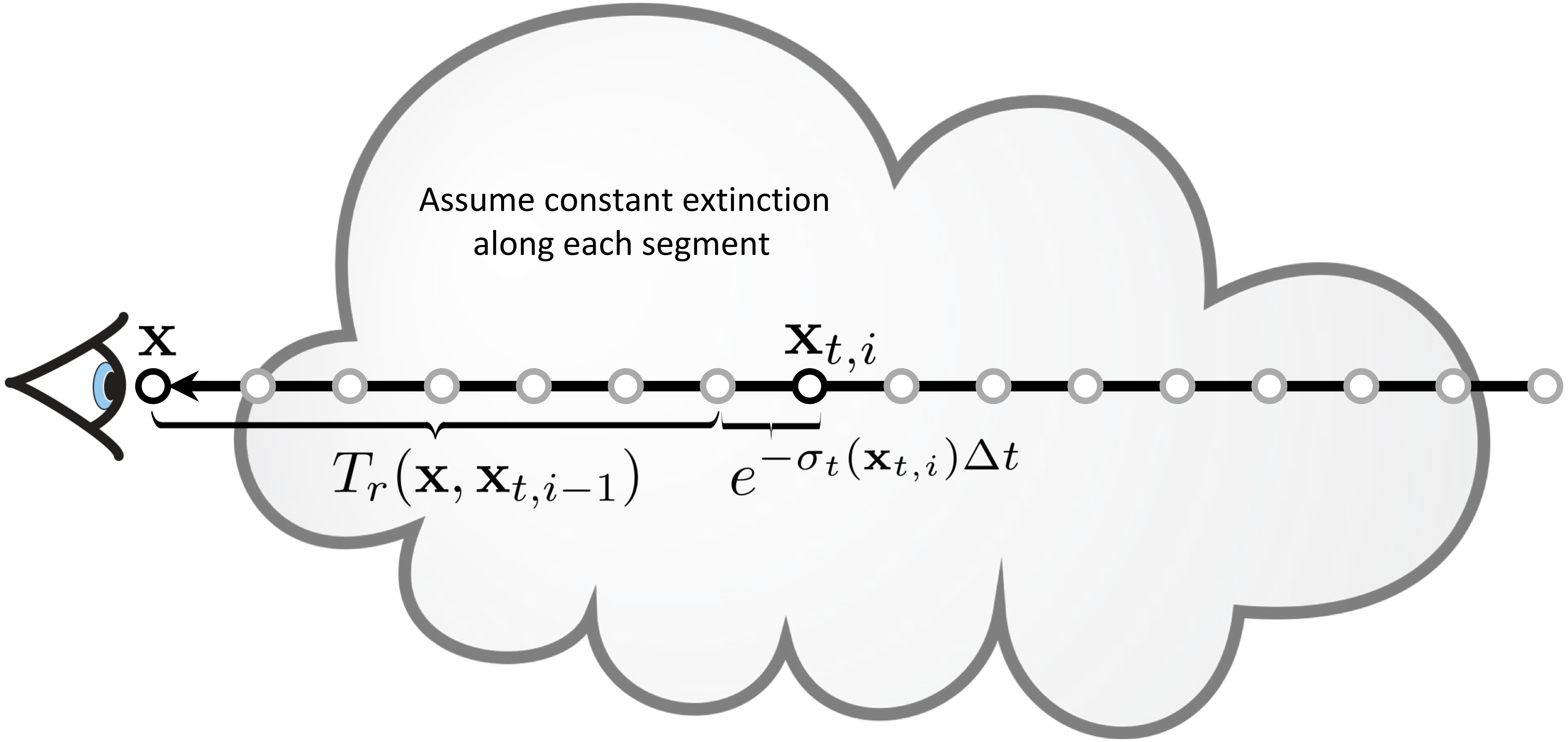
Homogeneous volume: $T_r(\mathbf{x},\mathbf{x}_{t,i}) = e^{-\sigma_t \|\mathbf{x},\mathbf{x}_{t,i}\|}$



Ray-Marching

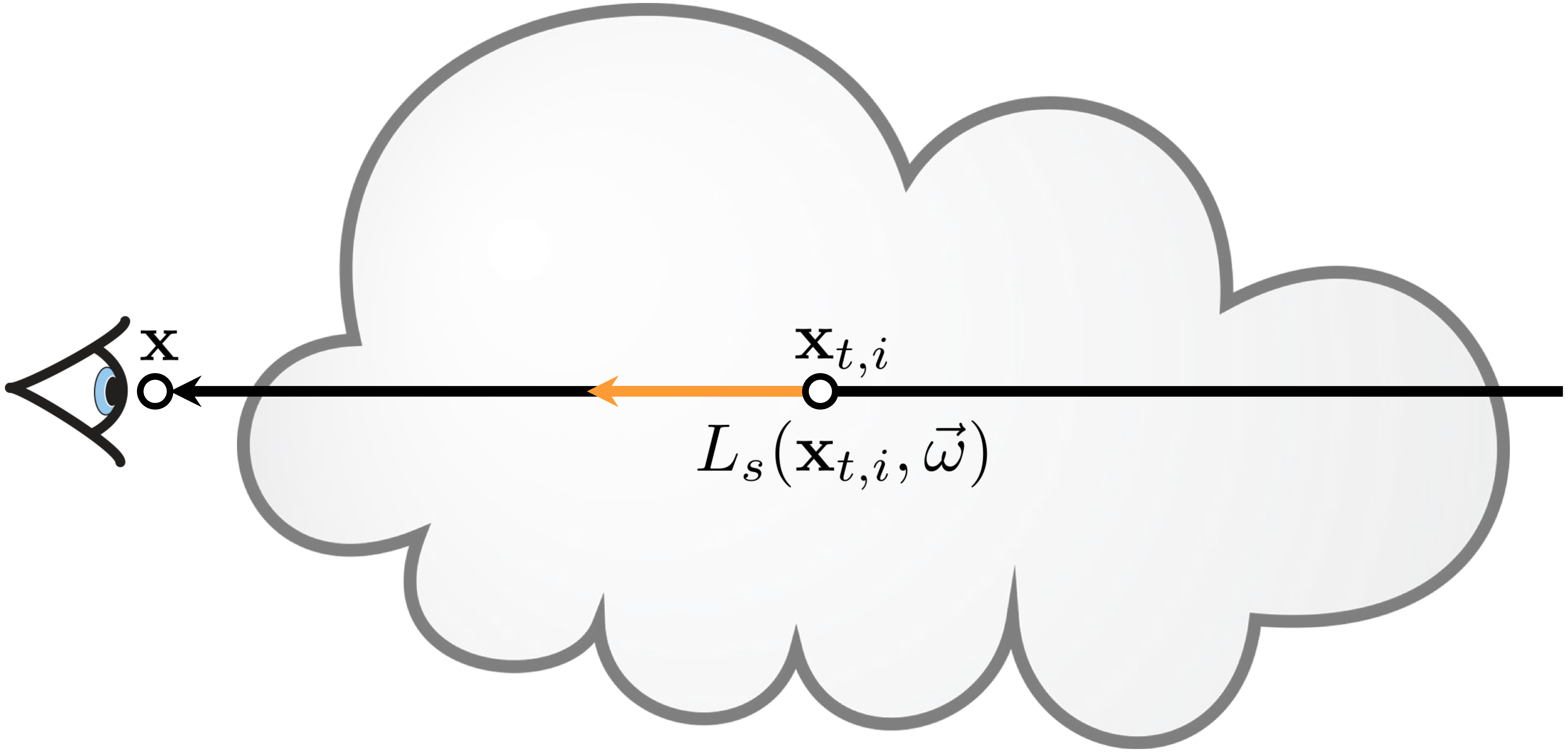
$$L(\mathbf{x},\vec{\omega}) \approx \sum_{i=1}^{N} T_r(\mathbf{x},\mathbf{x}_{t,i})\sigma_s(\mathbf{x}_{t,i})L_s(\mathbf{x}_{t,i},\vec{\omega})\Delta t$$

Heterogeneous volume: $T_r(\mathbf{x},\mathbf{x}_{t,i}) = T_r(\mathbf{x},\mathbf{x}_{t,i-1})\, e^{-\sigma_t(\mathbf{x}_{t,i})\Delta t}$



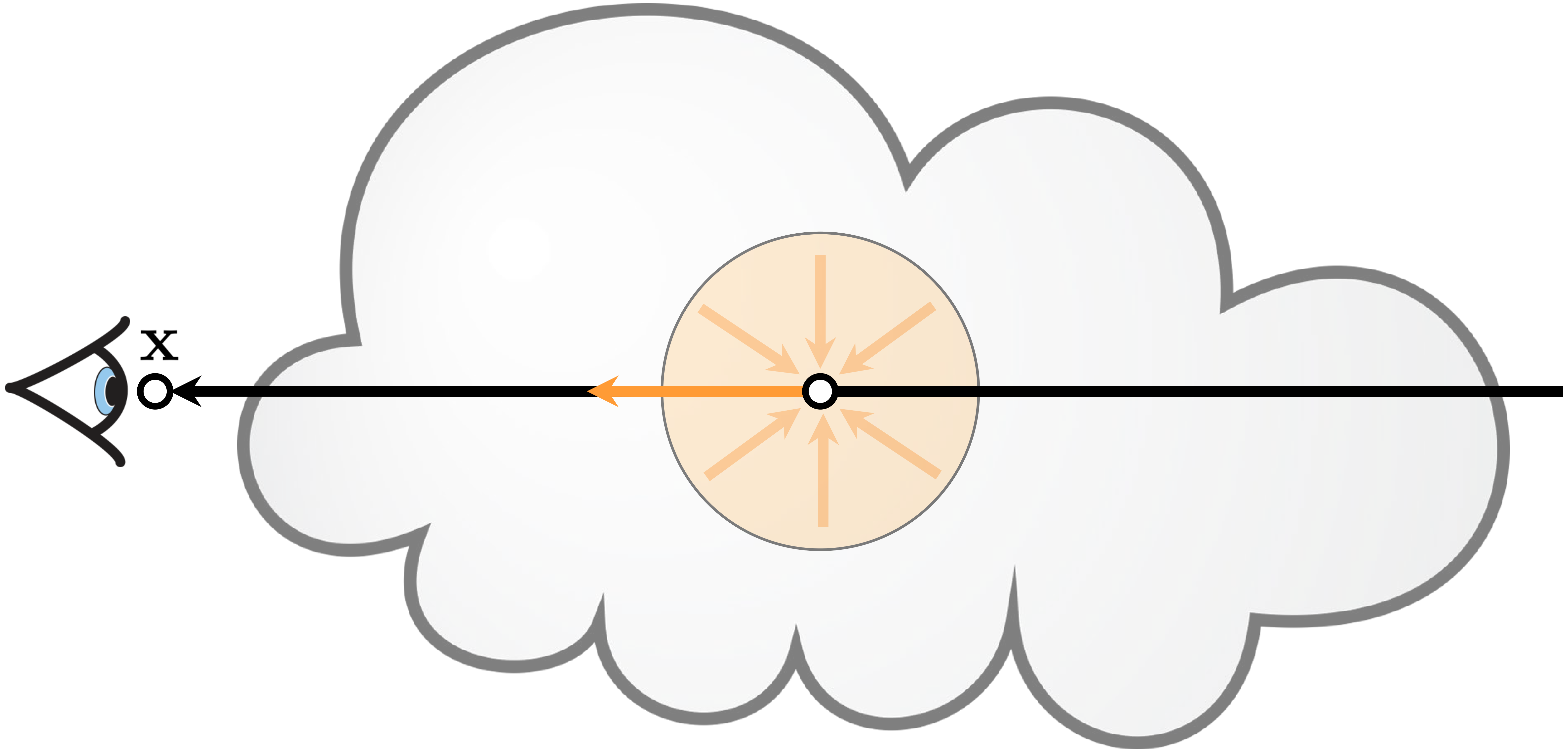
Ray-Marching

$$L(\mathbf{x}, \vec{\omega}) \approx \sum_{i=1}^{N} T_r(\mathbf{x}, \mathbf{x}_{t,i}) \sigma_s(\mathbf{x}_{t,i}) \boxed{L_s(\mathbf{x}_{t,i}, \vec{\omega})} \Delta t$$



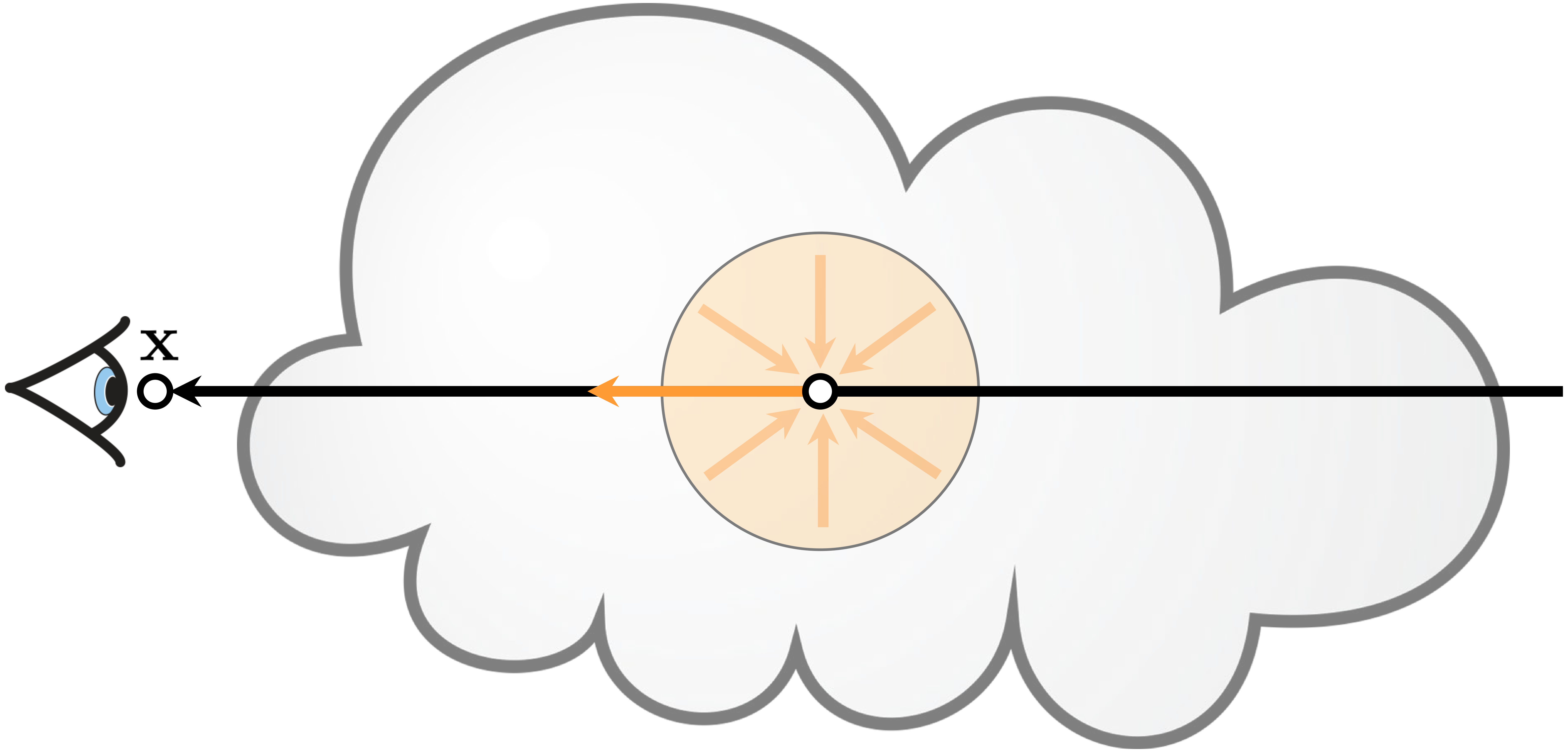
Ray-Marching

$$L_s(\mathbf{x}_t, \vec{\omega}) = \int_{S^2} f_p(\mathbf{x}_t, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}') d\vec{\omega}'$$



Ray-Marching

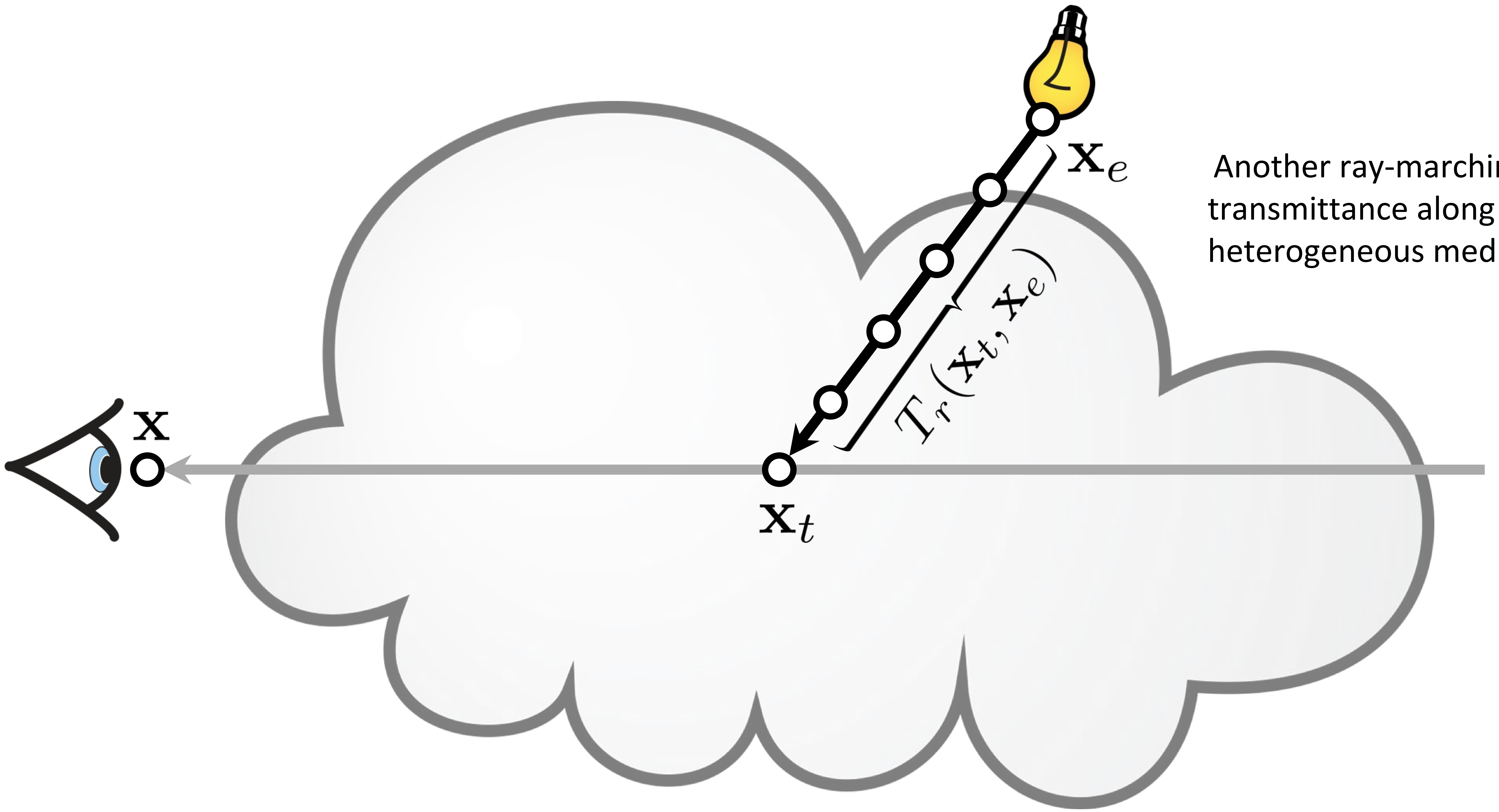
$$L_s(\mathbf{x}_t, \vec{\omega}) \approx \frac{1}{M} \sum_{j=0}^{M} \frac{f_p(\mathbf{x}_t, \vec{\omega}'_j, \vec{\omega}) L_i(\mathbf{x}_t, \vec{\omega}'_j)}{p(\vec{\omega}'_j)}$$



Ray-Marching

Single scattering:

$$L_i(\mathbf{x}_t, \vec{\omega}) = T_r(\mathbf{x}_t, \mathbf{x}_e)\, L_e(\mathbf{x}_e, -\vec{\omega})$$

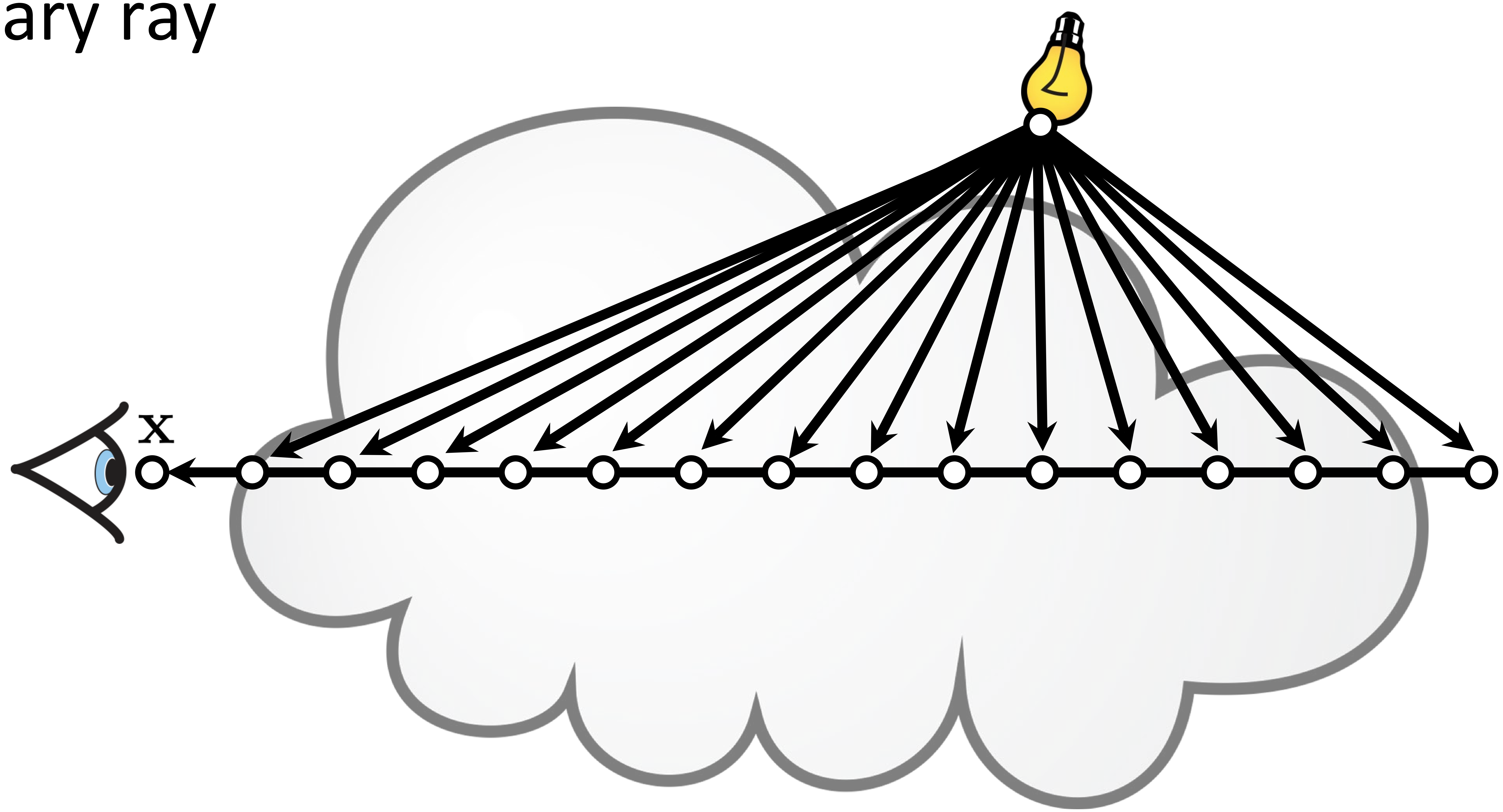


Another ray-marching needed to estimate the transmittance along the connection ray (in heterogeneous media)

Ray-Marching in Heterogeneous Media

Marching towards the light source

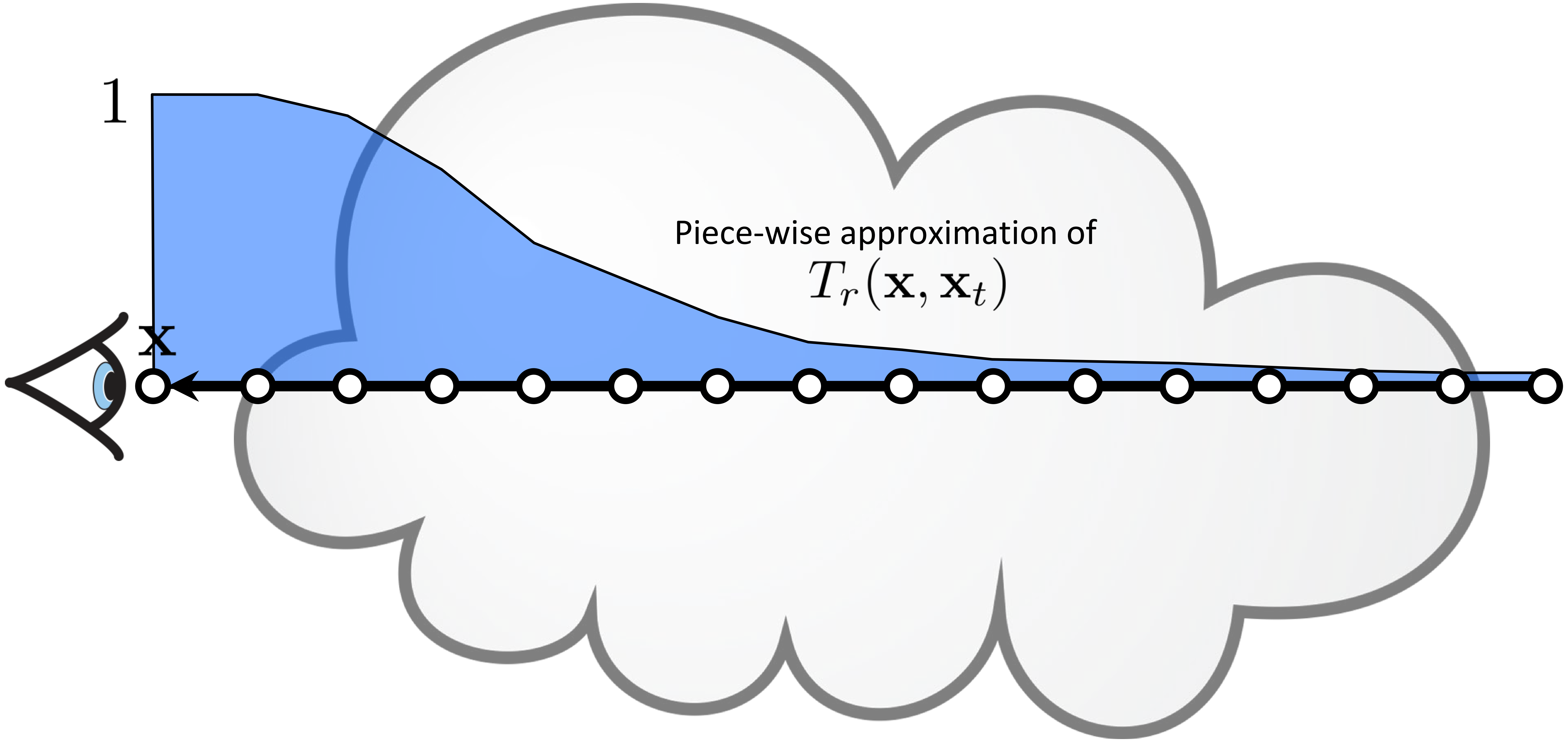
- Connections are expensive, many, and uniformly distributed along the primary ray



Decoupled Transmittance and In-scattering

1. Ray-march and cache transmittance

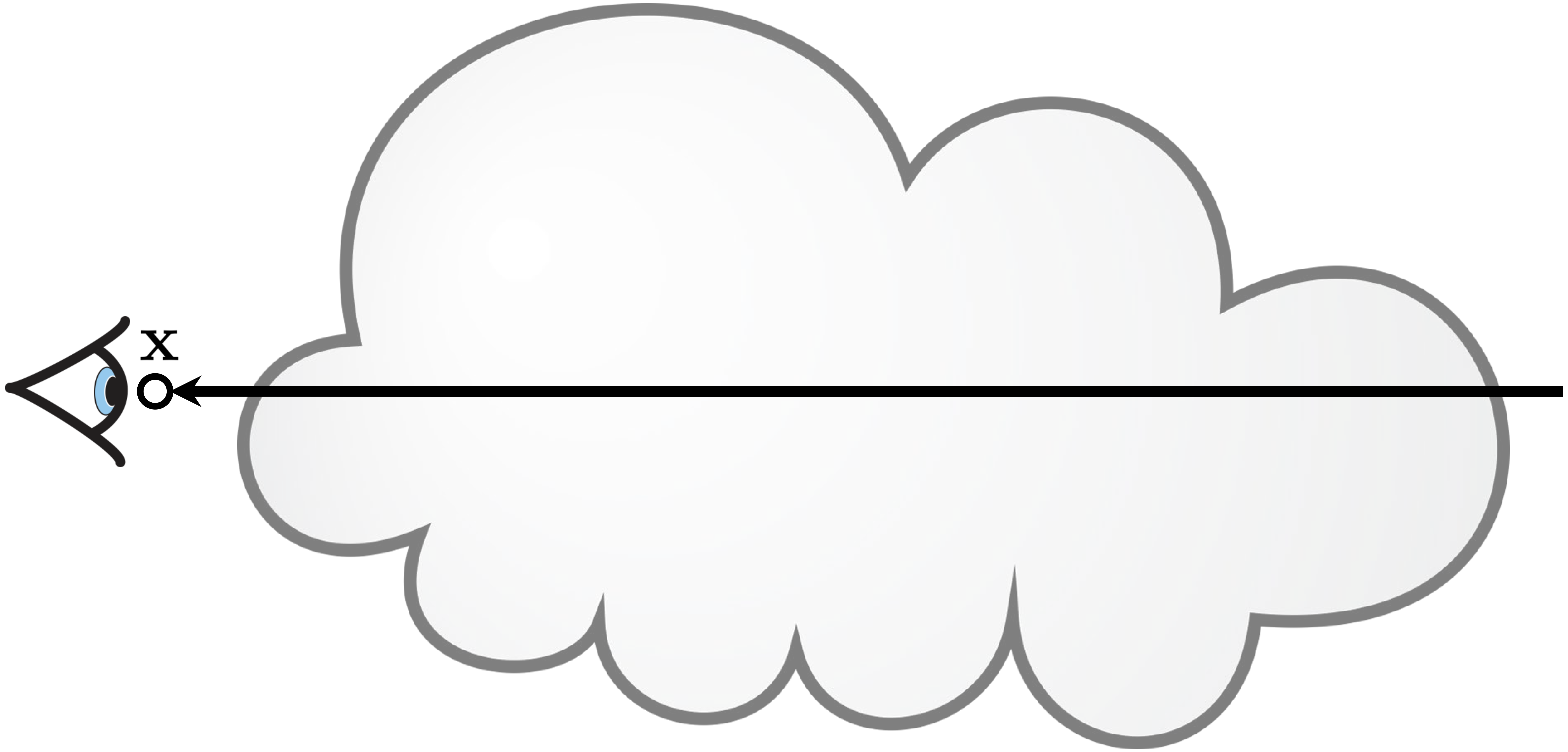
- Choose step-size w.r.t. frequency content to accurately capture variations



Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

- Distribute samples ∝ (part of) the integrand



Decoupled Transmittance and In-scattering

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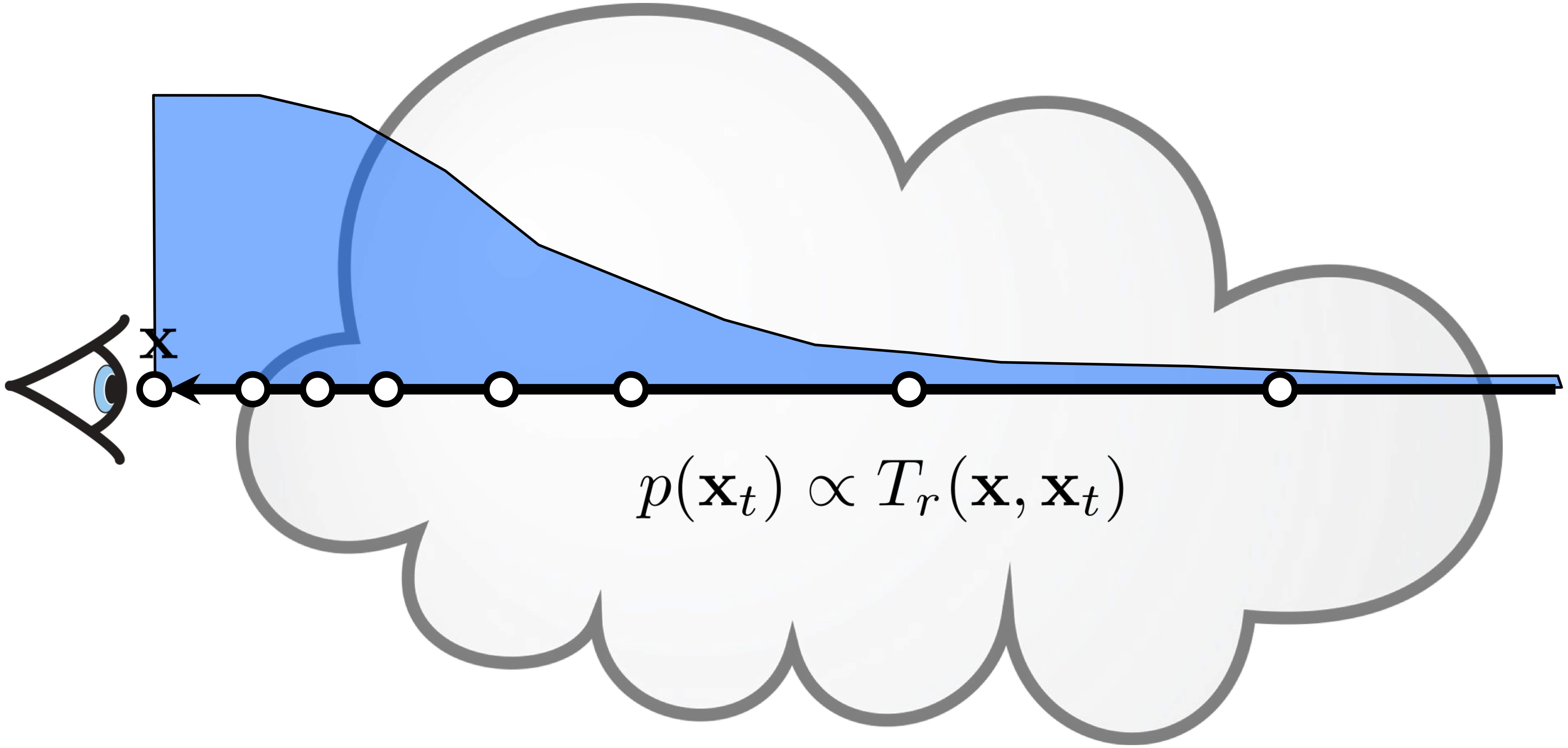
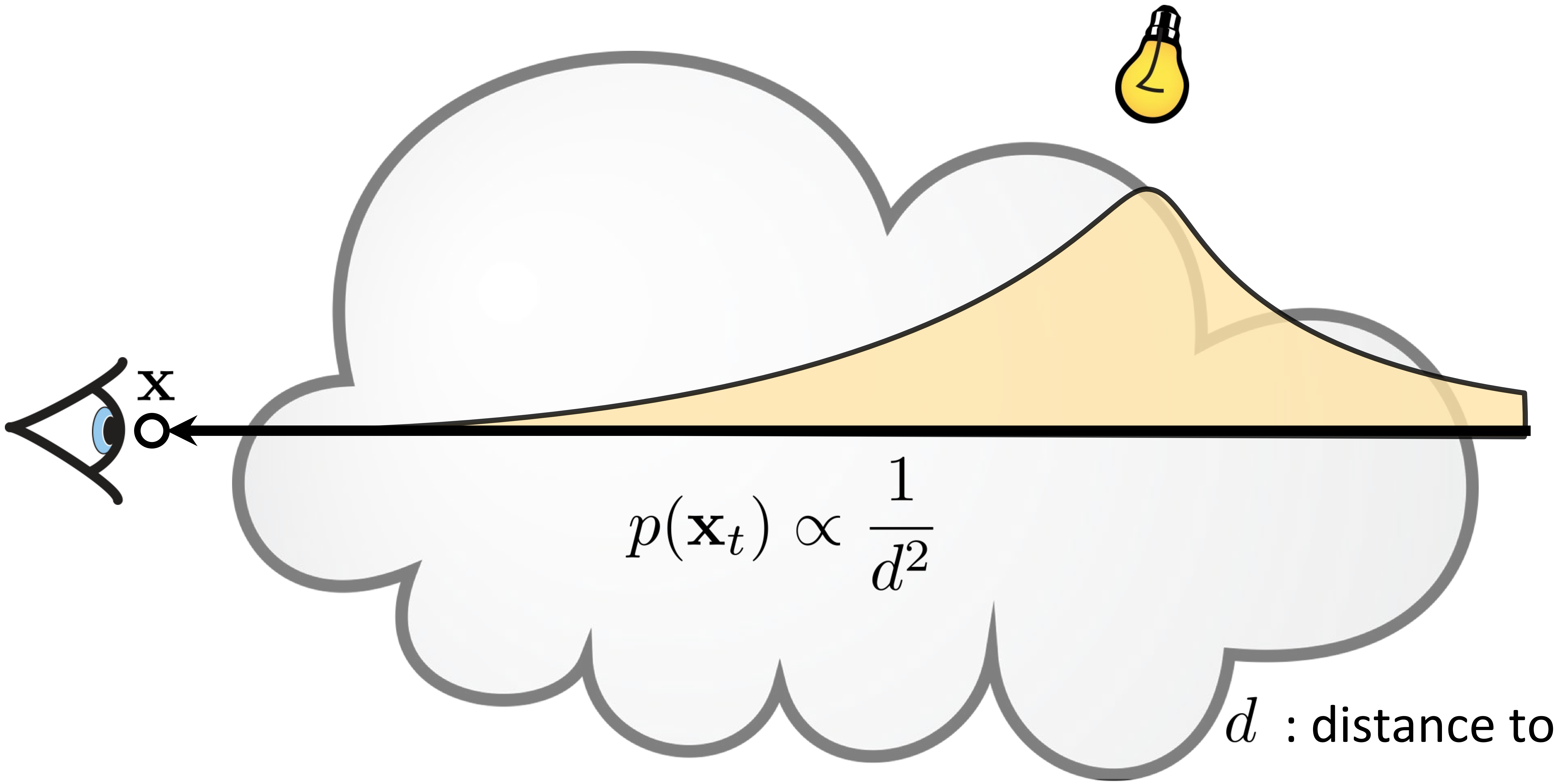


Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

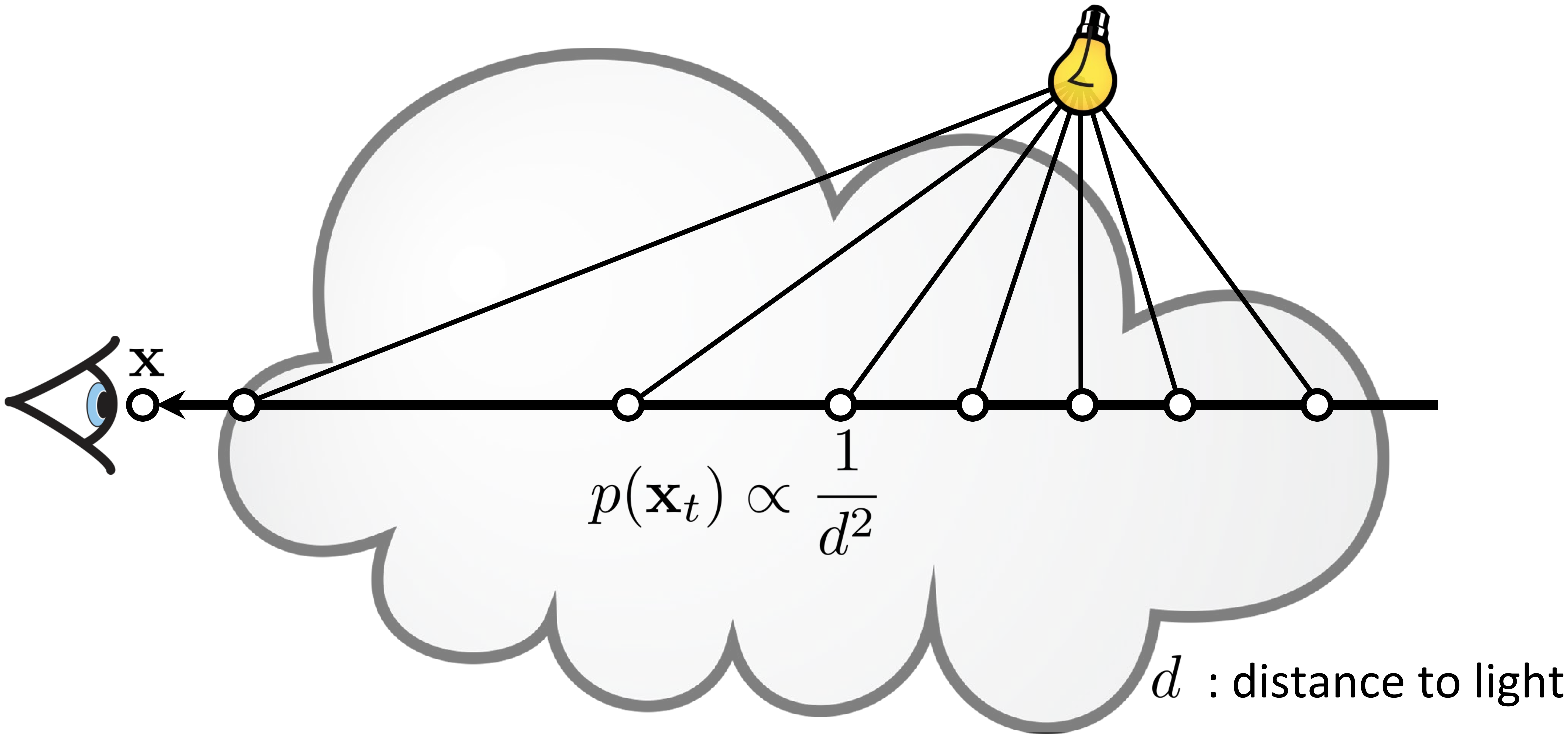
- Distribute samples ∝ (part of) the integrand



Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

- Distribute samples ∝ (part of) the integrand



Decoupled Transmittance and In-scattering

2. Estimate in-scattering using MC integration

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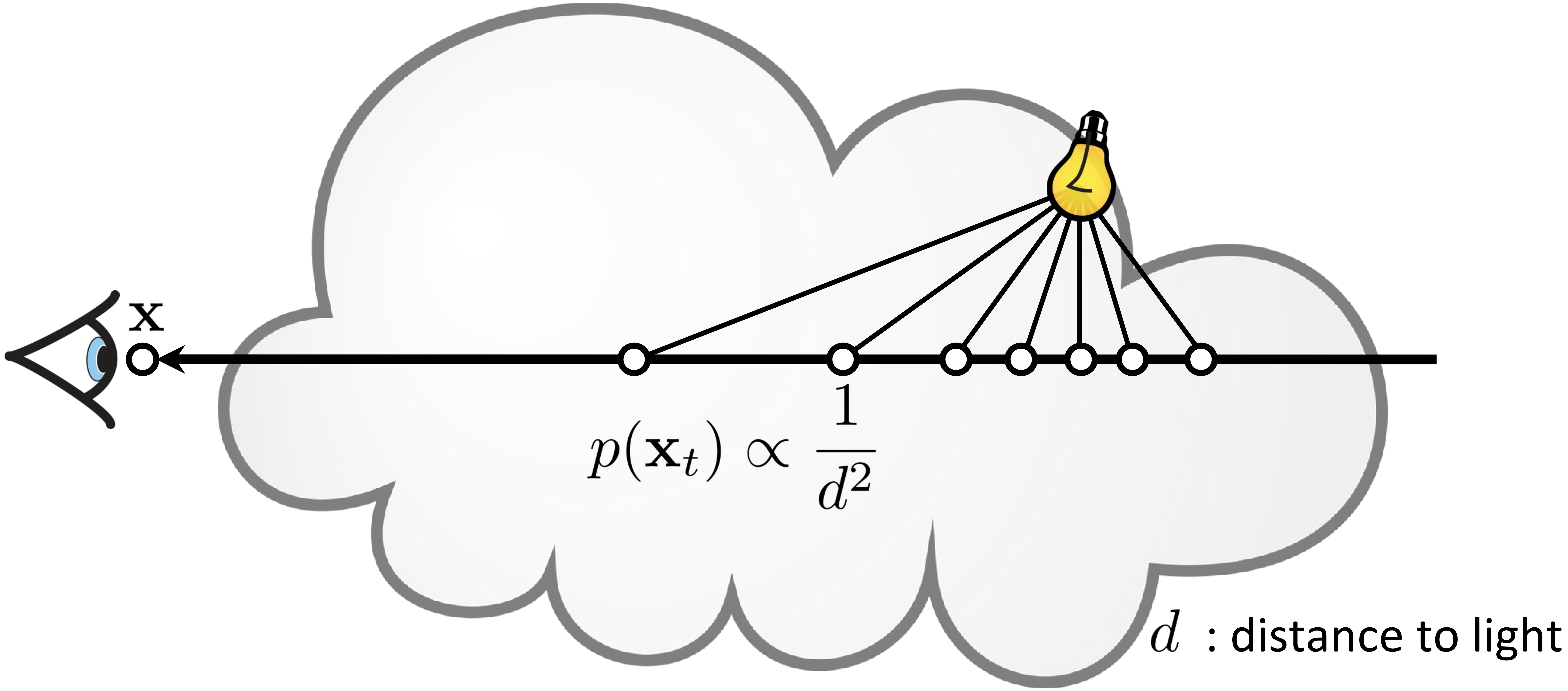
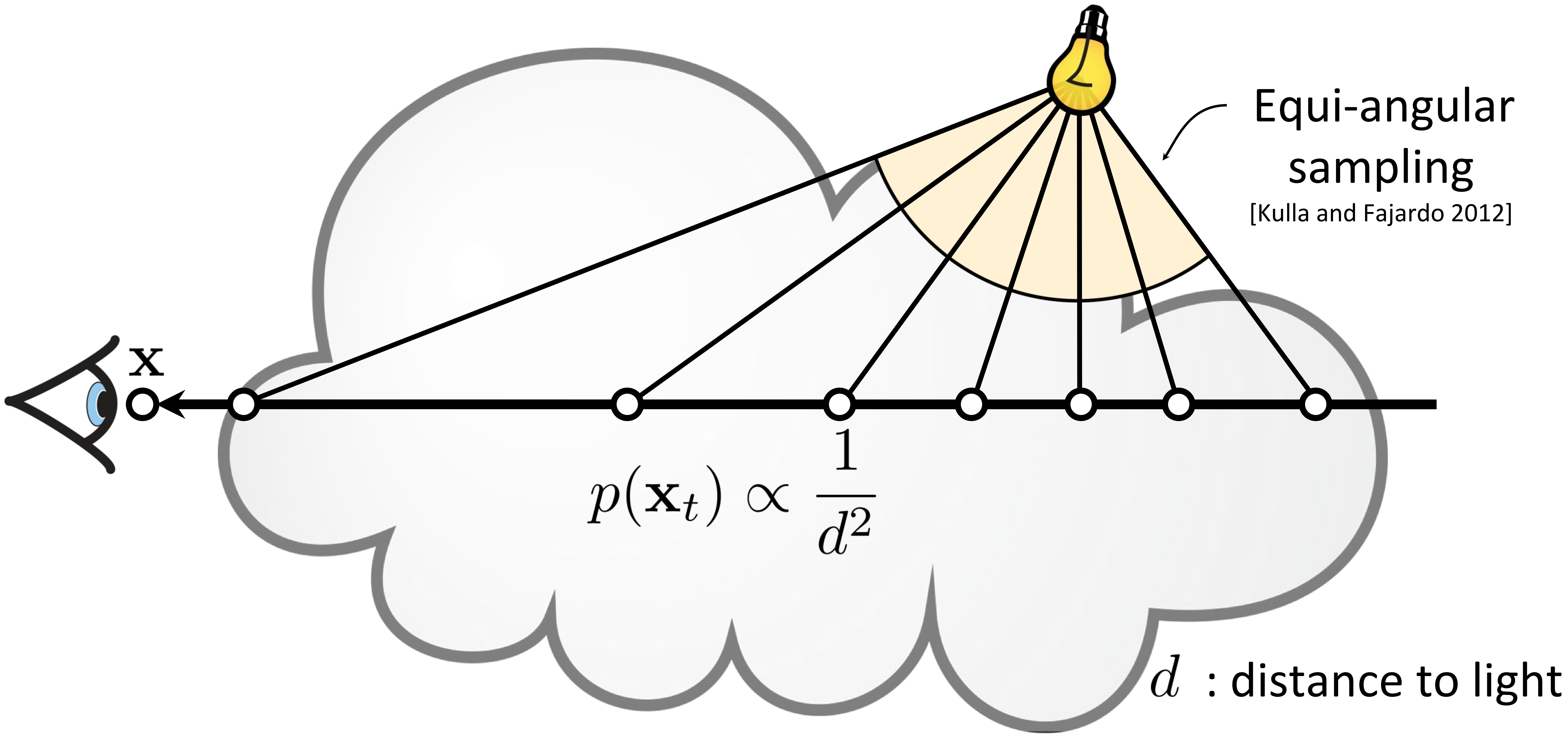


Decoupled Transmittance and In-scattering

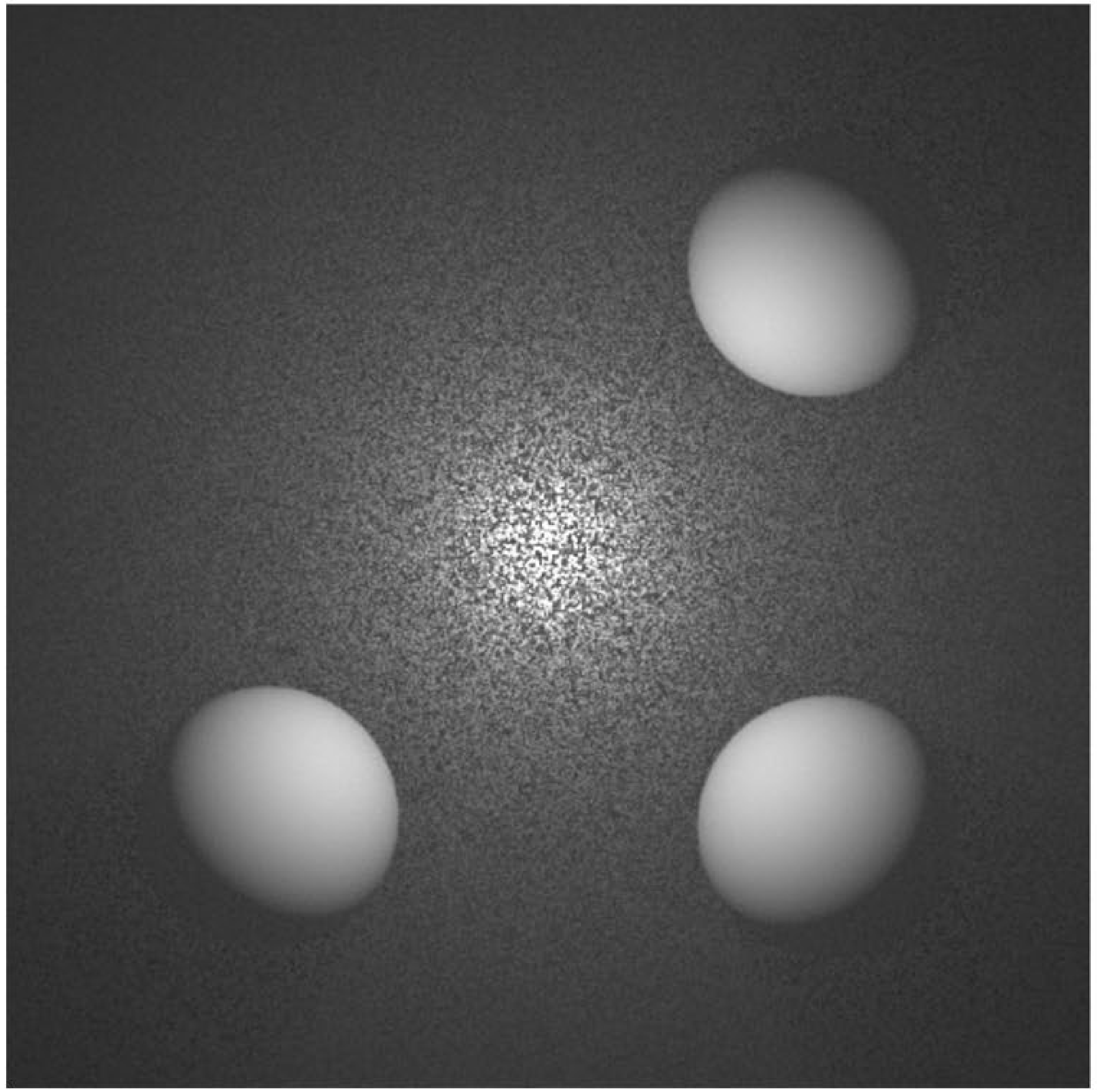
2. Estimate in-scattering using MC integration

- Distribute samples ∝ (part of) the integrand

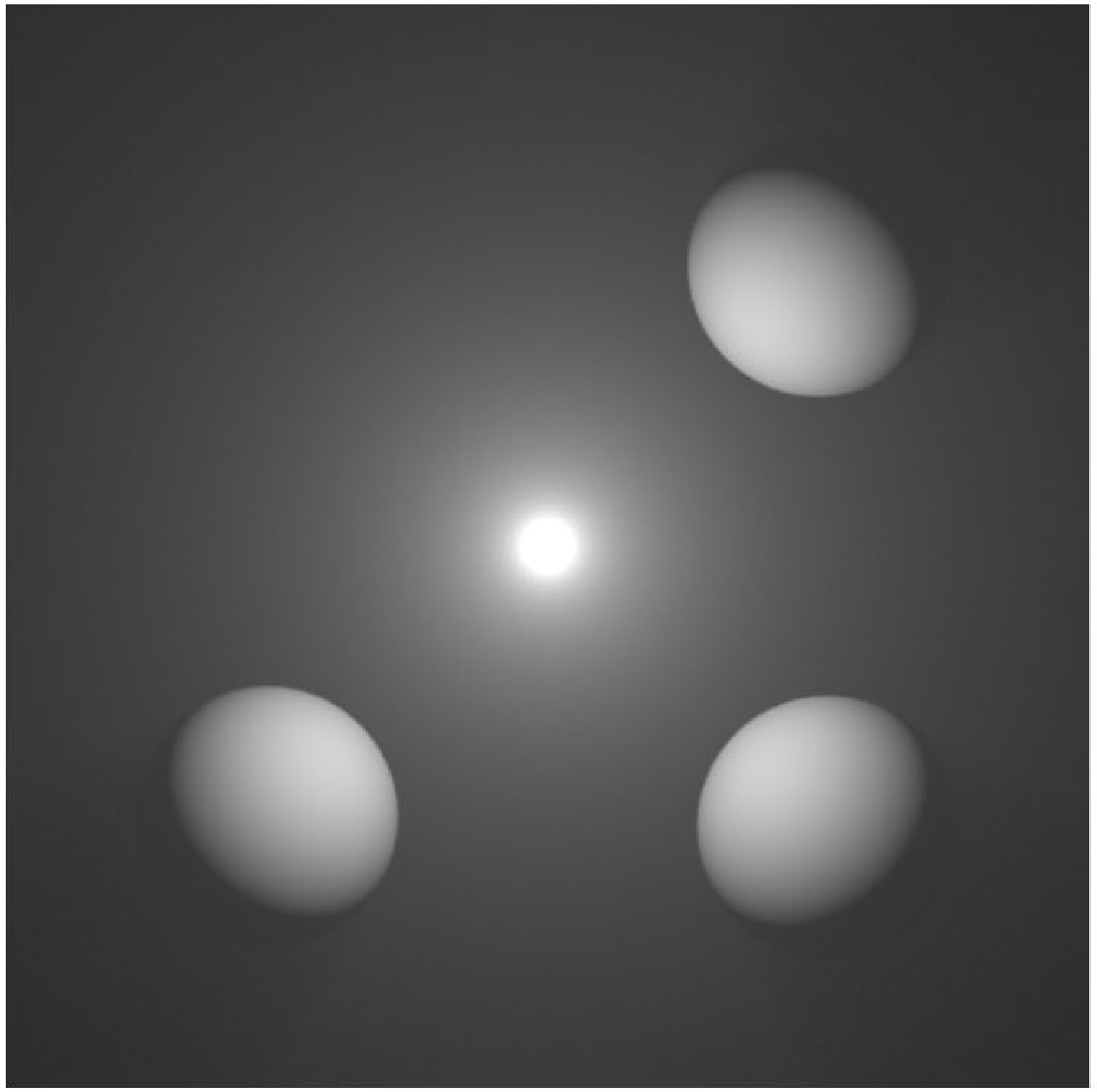


Decoupled Transmittance and In-scattering

Ray-marching



Equiangular sampling

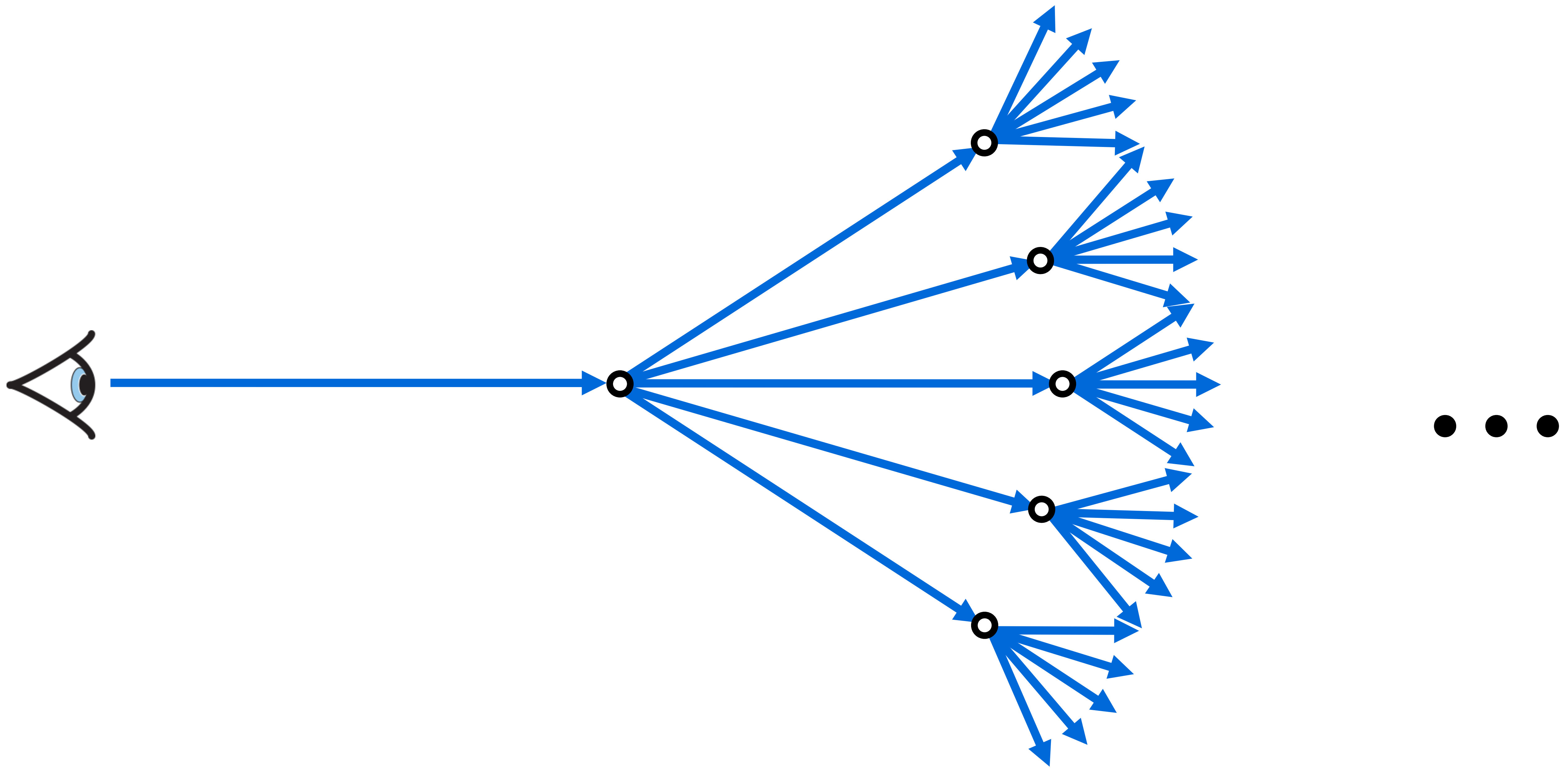


Images courtesy of Kulla and Fajardo

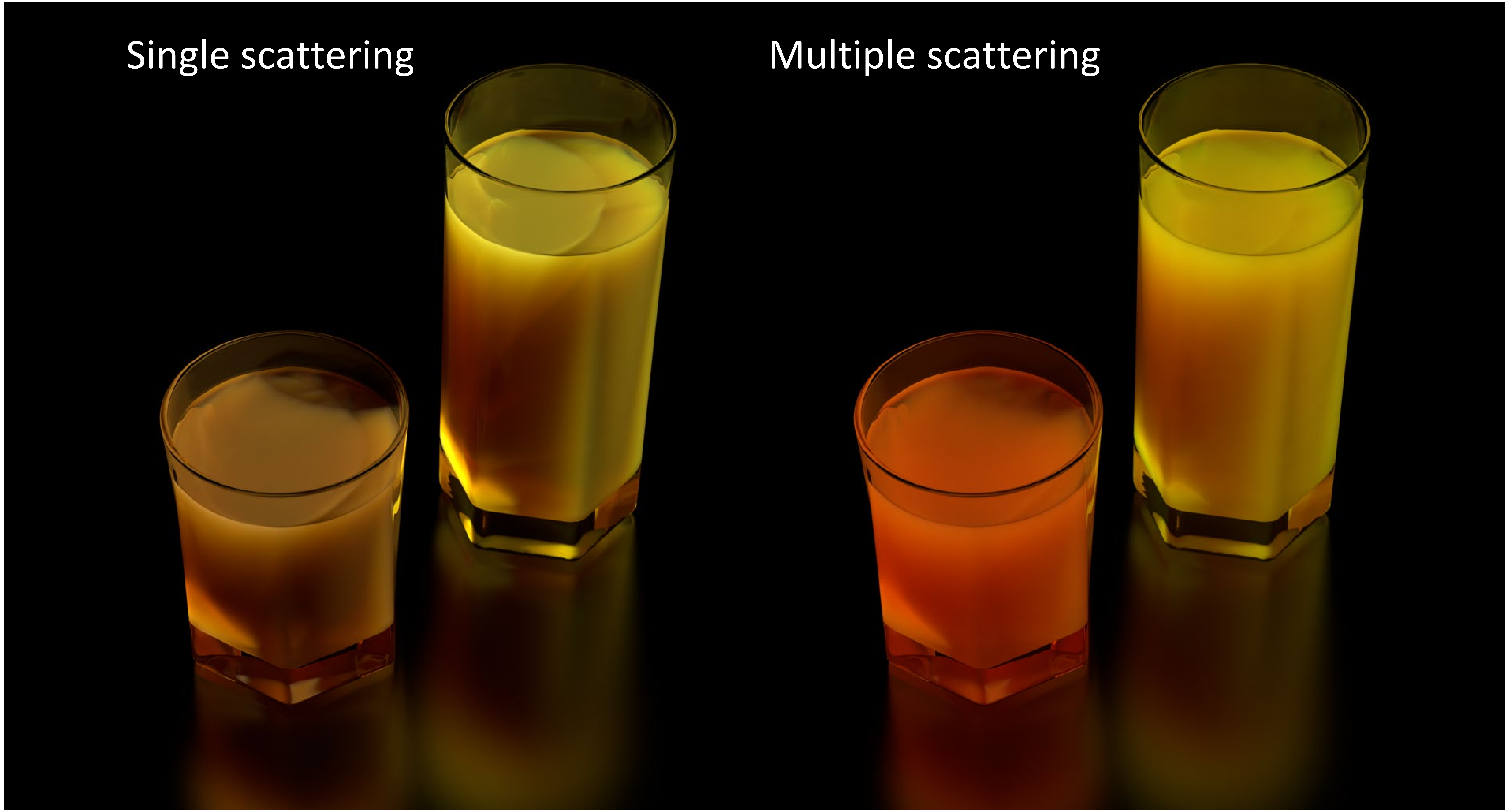
Multiple Bounces

Same concept as in recursive Monte Carlo ray tracing, but taking into account volumetric scattering

Exponential growth:



Visual Break



Volumetric Path Tracing

Volumetric Path Tracing

Motivation:

- Same as with standard path tracing: avoid the exponential growth

Paths can:

- Reflect/refract off surfaces
- Scatter inside a volume

Volume Rendering Equation

Accumulated emitted radiance

$$L(\mathbf{x}, \vec{\omega}) = \boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega})\,\mathrm{d}t}$$
$$+ \boxed{\int_0^z T_r(\mathbf{x}, \mathbf{x}_t)\sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega})\,\mathrm{d}t}$$
$$+ \boxed{T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})}$$

Accumulated in-scattered radiance

Attenuated background radiance

Volume Rendering Equation

Accumulated emitted + in-scattered radiance

$$L(\mathbf{x}, \vec{\omega}) = \int_0^z T_r(\mathbf{x}, \mathbf{x}_t) \left[\sigma_a(\mathbf{x}_t) L_e(\mathbf{x}_t, \vec{\omega}) + \sigma_s(\mathbf{x}_t) L_s(\mathbf{x}_t, \vec{\omega}) \right] \, \mathrm{d}t$$
$$+ T_r(\mathbf{x}, \mathbf{x}_z) L(\mathbf{x}_z, \vec{\omega})$$

Attenuated background radiance

Volume Rendering Equation

$$L(\mathbf{x},\vec{\omega}) = \int_0^z T_r(\mathbf{x},\mathbf{x}_t)\left[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})\right]\,\mathrm{d}t$$
$$+ T_r(\mathbf{x},\mathbf{x}_z)L(\mathbf{x}_z,\vec{\omega})$$

1-Sample Monte Carlo Estimator

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)}\left[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)L_s(\mathbf{x}_t,\vec{\omega})\right]$$
$$+ \frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z,\vec{\omega})$$

$p(t)$ - probability *density* of distance t

$P(z)$ - *probability* of exceeding distance z

1-Sample Monte Carlo Estimator

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)}\left[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)\frac{f_p(\vec{\omega},\vec{\omega}_i)L(\mathbf{x}_t,\vec{\omega}_i)}{p(\vec{\omega}_i)}\right] + \frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z,\vec{\omega})$$

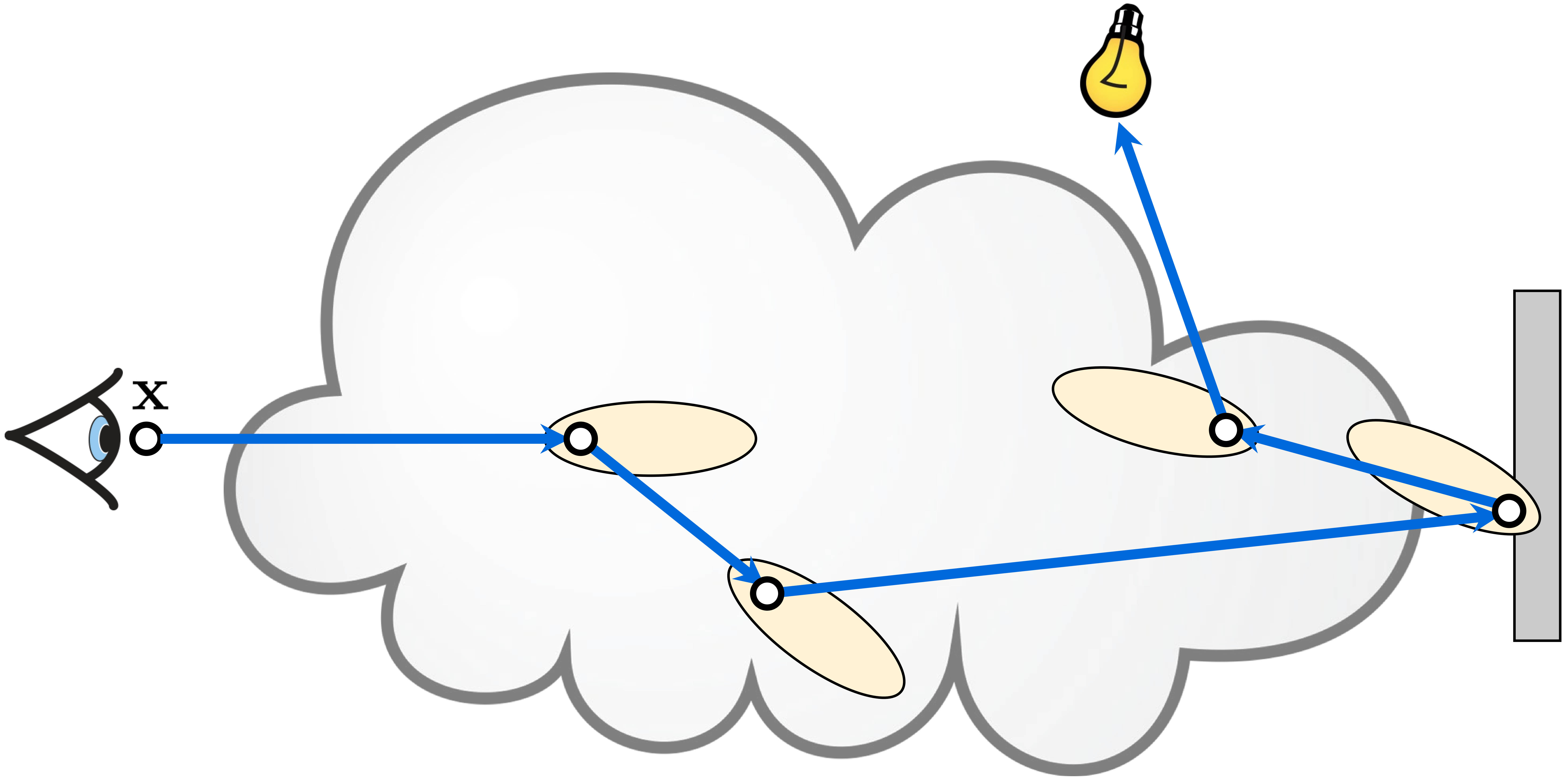
$p(t)$ - probability *density* of distance t

$P(z)$ - *probability* of exceeding distance z

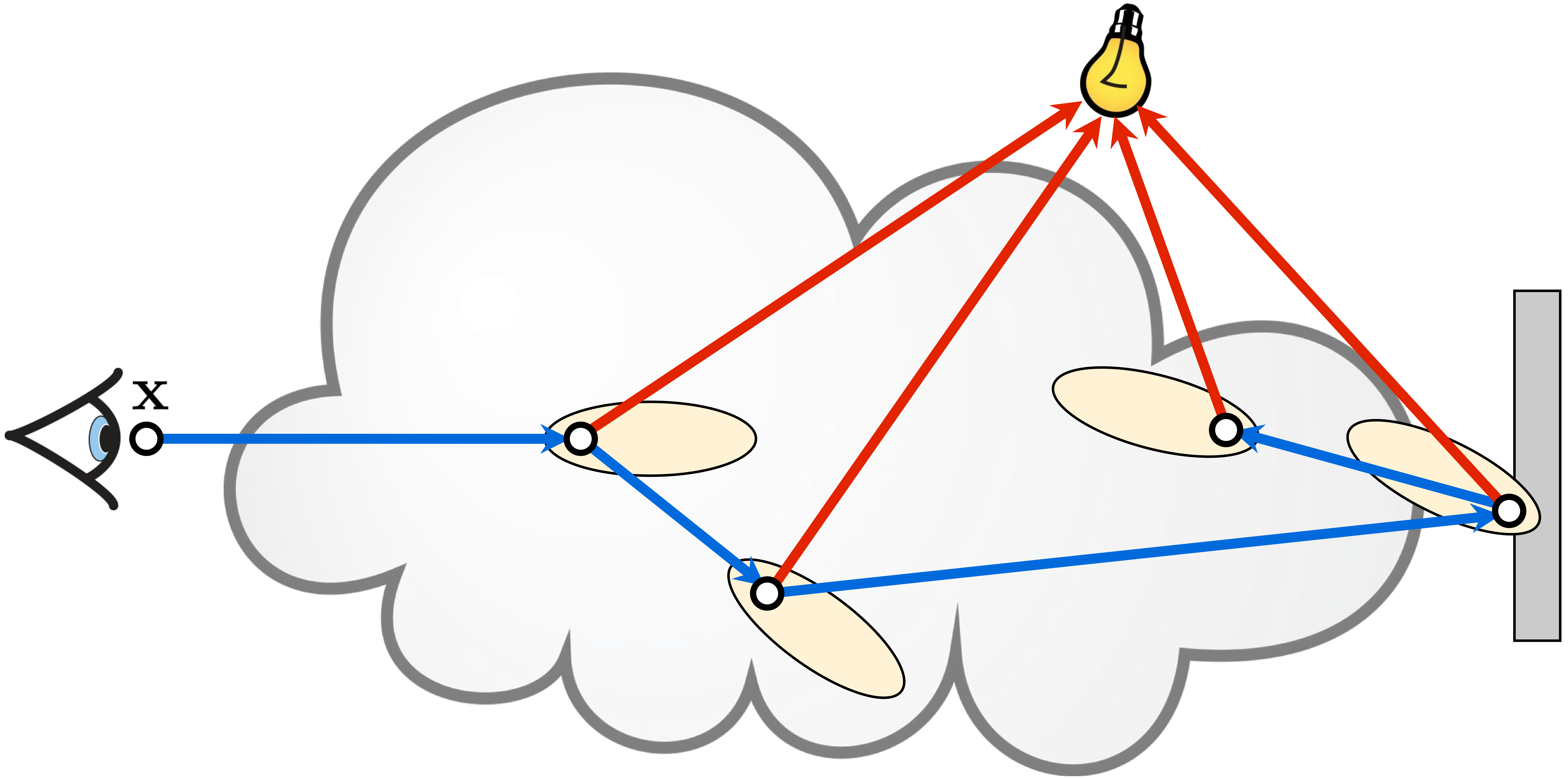
$p(\vec{\omega}_i)$ - probability *density* of direction $\vec{\omega}_i$

Volumetric Path Tracing

1. Sample distance to next interaction
2. Scatter in the volume or bounce off a surface



Volumetric Path Tracing with NEE



Sampling the Phase Function

Isotropic:

- Uniform sphere sampling

Henyey-Greenstein:

- Using the inversion method we can derive

$$\cos\theta = \frac{1}{2g}\left(1 + g^2 - \left(\frac{1-g^2}{1-g+2g\xi_1}\right)^2\right)$$

$$\phi = 2\pi\xi_2$$

- PDF is the value of the HG phase function

Free-path Sampling

Free-path (or free-flight distance):

- Distance to the next interaction within the medium
- Dense media (e.g. milk): short mean-free path
- Thin media (e.g. atmosphere): long mean-free path

Ideally, we want to sample proportional to (part of) integrand, e.g. transmittance:

$$p(\mathbf{x}_t | (\mathbf{x}, \vec{\omega})) \propto T_r(\mathbf{x}, \mathbf{x}_t)$$

$$p(t) \propto T_r(t)$$

simplified notation for brevity

Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

- PDF: $p(t) \propto e^{-\sigma_t t}$

$$p(t) = \frac{e^{-\sigma_t t}}{\int_0^\infty e^{-\sigma_t s} ds} = \sigma_t e^{-\sigma_t t}$$

- CDF: $P(t) = \int_0^t \sigma_t e^{-\sigma_t s} ds = 1 - e^{-\sigma_t t}$

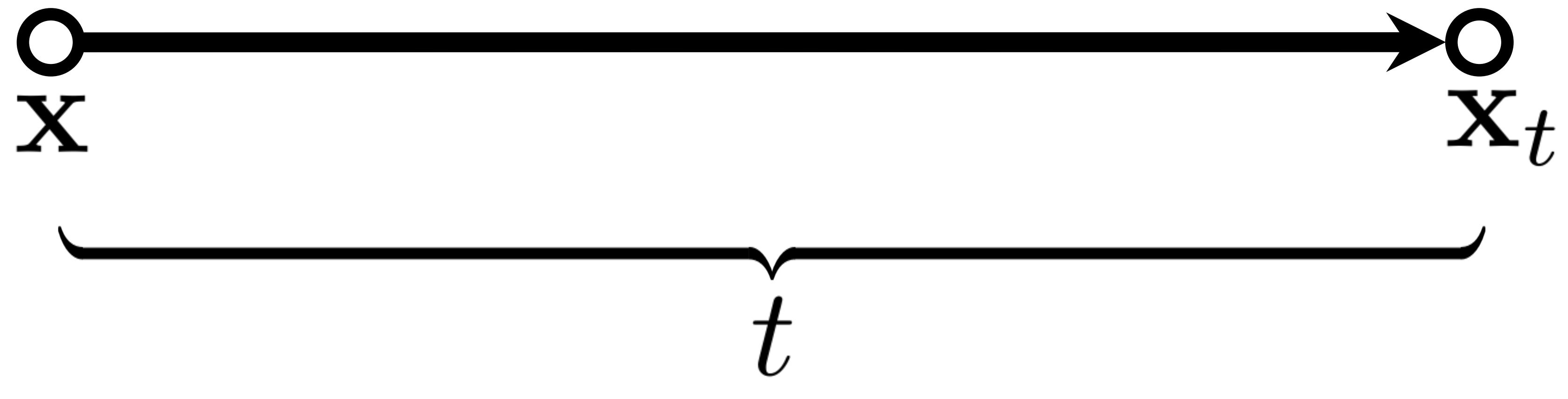
- Inverted CDF: $P^{-1}(\xi) = -\frac{\ln(1-\xi)}{\sigma_t}$

Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

- Generate random number ξ
- Sample distance $t = -\frac{\ln(1-\xi)}{\sigma_t}$
- Compute PDF $p(t) = \sigma_t e^{-\sigma_t t}$



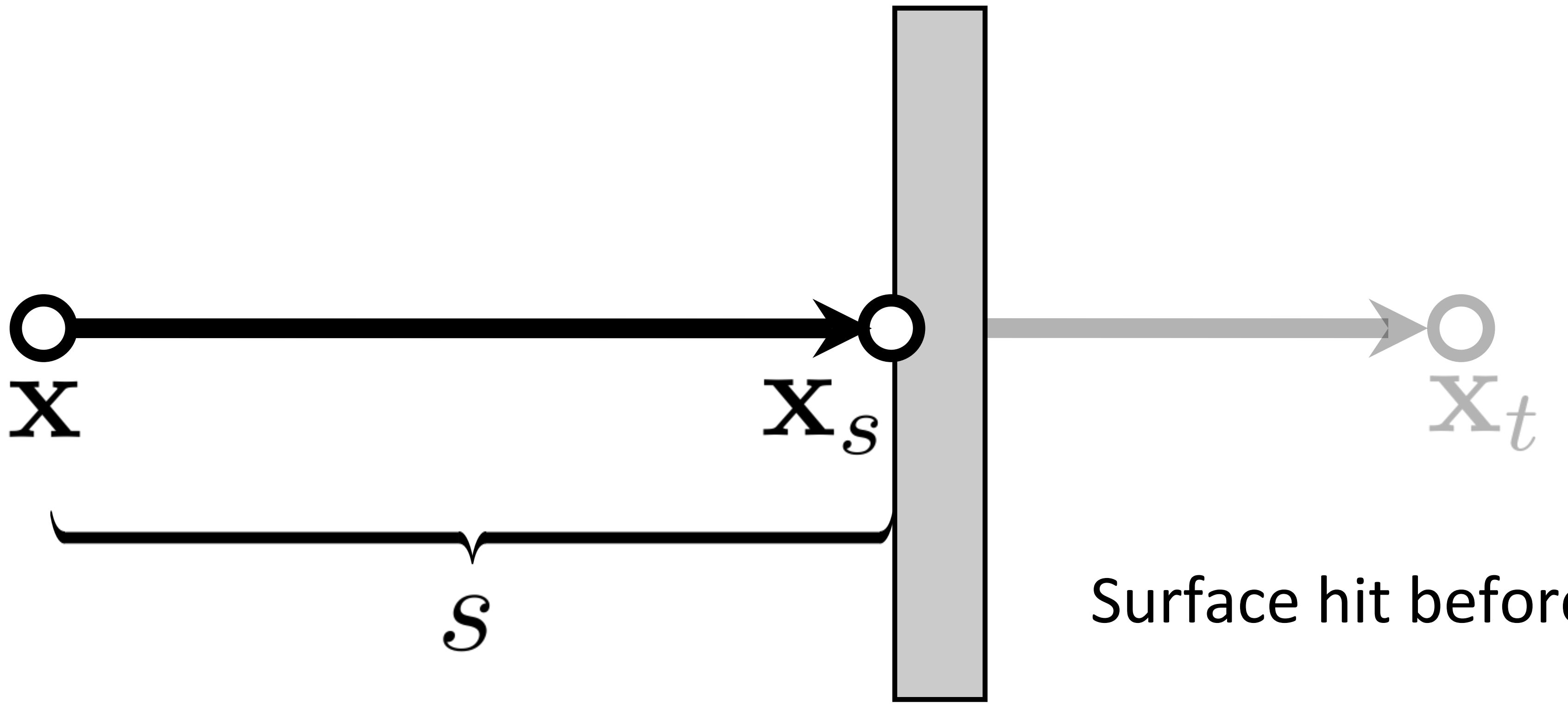
Free-path Sampling

Homogeneous media: $T_r(t) = e^{-\sigma_t t}$

Recipe:

- Generate random number ξ
- Sample distance $t = -\frac{\ln(1-\xi)}{\sigma_t} = s$
- Compute PDF $p(t) = \sigma_t e^{-\sigma_t t} = e^{-\sigma_t s}$

Note: This is now a probability, not a probability density!



Surface hit before reaching t

Volumetric PT for Homogeneous Volumes

```
Color vPT(x, ω)
tmax = nearestSurface(x, ω)
t = -log(1 - randf()) / σt   // Sample free path
if t < tmax: // Volume interaction
x += t * ω
pdf_t = σt * exp(-σt * t)
(ω', pdf_ω') = samplePF(ω)
return Tr(t) / pdf_t * (σa * Le(x, ω) + σs * PF(ω, ω') * vPT(x, ω') / pdf_ω')
else: // Surface interaction
x += tmax * ω
Pr_tmax = exp(-σt * tmax)
(ω', pdf_ω') = sampleBRDF(n, ω)
return Tr(tmax) / Pr_tmax * (Le(x, ω) + BRDF(ω, ω') * vPT(x, ω') / pdf_ω')
```

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)}\left[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)\frac{f_p(\vec{\omega},\vec{\omega}_i)L(\mathbf{x}_t,\vec{\omega}_i)}{p(\vec{\omega}_i)}\right] + \frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z,\vec{\omega})$$

Volumetric PT for Homogeneous Volumes

```
Color vPT(x, ω)
tmax = nearestSurface(x, ω)
t = -log(1 - randf()) / σt   // Sample free path
if t < tmax: // Volume interaction
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pdf_t = σt * exp(-σt * t)
(ω', pdf_ω') = samplePF(ω)
// Note: transmittance and PF cancel out with PDFs except for a constant factor 1/σt
return Tr(t) / pdf_t * (σa * Le(x, ω) + σs * PF(ω, ω') * vPT(x, ω') / pdf_ω')
else: // Surface interaction
x += tmax * ω
Pr_tmax = exp(-σt * tmax)
(ω', pdf_ω') = sampleBRDF(n, ω)
// Note: transmittance and prob of sampling the distance cancel out
return Tr(tmax) / Pr_tmax * (Le(x, ω) + BRDF(ω, ω') * vPT(x, ω') / pdf_ω')
```

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)}\left[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)\frac{f_p(\vec{\omega},\vec{\omega}_i)L(\mathbf{x}_t,\vec{\omega}_i)}{p(\vec{\omega}_i)}\right] + \frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z,\vec{\omega})$$

Volumetric PT for Homogeneous Volumes

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Color vPT(x, ω)
tmax = nearestSurface(x, ω)
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x += t * ω
pdf_t = σt * exp(-σt * t)
(ω', pdf_ω') = samplePF(ω)
// Note: transmittance and PF cancel out with PDFs except for a constant factor 1/σt
return σa/σt * Le(x, ω) + σs/σt * vPT(x, ω')
else: // Surface interaction
x += tmax * ω
Pr_tmax = exp(-σt * tmax)
(ω', pdf_ω') = sampleBRDF(n, ω)
// Note: transmittance and prob of sampling the distance cancel out
return Le(x, ω) + BRDF(ω, ω') * vPT(x, ω') / pdf_ω'
```

$$\langle L(\mathbf{x},\vec{\omega})\rangle = \frac{T_r(\mathbf{x},\mathbf{x}_t)}{p(t)}\left[\sigma_a(\mathbf{x}_t)L_e(\mathbf{x}_t,\vec{\omega}) + \sigma_s(\mathbf{x}_t)\frac{f_p(\vec{\omega},\vec{\omega}_i)L(\mathbf{x}_t,\vec{\omega}_i)}{p(\vec{\omega}_i)}\right] + \frac{T_r(\mathbf{x},\mathbf{x}_z)}{P(z)}L(\mathbf{x}_z,\vec{\omega})$$

What about heterogeneous media?

Free-path Sampling

Heterogeneous media: $T_r(t) = e^{\int_0^t -\sigma_t(s)ds}$

- Closed-form solutions exist only for simple media
  - e.g. linearly or exponentially varying extinction
- Other solutions:
  - Regular tracking (3D DDA)
  - Ray marching
  - Delta tracking

Free-path Sampling

How to sample the flight distance to the next interaction?

Random variable representing flight distance

$$T(t) = \mathrm{e}^{-\int_0^t \sigma_\mathrm{t}(s)\,\mathrm{d}s} = P(X > t)$$

CDF

$$P(X \leq t) = F(t)$$

Partition of unity

$$F(t) = 1 - T(t)$$

Recipe for generating samples

Free-path Sampling

Cumulative distribution function (**CDF**)

$$F(t) = 1 - T(t) = 1 - \mathrm{e}^{-\tau(t)}$$

Probability density function (**PDF**)

$$p(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(1 - \mathrm{e}^{-\tau(t)}\right) = \sigma_{\mathrm{t}}(t)\mathrm{e}^{-\tau(t)}$$

Inverted cumulative distr. function (**CDF^{-1}**)

$$\xi = 1 - \mathrm{e}^{-\tau(t)}$$

Solve for t

$$\int_0^t \sigma_{\mathrm{t}}(s)\,\mathrm{d}s = -\ln(1-\xi)$$

Approaches for finding t:

1) **ANALYTIC (closed-form CDF^{-1})**
2) **SEMI-ANALYTIC (regular tracking)**
3) **APPROXIMATE (ray marching)**

Regular Tracking (Semi-Analytic)

For piecewise-simple (e.g. piecewise-constant), summation replaces integration

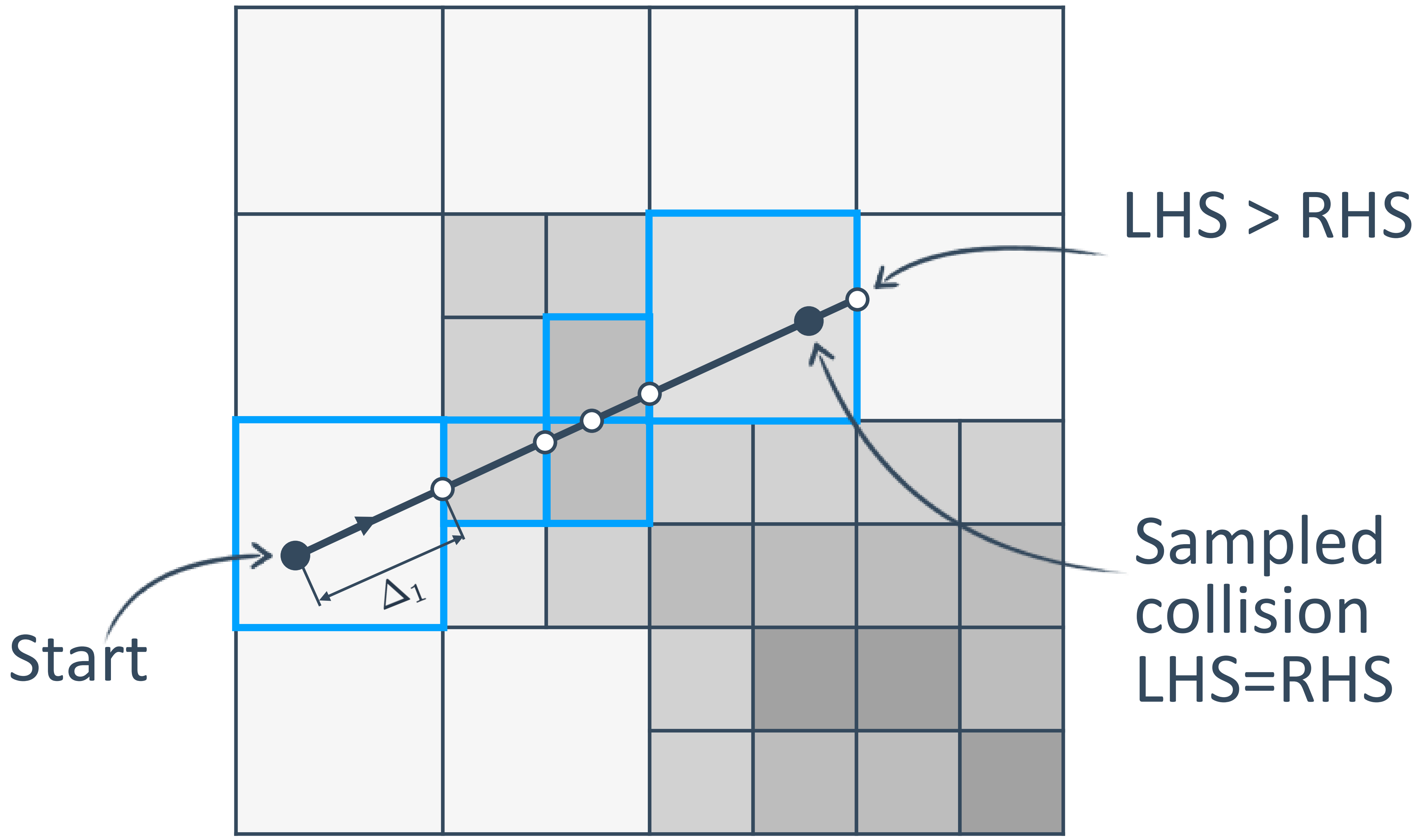
$$\int_0^t \sigma_t(s)\,ds = -\ln(1-\xi)$$

$$\sum_{i=1}^{k} \sigma_{t,i}\,\Delta_i = -\ln(1-\xi)$$

Regular tracking:

1) Draw a random number ξ
2) While LHS < RHS
   move to the next intersection
3) Find the exact location
   in the last segment analytically

(Hierarchical) voxel grid



Ray Marching

Find the collision distance approximately

$$\int_0^t \sigma_t(s)\,ds = -\ln(1-\xi)$$

$$\not\approx$$

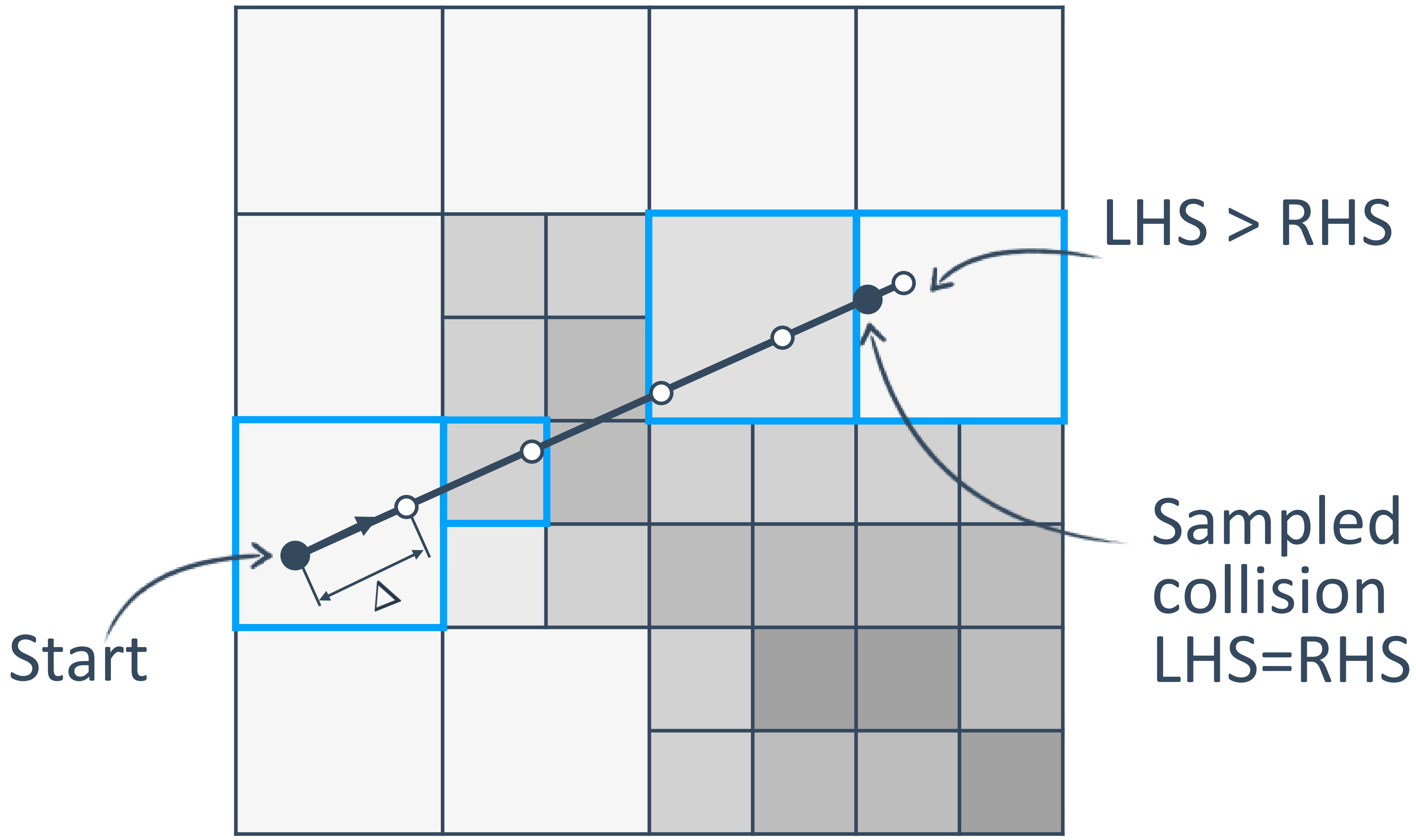
$$\sum_{i=1}^{k} \sigma_{t,i}\,\Delta = -\ln(1-\xi)$$

Constant step

Ray marching:

1) Draw a random number ξ
2) While LHS < RHS
   make a (fixed-size) step
3) Find the exact location
   in the last segment analytically

(Hierarchical) voxel grid



Ray Marching

Find the collision distance approximately

$$\int_0^t \sigma_t(s)\,\mathrm{d}s = -\ln(1-\xi)$$

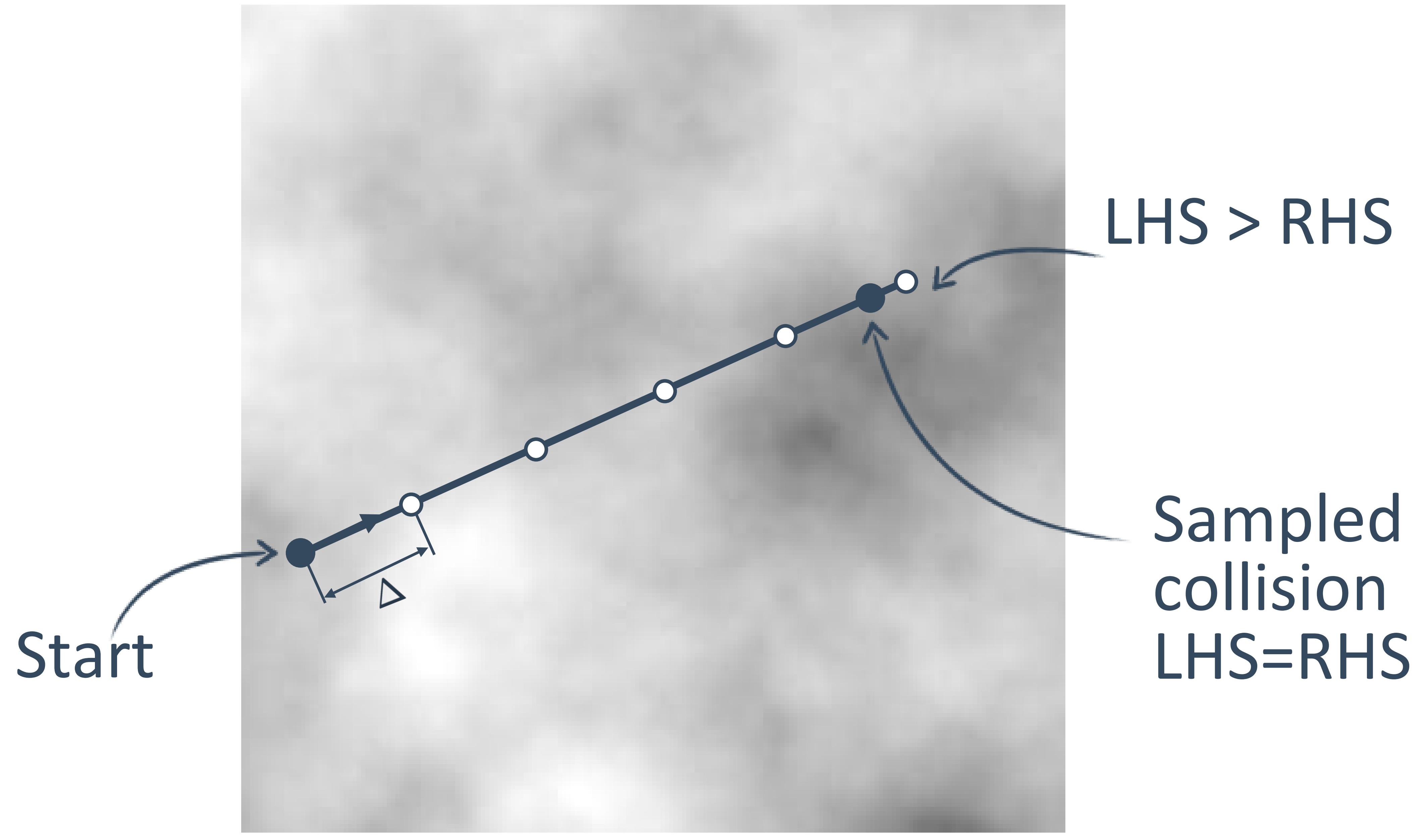
$$\sum_{i=1}^{k} \sigma_{t,i}\,\Delta = -\ln(1-\xi)$$

Constant step

Ray marching:

1) Draw a random number ξ
2) While LHS < RHS
   make a (fixed-size) step
3) Find the exact location
   in the last segment analytically

General volume



Ray Marching

Find the collision distance approximately

$$\int_0^t \sigma_t(s)\, ds = -\ln(1-\xi)$$

$$\neq$$

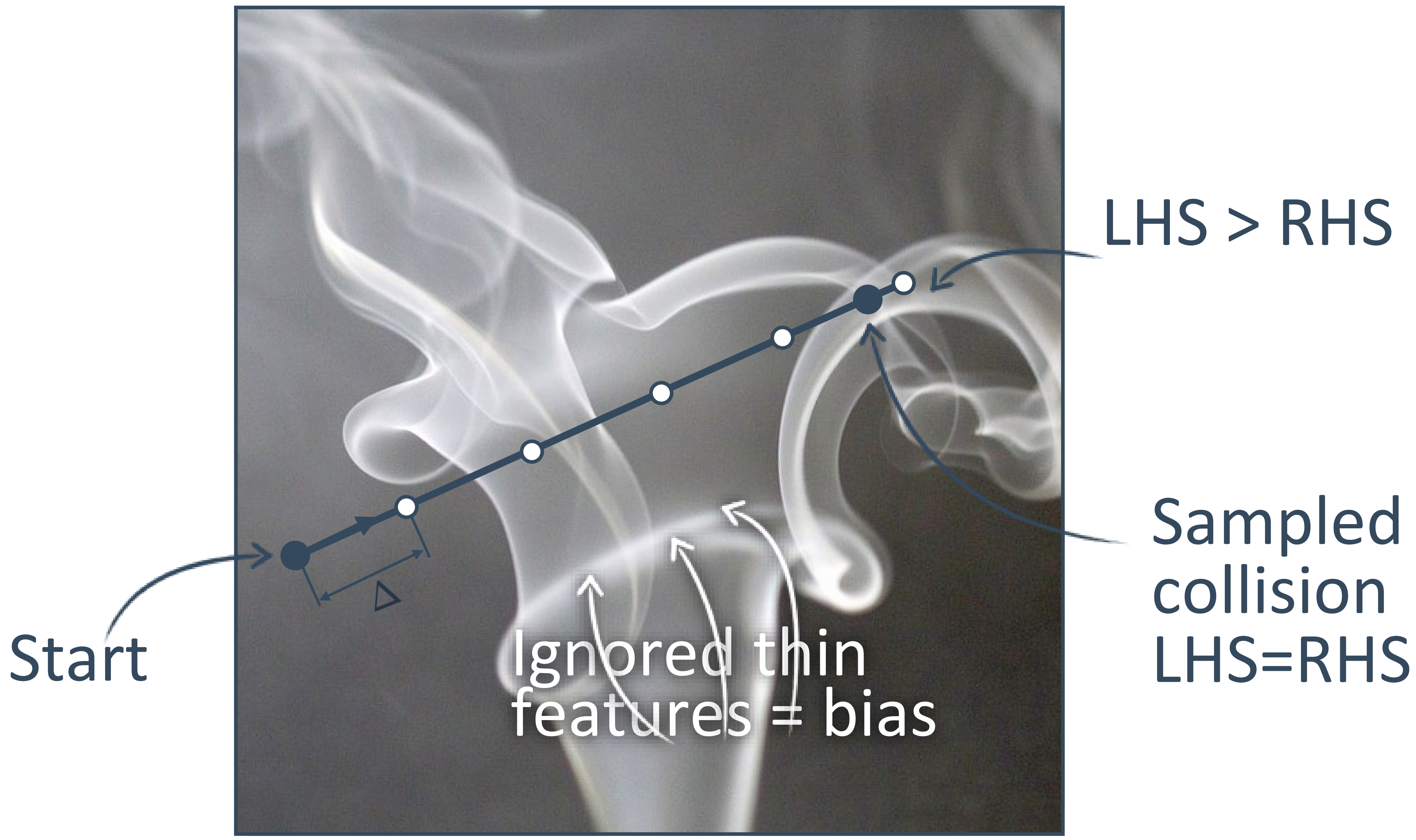
$$\sum_{i=1}^{k} \sigma_{t,i}\, \Delta = -\ln(1-\xi)$$

Constant step

Ray marching:

1) Draw a random number ξ
2) While LHS < RHS
   make a (fixed-size) step
3) Find the exact location
   in the last segment analytically

General volume



Free-path Sampling

ANALYTIC CDF^{-1}	REGULAR TRACKING	RAY MARCHING
▶ Efficient & simple, limited to few volumes	▶ Iterative, inefficient if free paths cross many boundaries	▶ Iterative, inaccurate (or inefficient) for media with high frequencies
▶ Simple volumes (e.g. homogeneous)	▶ Piecewise-simple volumes	▶ Any volume
▶ Unbiased	▶ Unbiased	▶ Biased

Common approach: sample optical thickness, find corresponding distance

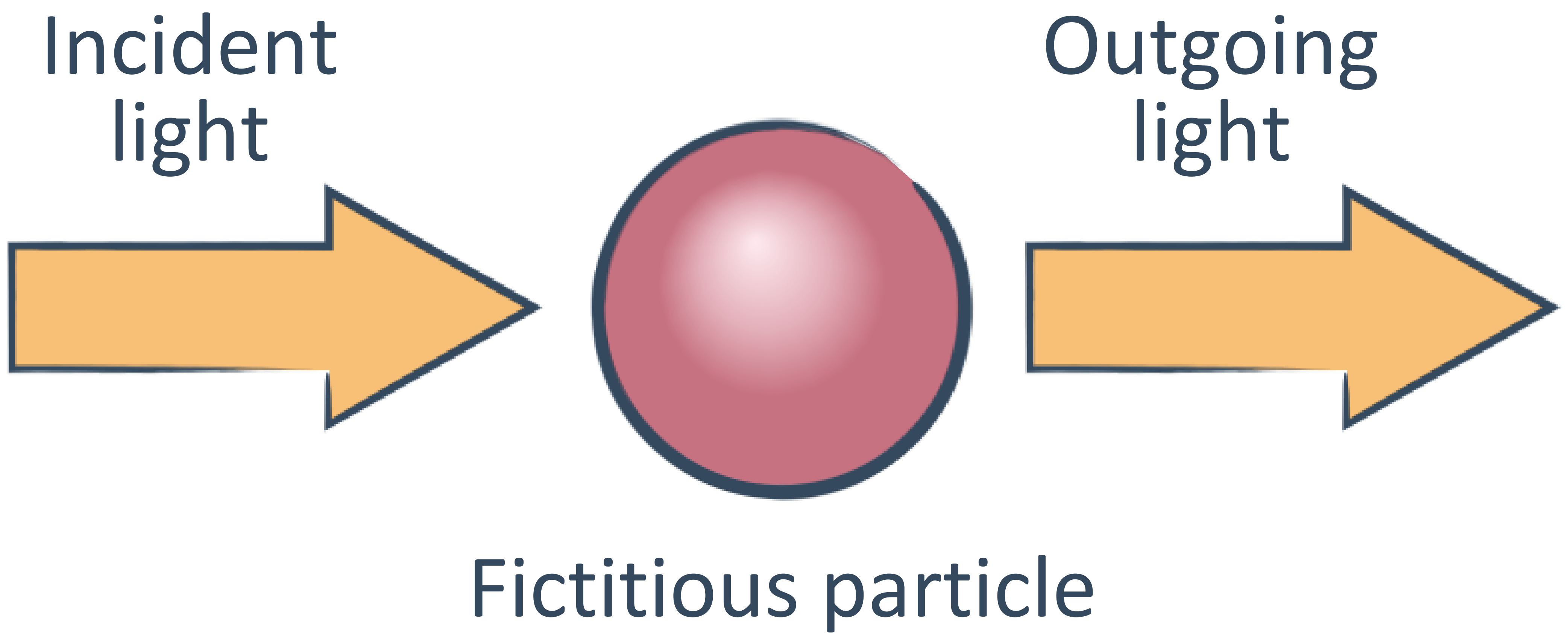
Delta Tracking

(a.k.a. Woodcock tracking, pseudo scattering, hole tracking, null-collision method,…)

Delta tracking idea

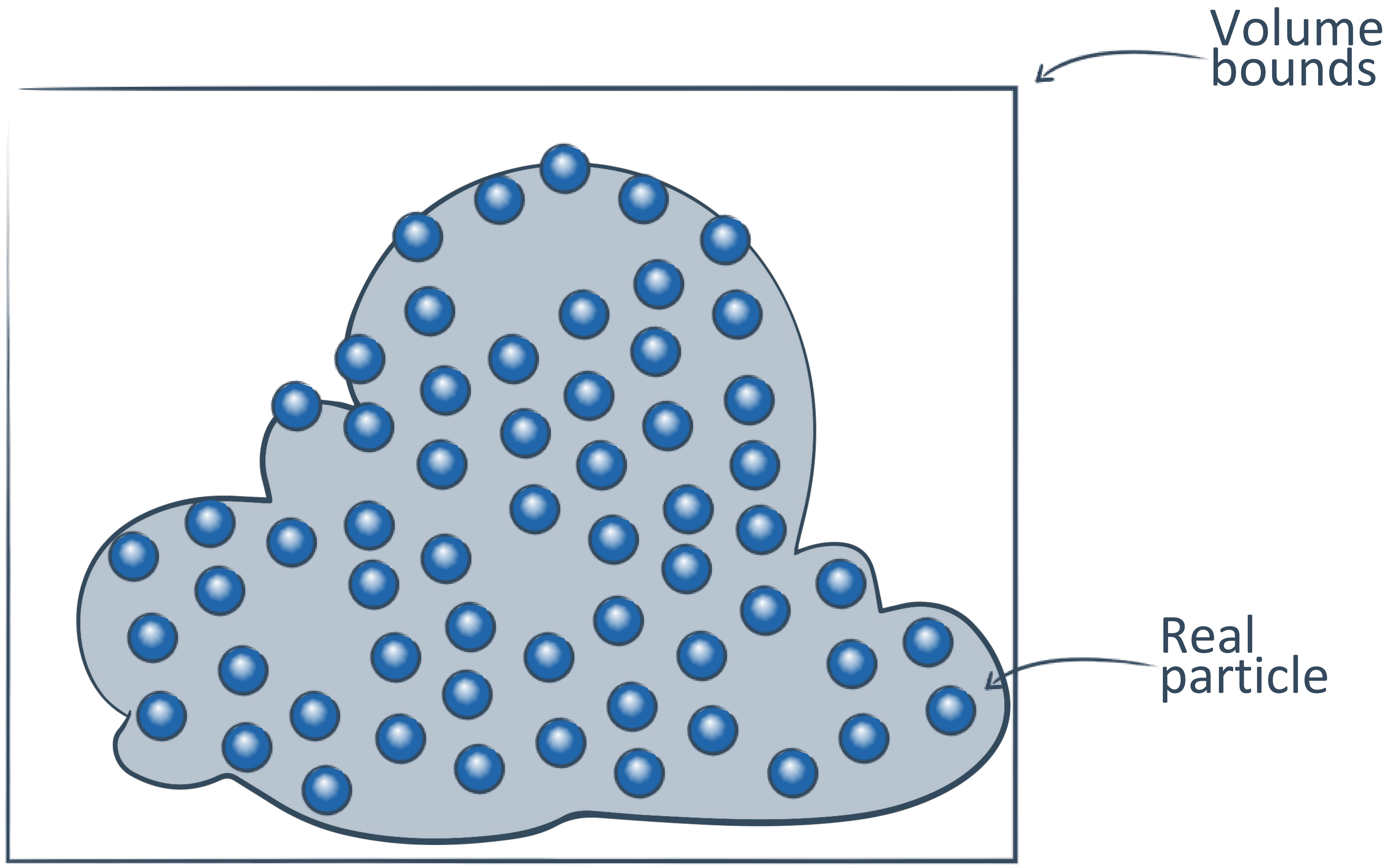
Add **FICTITIOUS MATTER** to homogenize medium

- albedo: $\alpha(\mathbf{x}) = 1$
- phase function: $f_p(\vec{\omega}, \vec{\omega}') = \delta(\vec{\omega} - \vec{\omega}')$

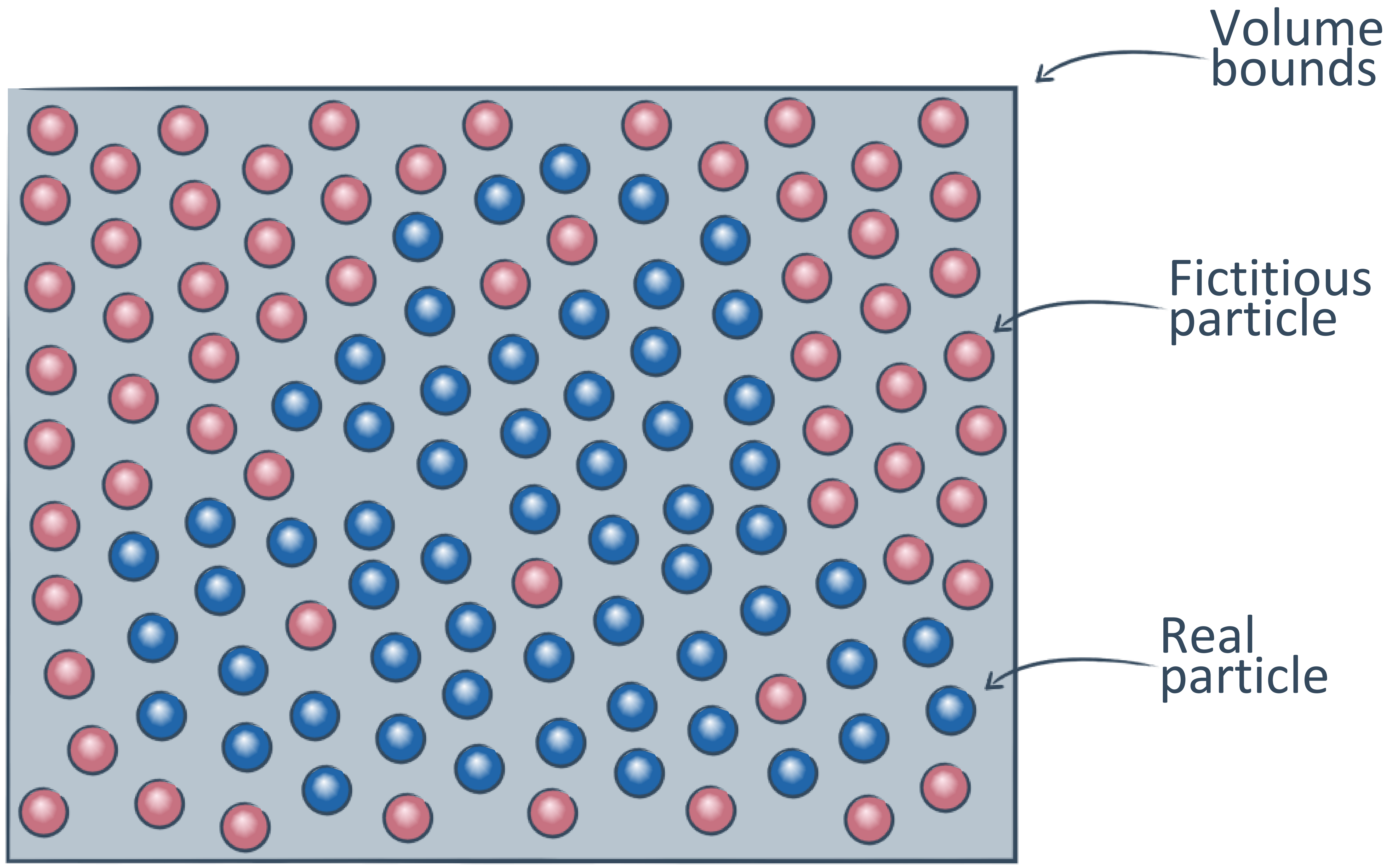


Presence of fictitious matter
does not impact light transport

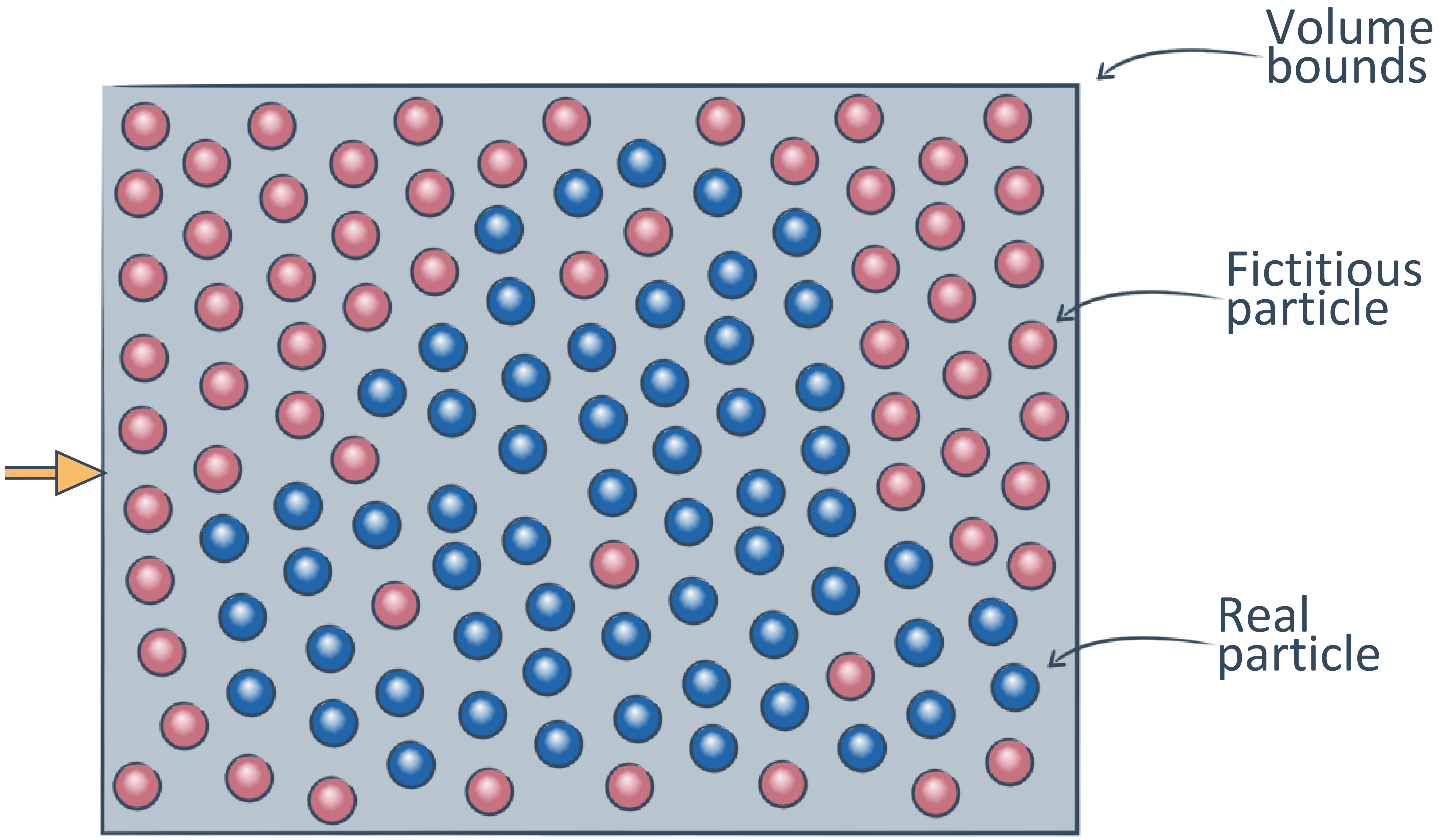
Homogenization



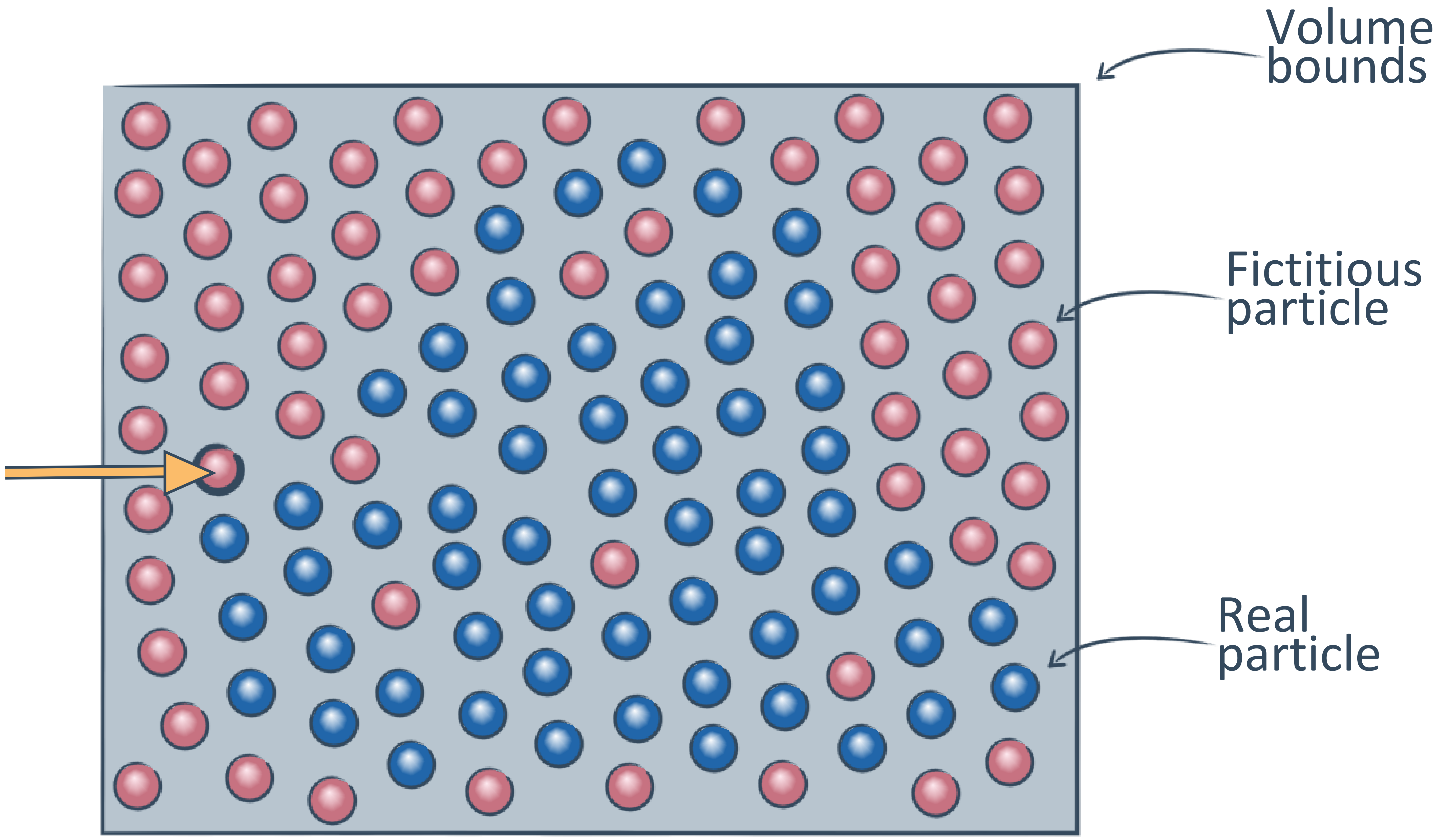
Homogenization



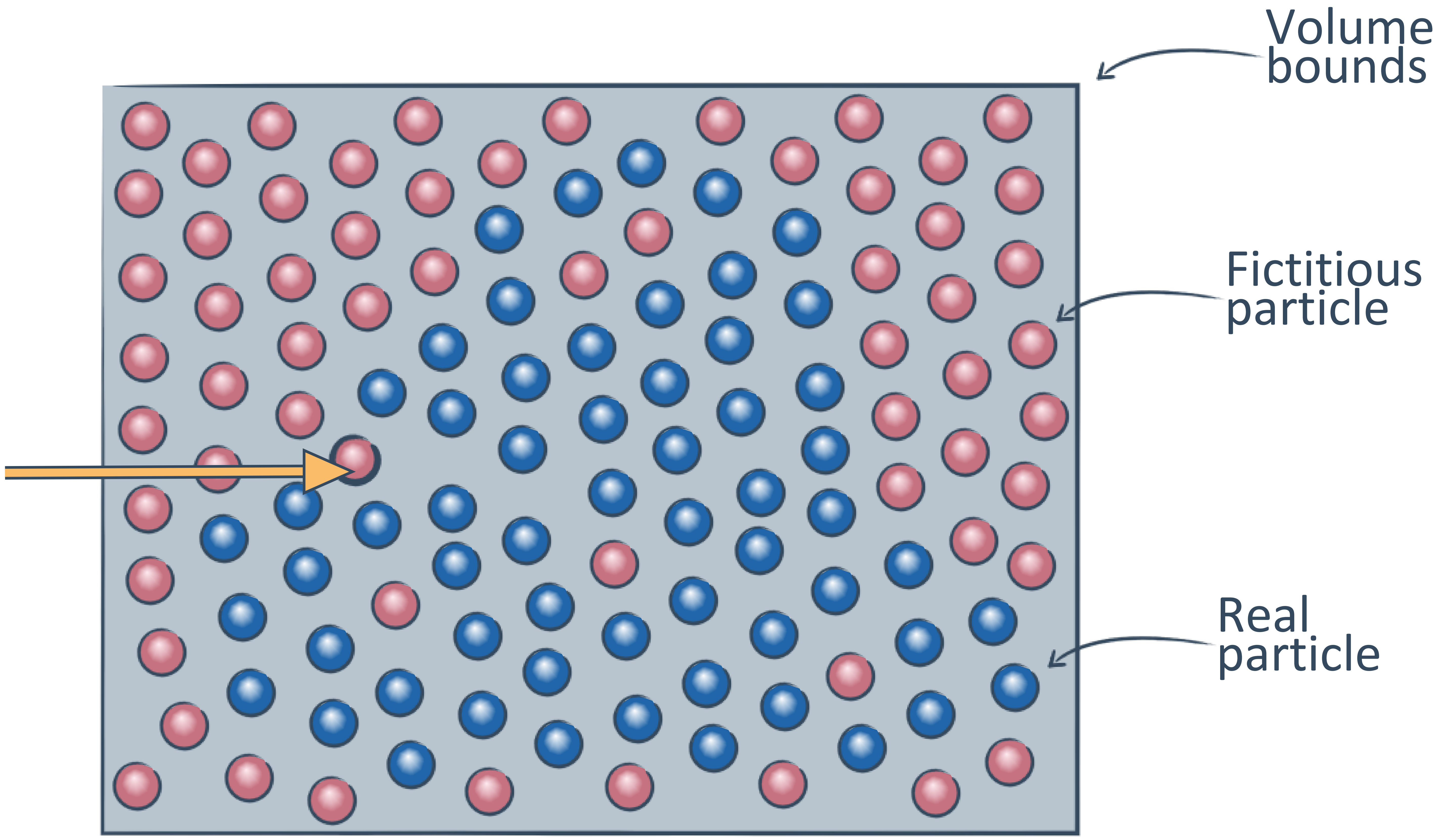
Homogenization



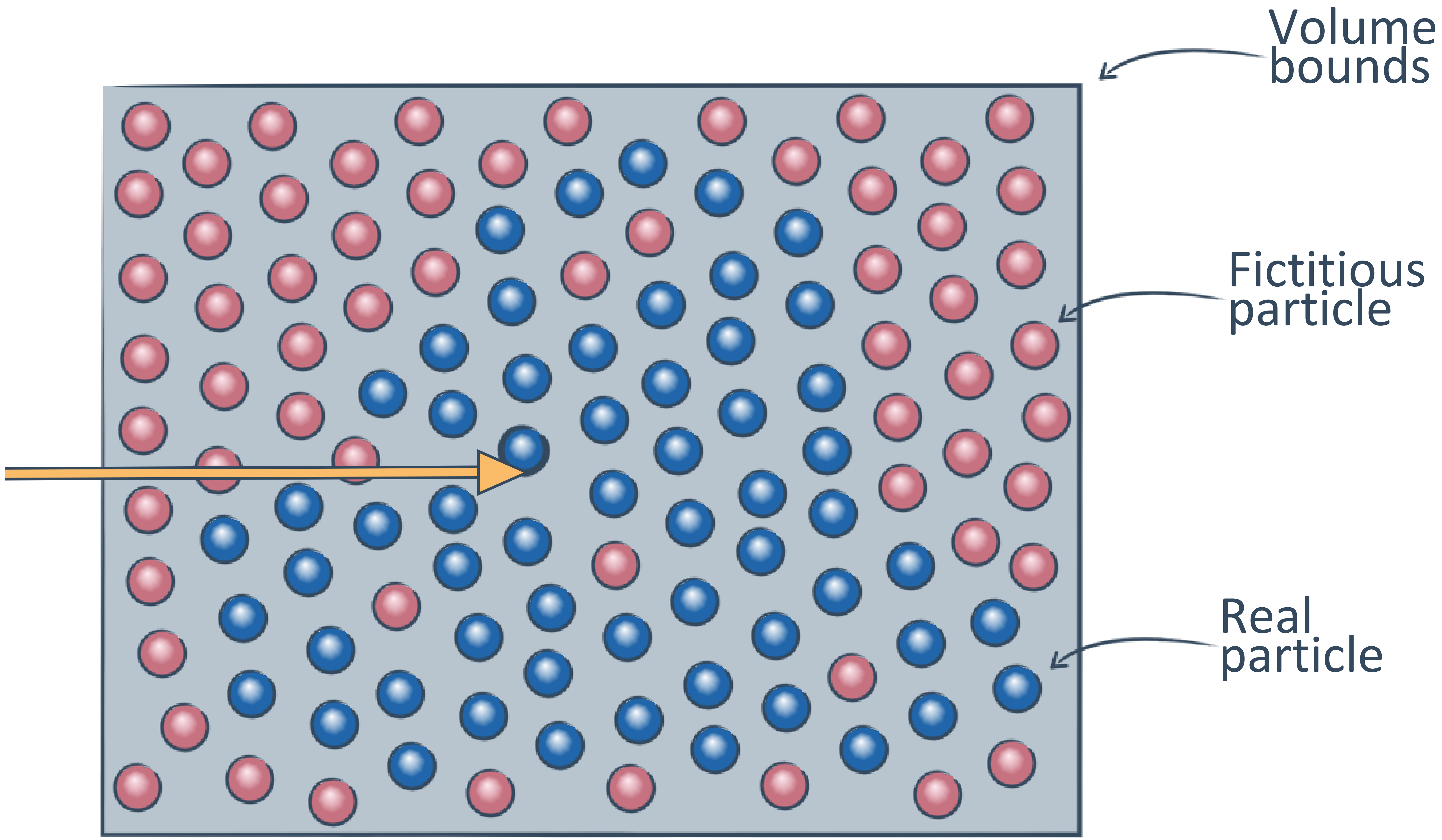
Homogenization



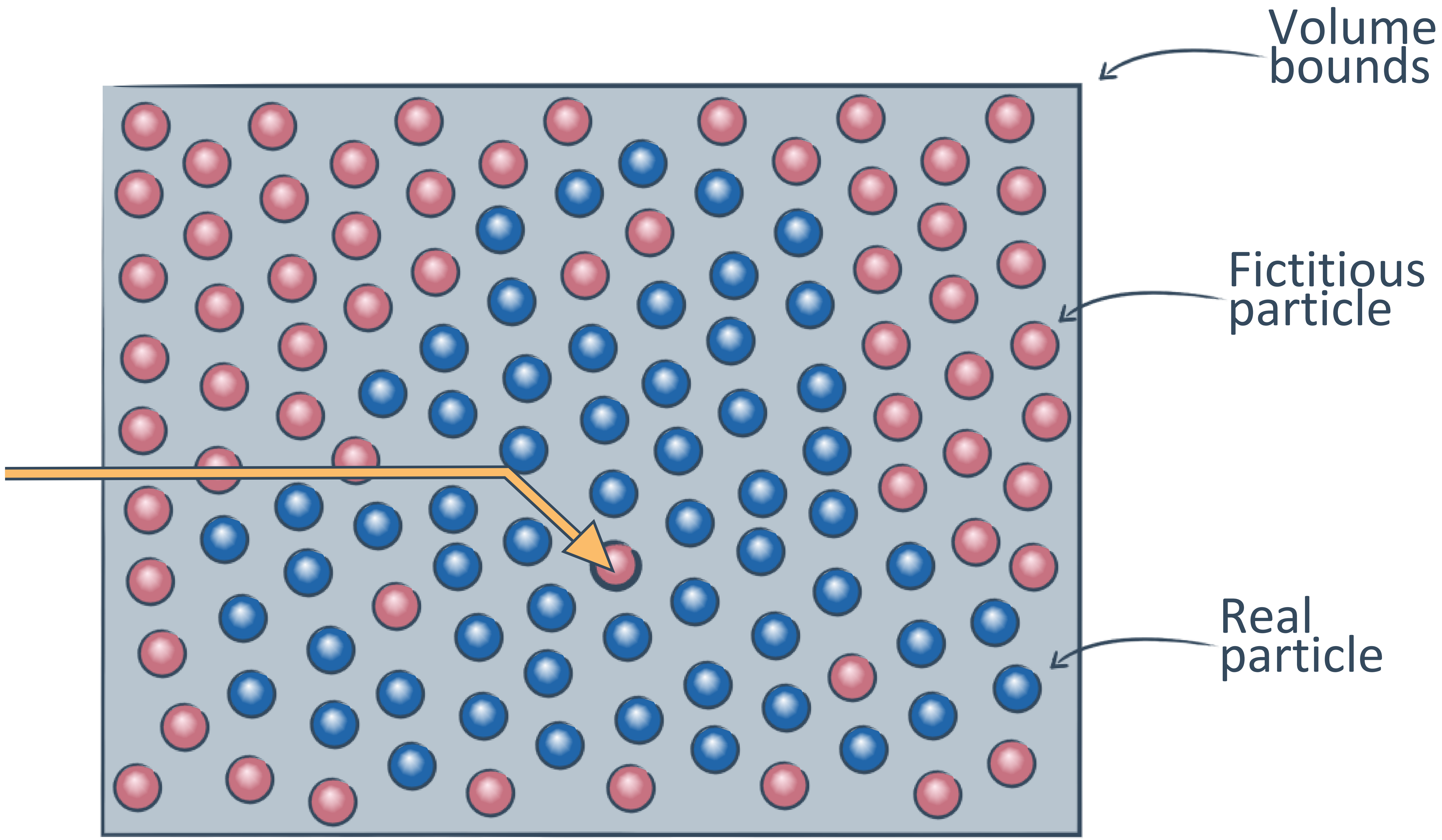
Homogenization



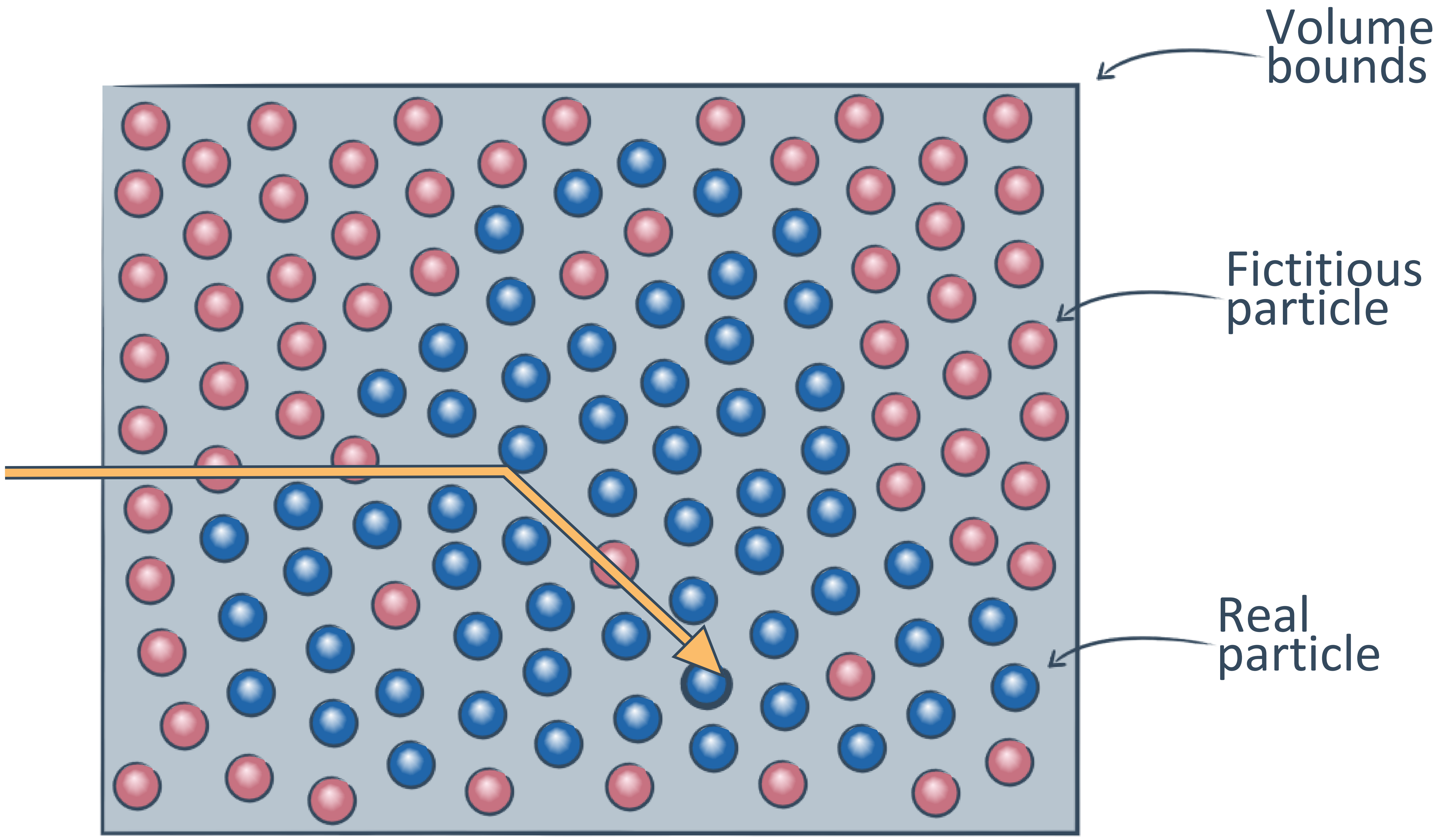
Homogenization



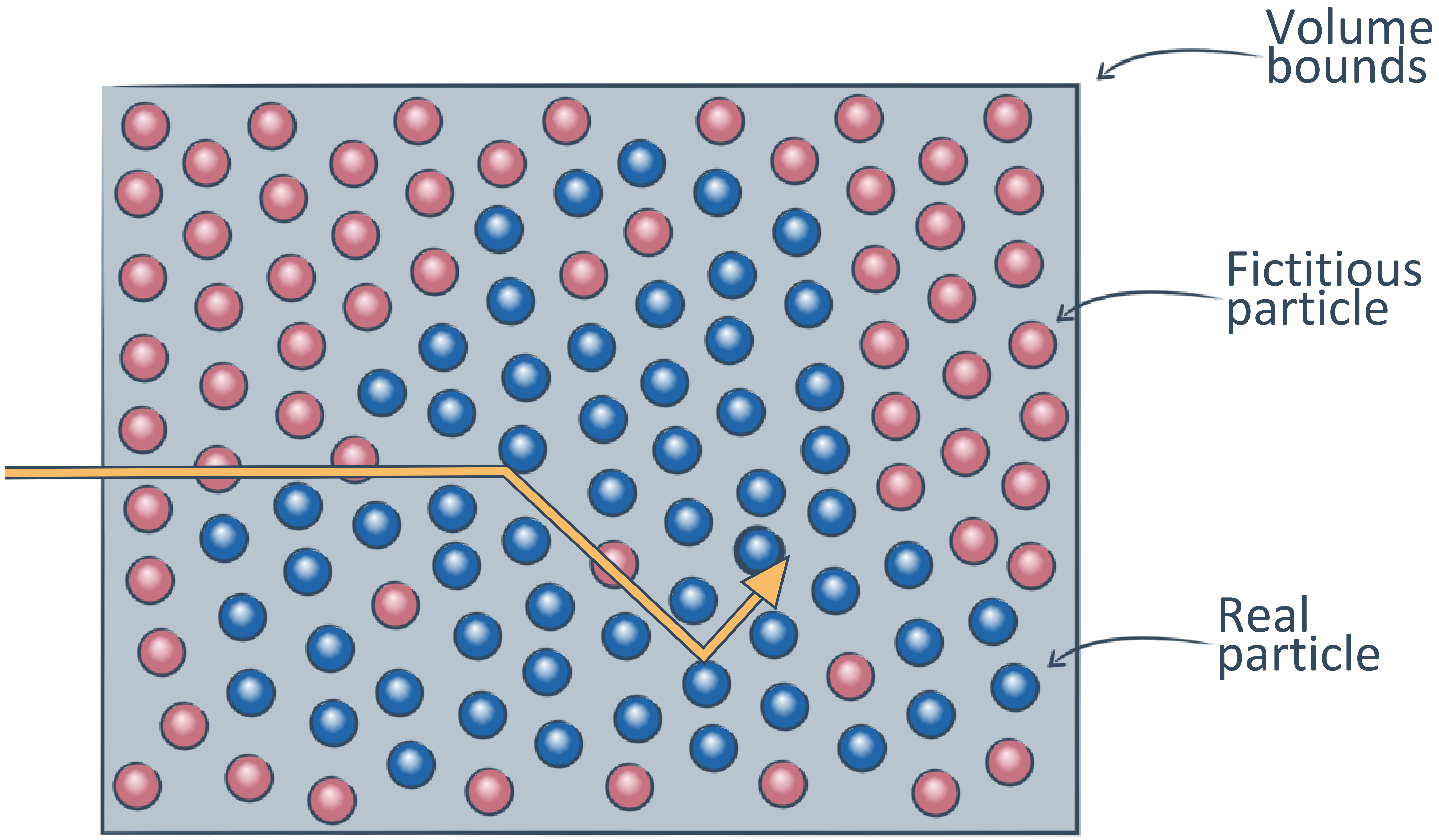
Homogenization



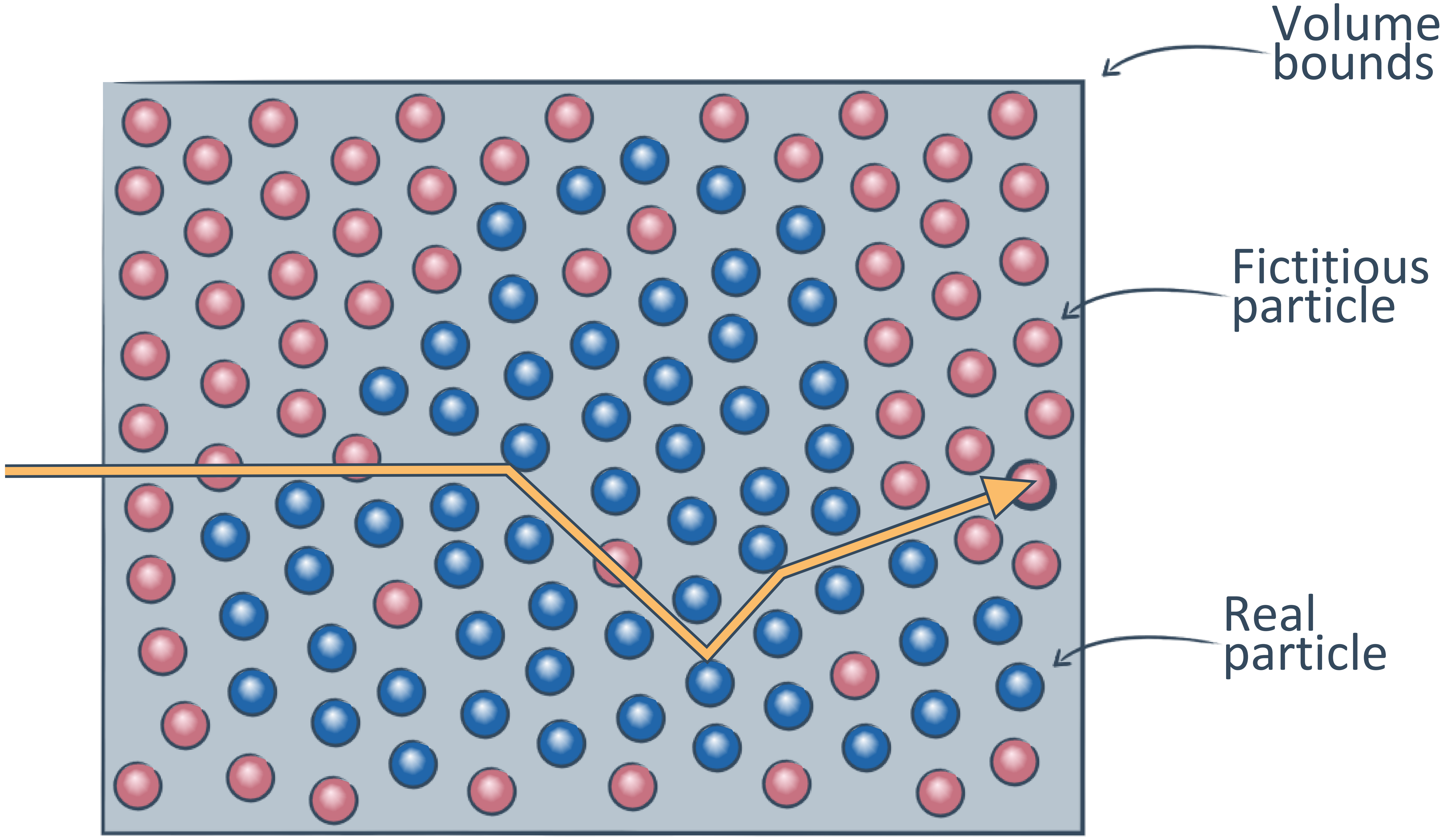
Homogenization



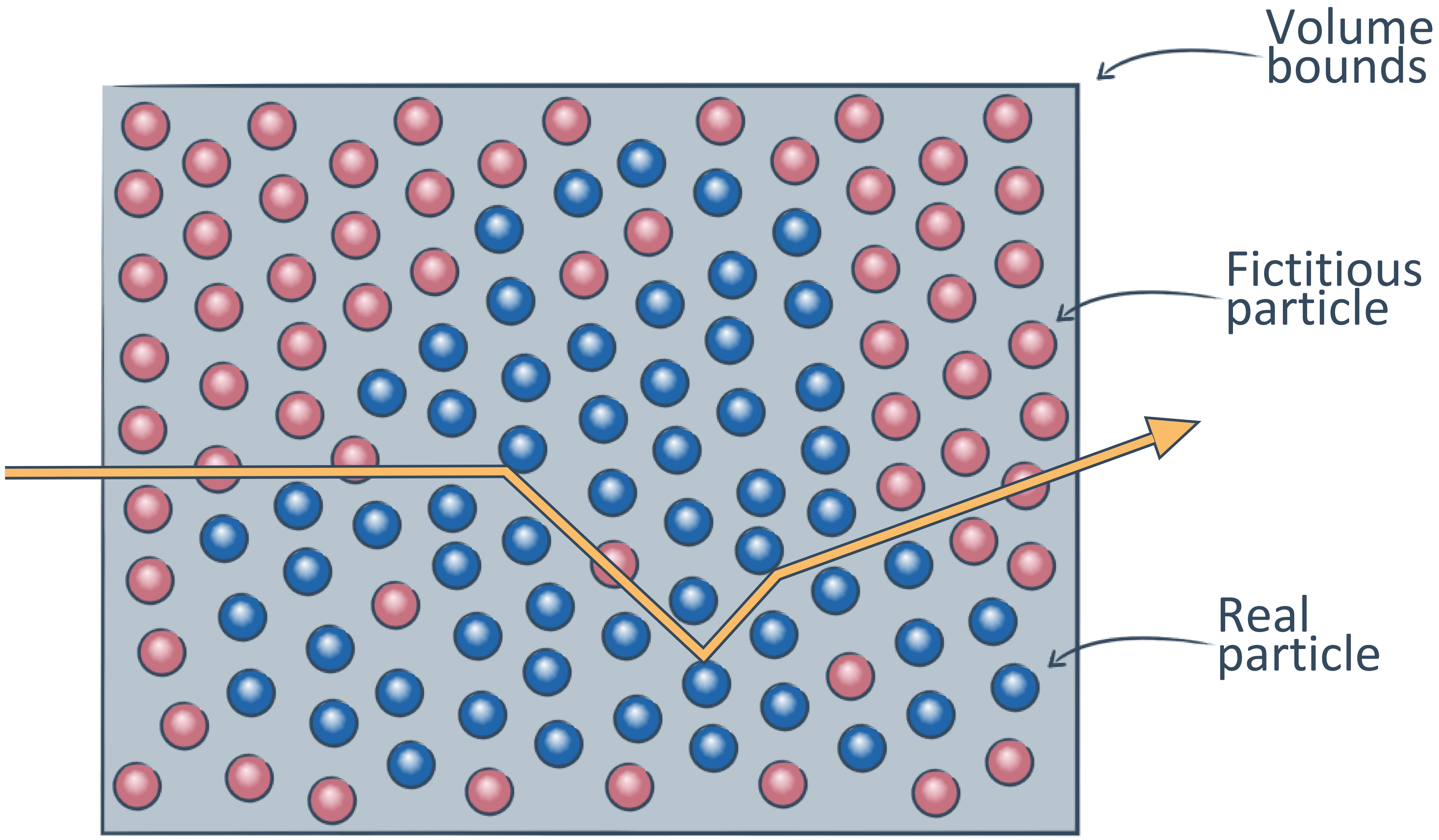
Homogenization



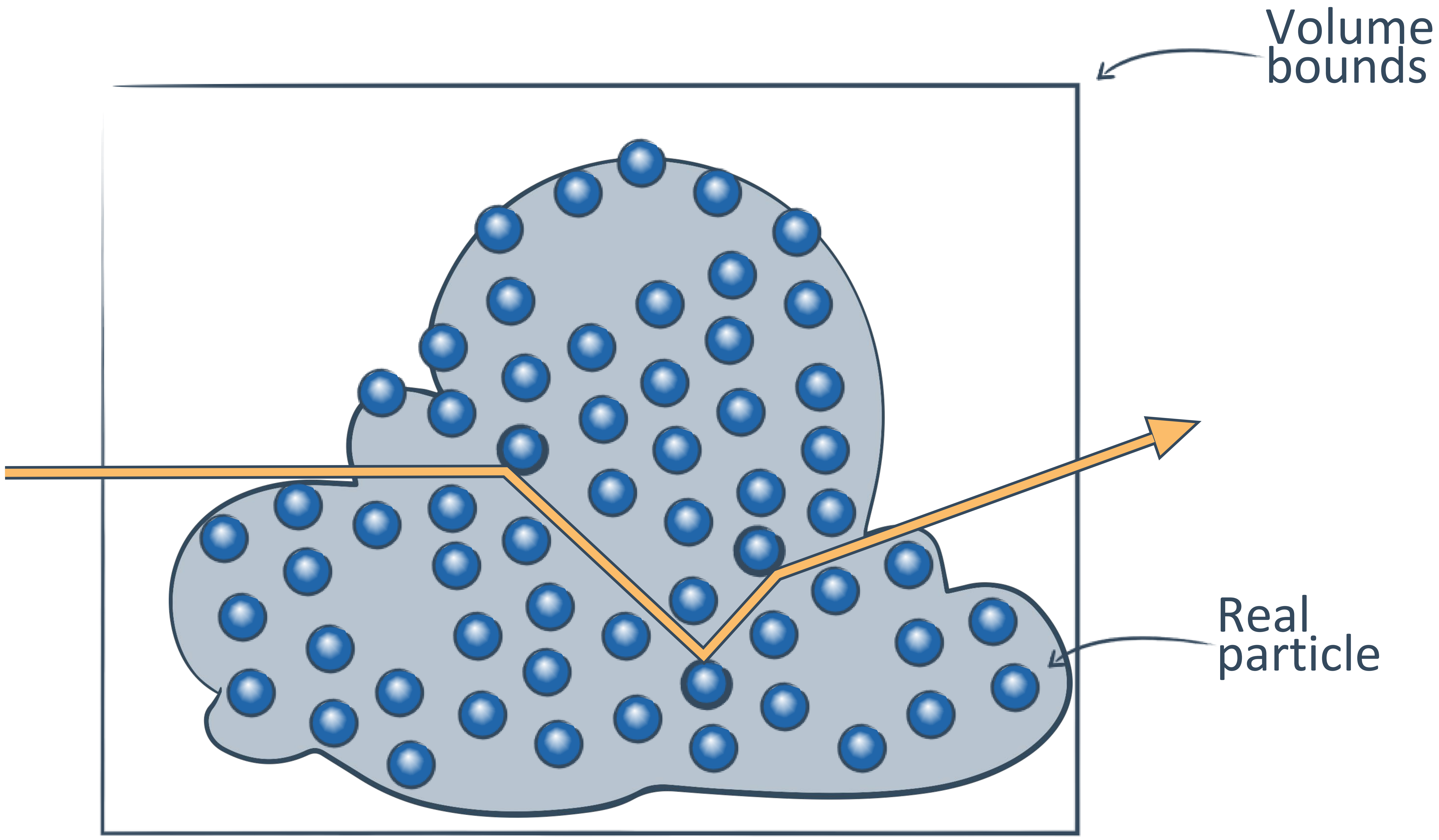
Homogenization



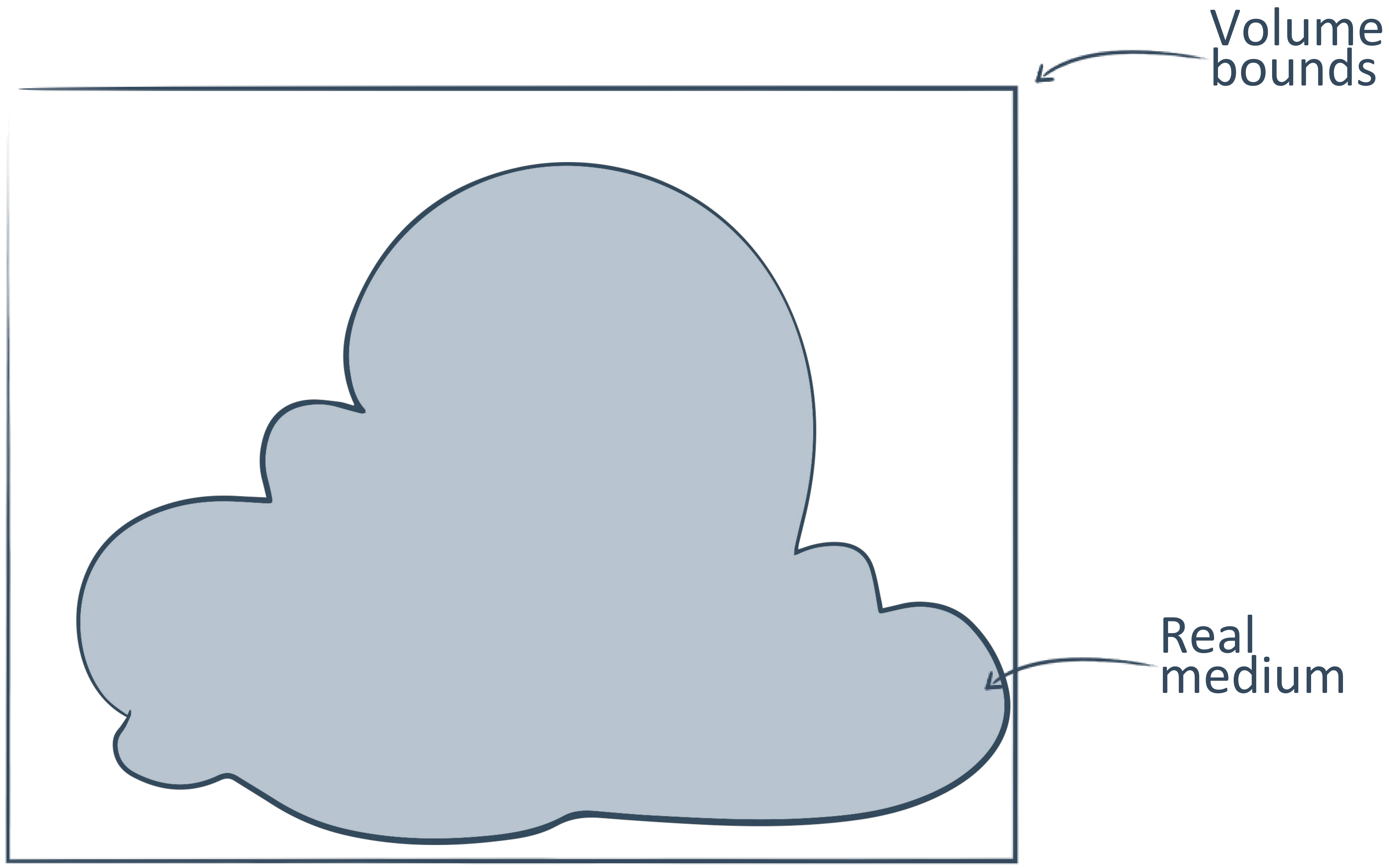
Homogenization



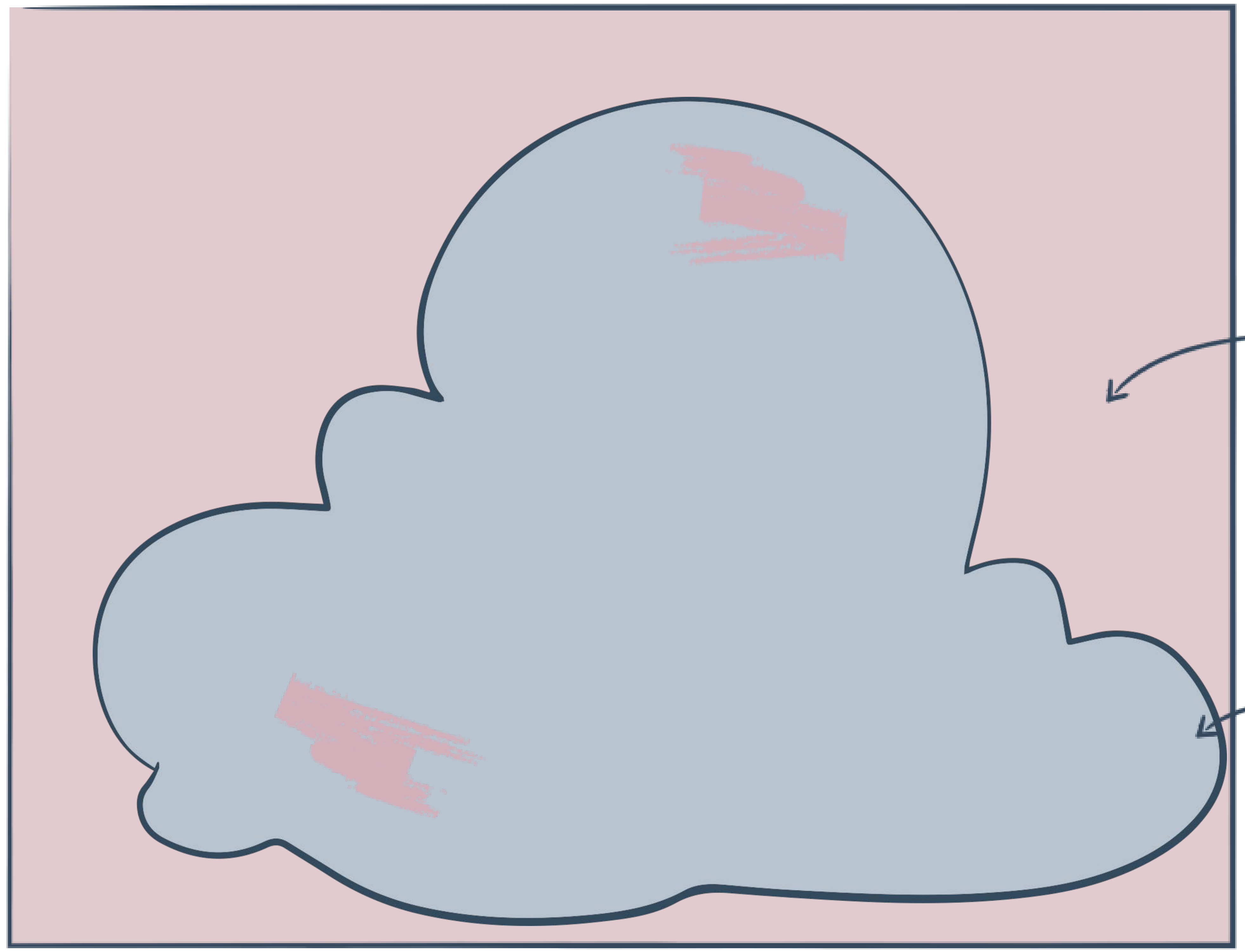
Homogenization



Stochastic Sampling



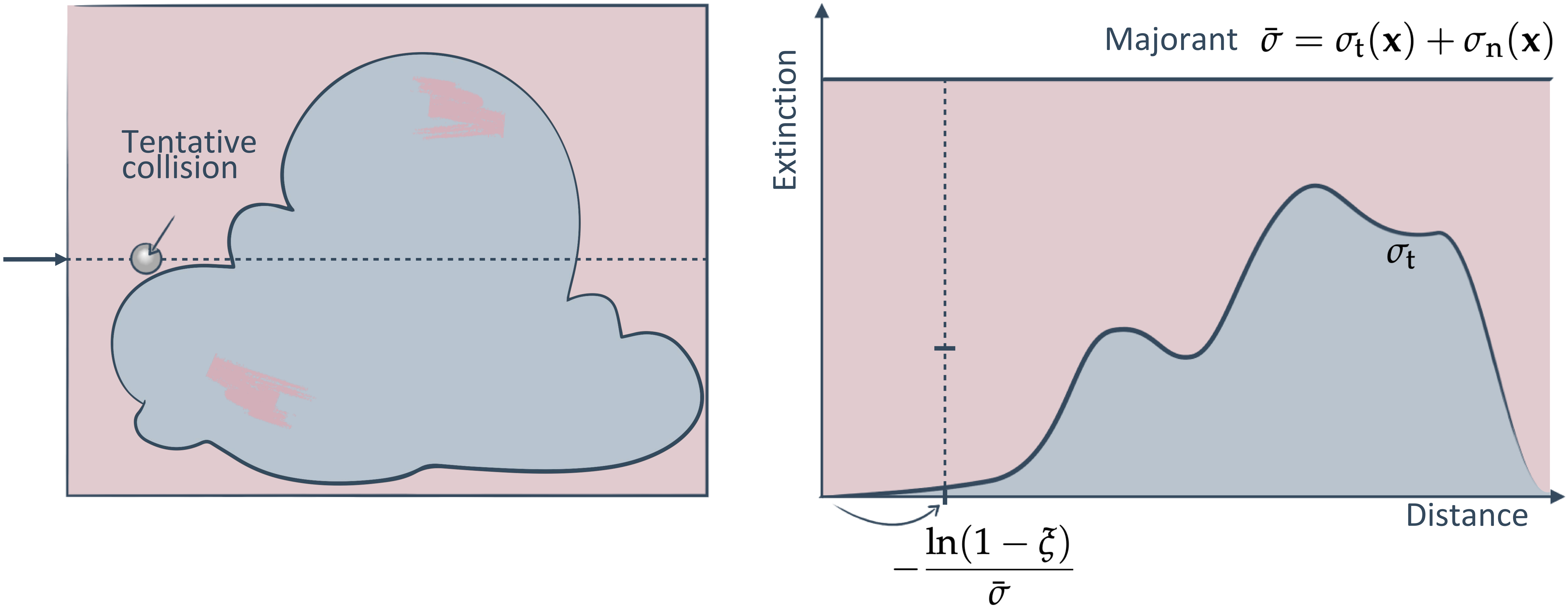
Stochastic Sampling



$$\bar{\sigma} = \sigma_{\mathrm{t}}(\mathbf{x}) + \sigma_{\mathrm{n}}(\mathbf{x})$$

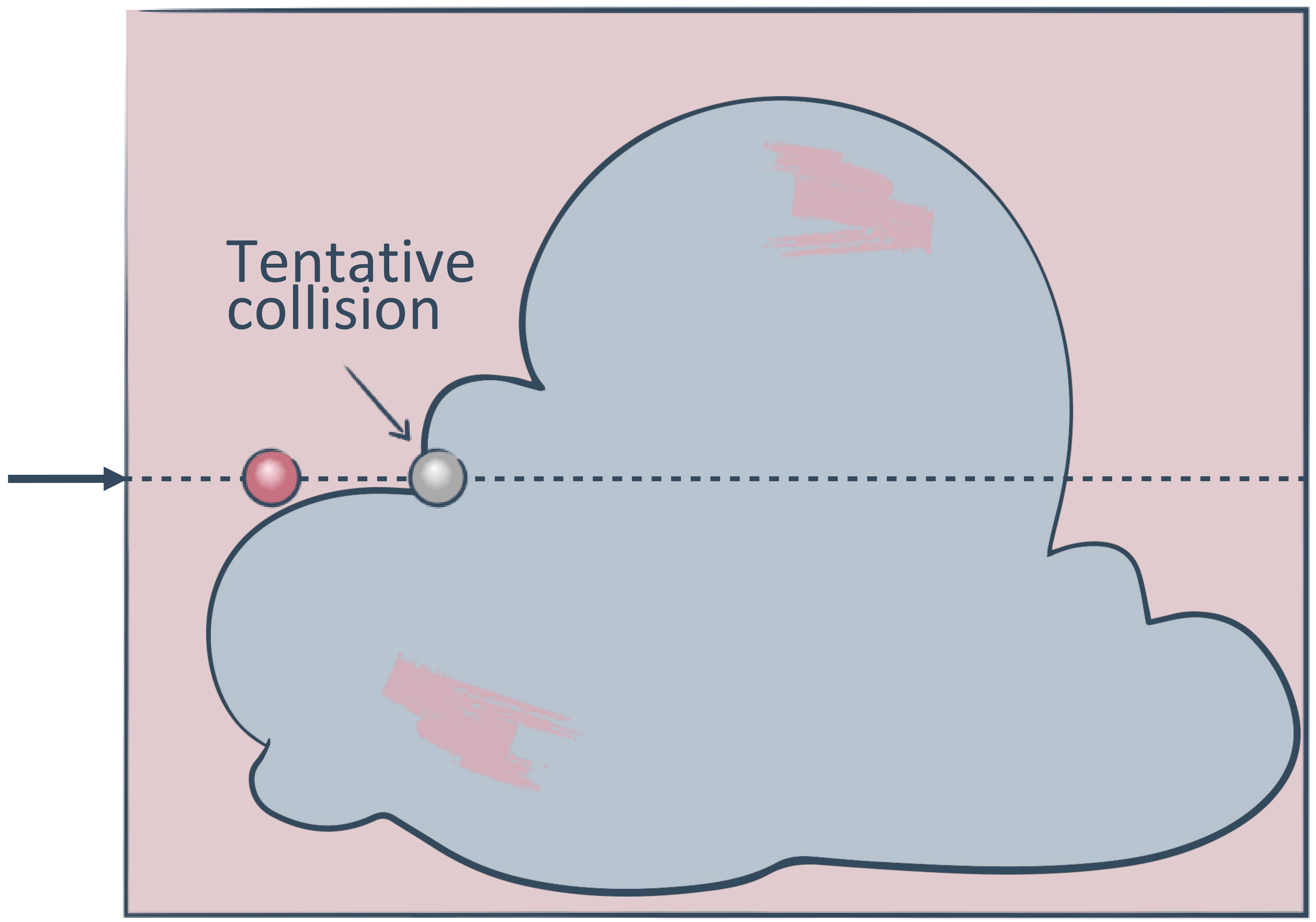
Stochastic Sampling

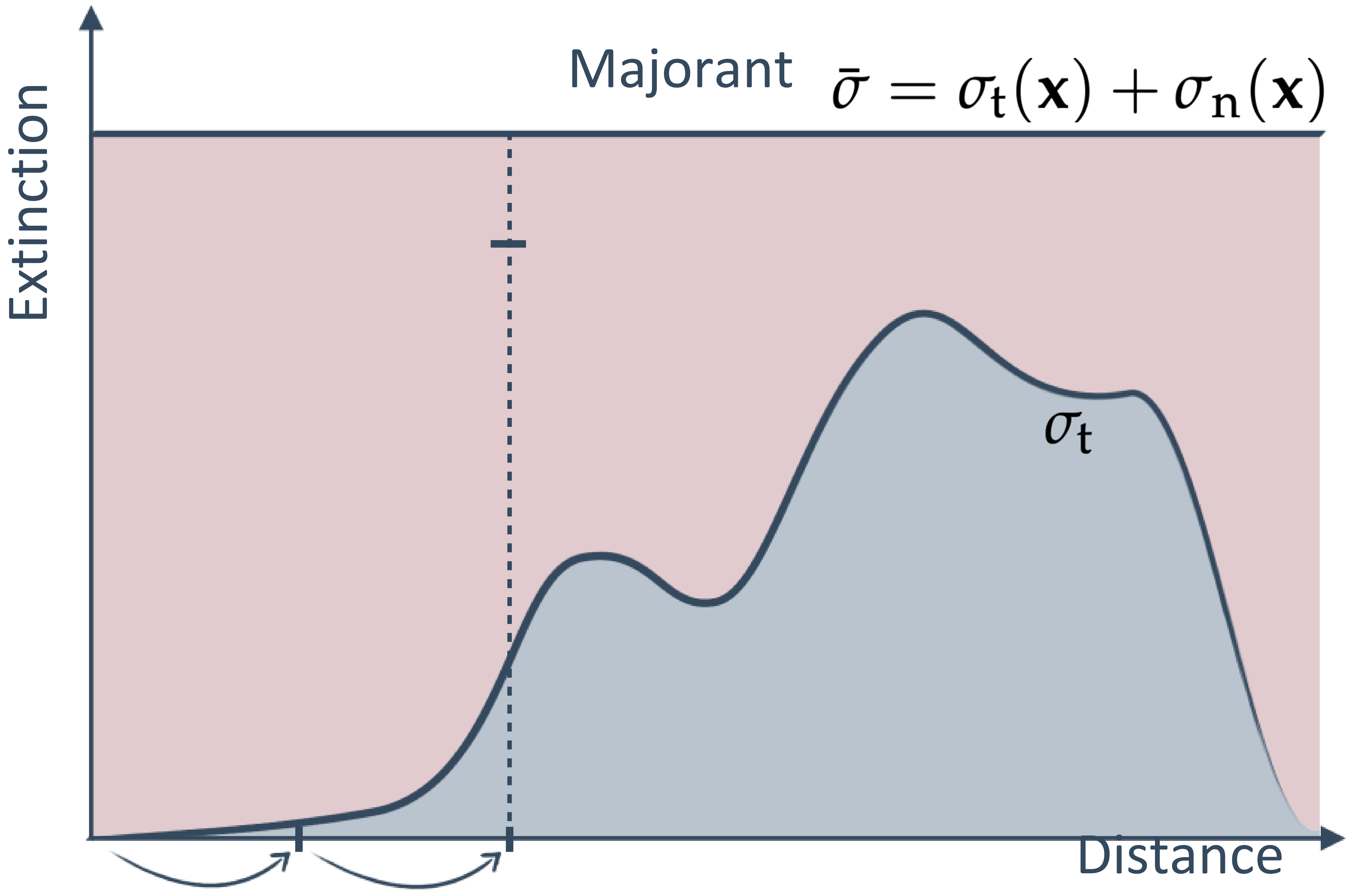
$$P_{\mathrm{r}}(\mathbf{x}) = \frac{\sigma_{\mathrm{t}}(\mathbf{x})}{\bar{\sigma}} \qquad P_{\mathrm{n}}(\mathbf{x}) = \frac{\sigma_{\mathrm{n}}(\mathbf{x})}{\bar{\sigma}}$$



Stochastic Sampling

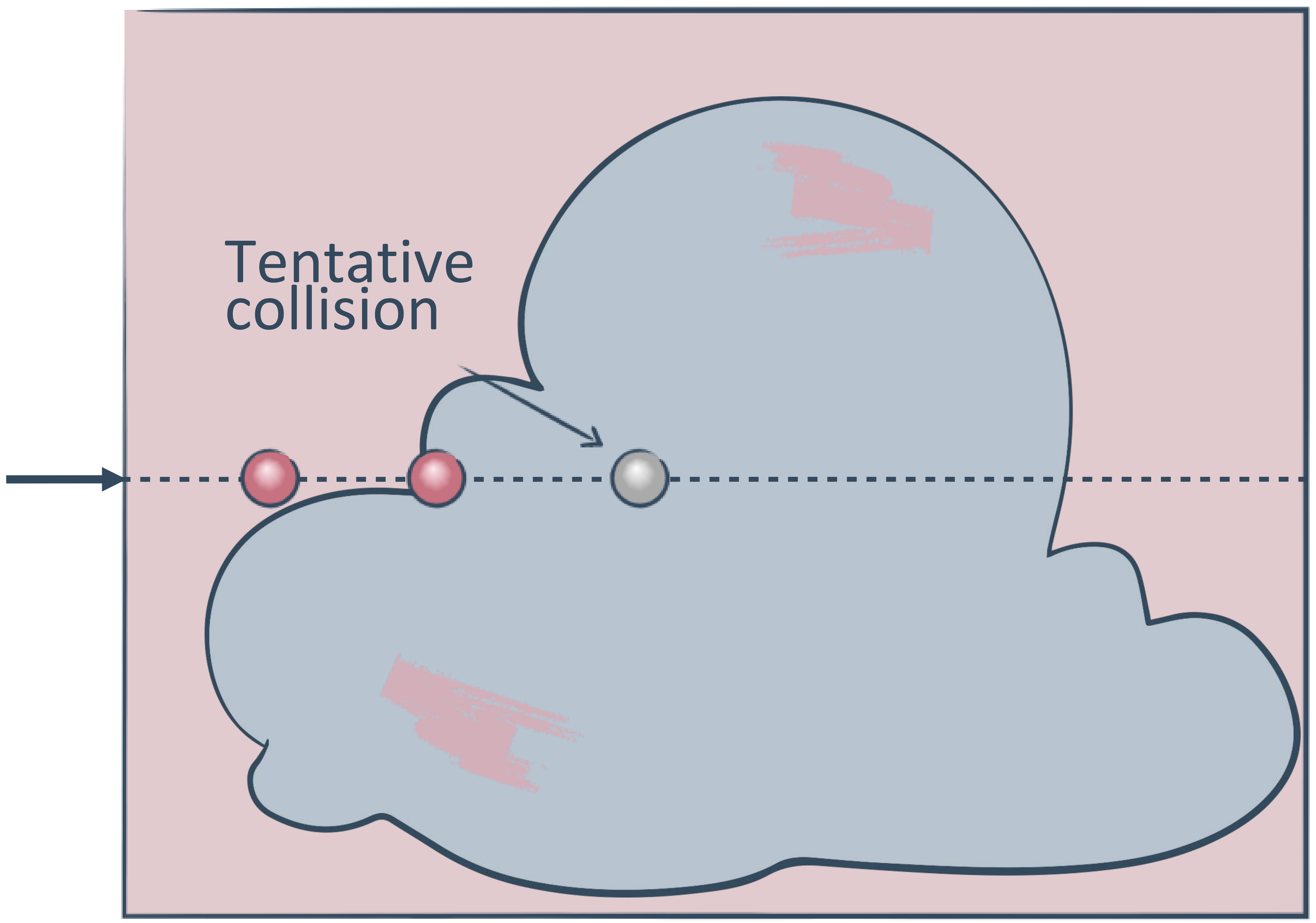
$$P_{\mathrm{r}}(\mathbf{x}) = \frac{\sigma_{\mathrm{t}}(\mathbf{x})}{\bar{\sigma}} \qquad P_{\mathrm{n}}(\mathbf{x}) = \frac{\sigma_{\mathrm{n}}(\mathbf{x})}{\bar{\sigma}}$$

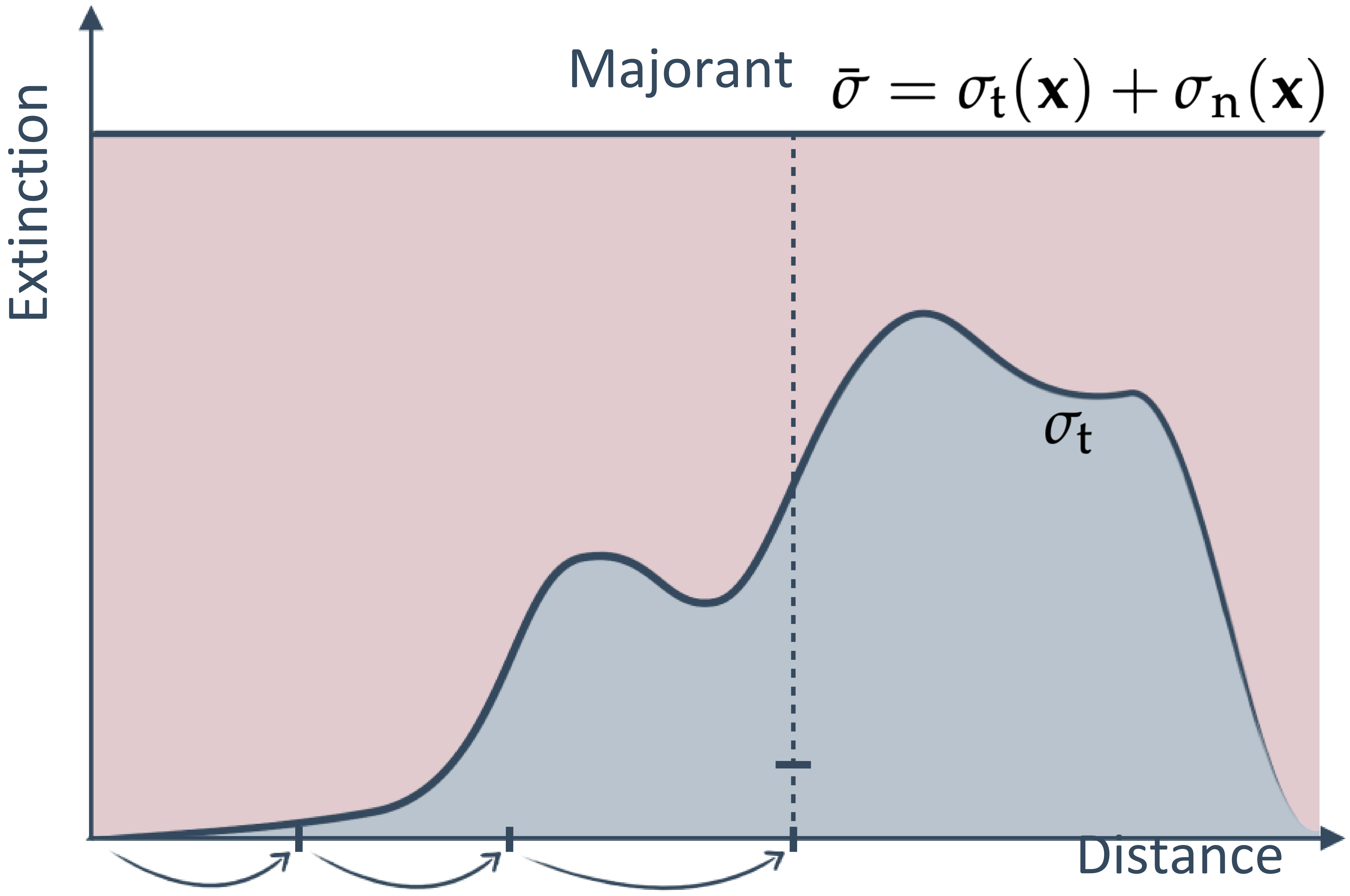




Stochastic Sampling

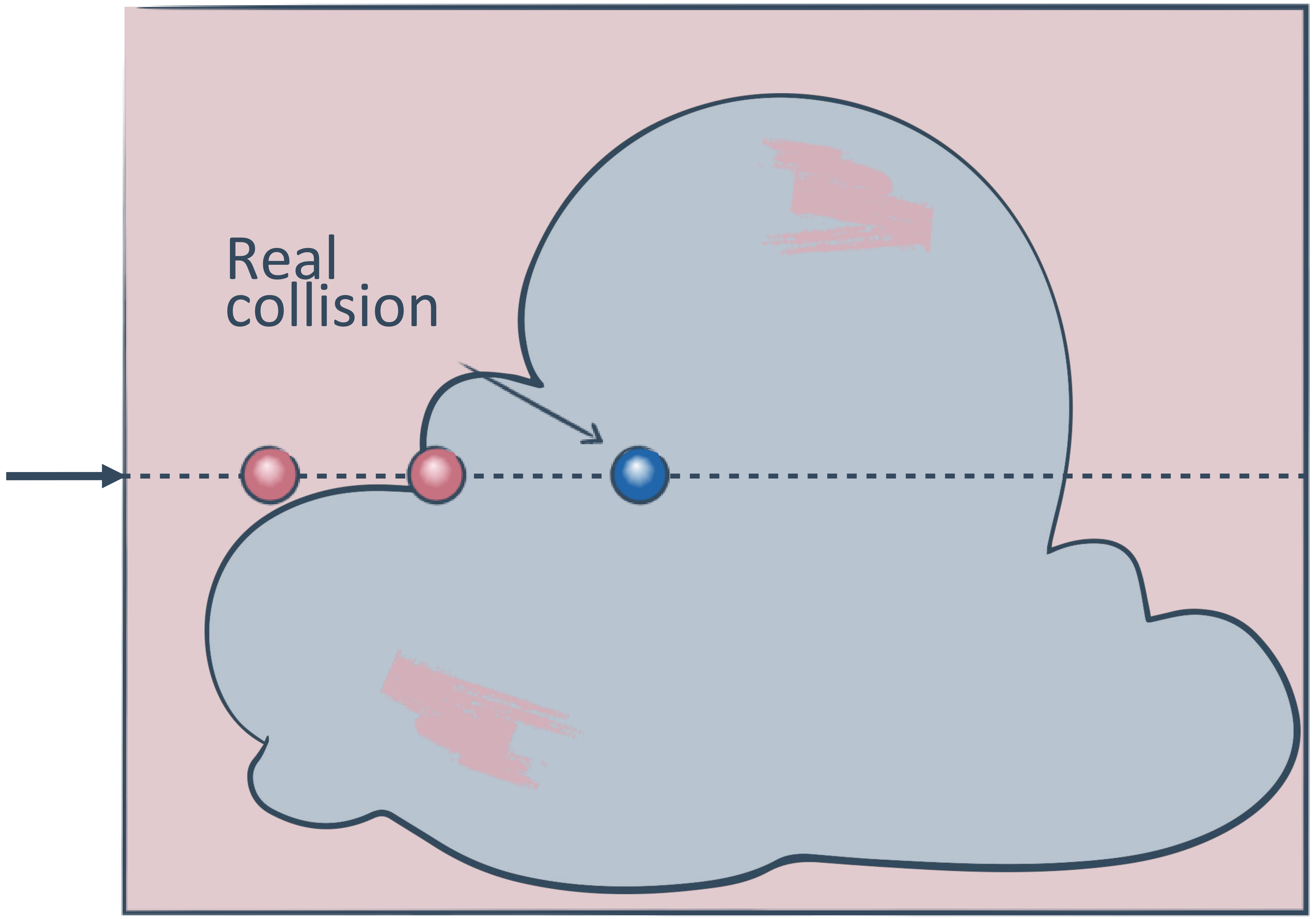
$$P_{\mathrm{r}}(\mathbf{x}) = \frac{\sigma_{\mathrm{t}}(\mathbf{x})}{\bar{\sigma}} \qquad P_{\mathrm{n}}(\mathbf{x}) = \frac{\sigma_{\mathrm{n}}(\mathbf{x})}{\bar{\sigma}}$$

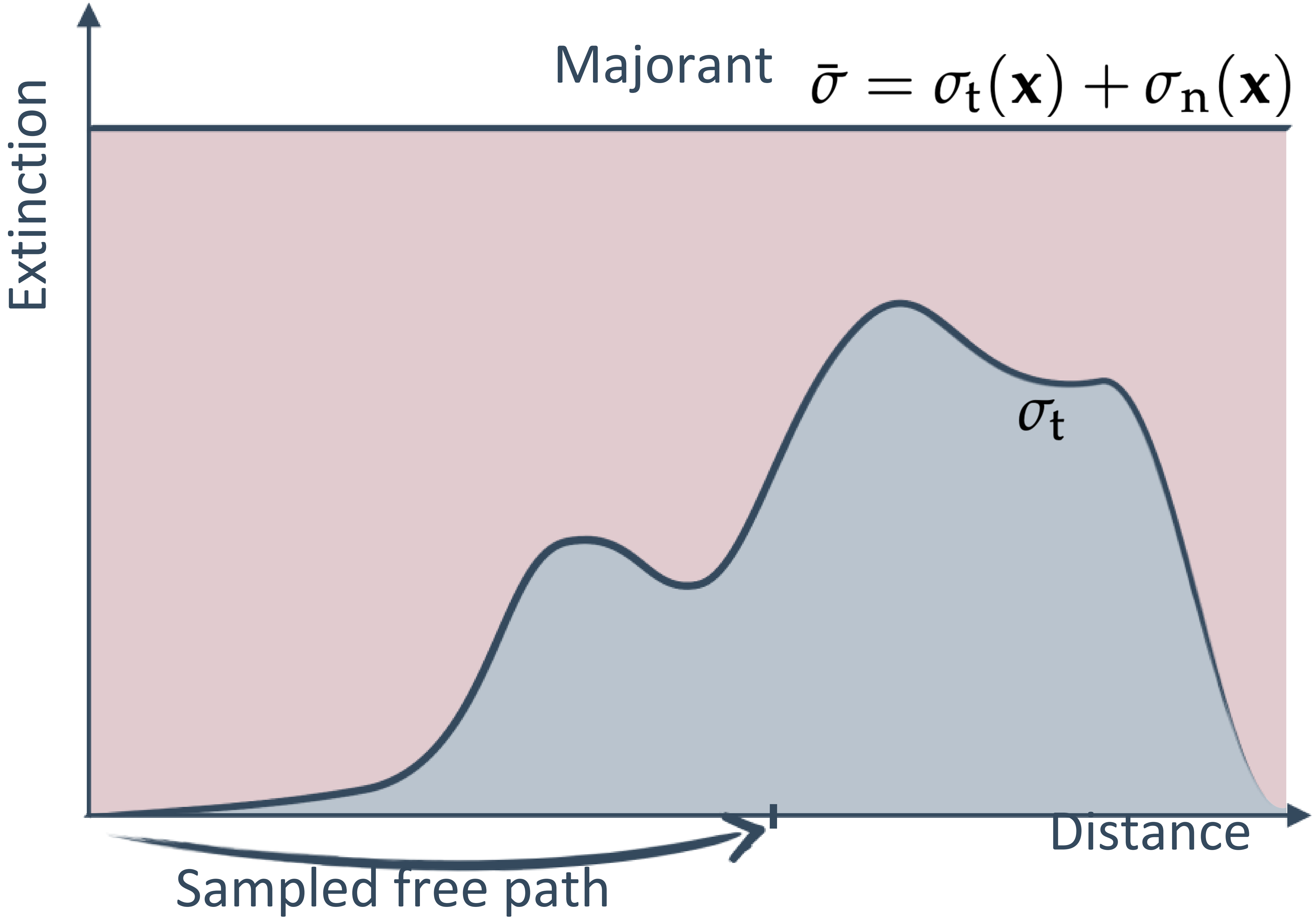




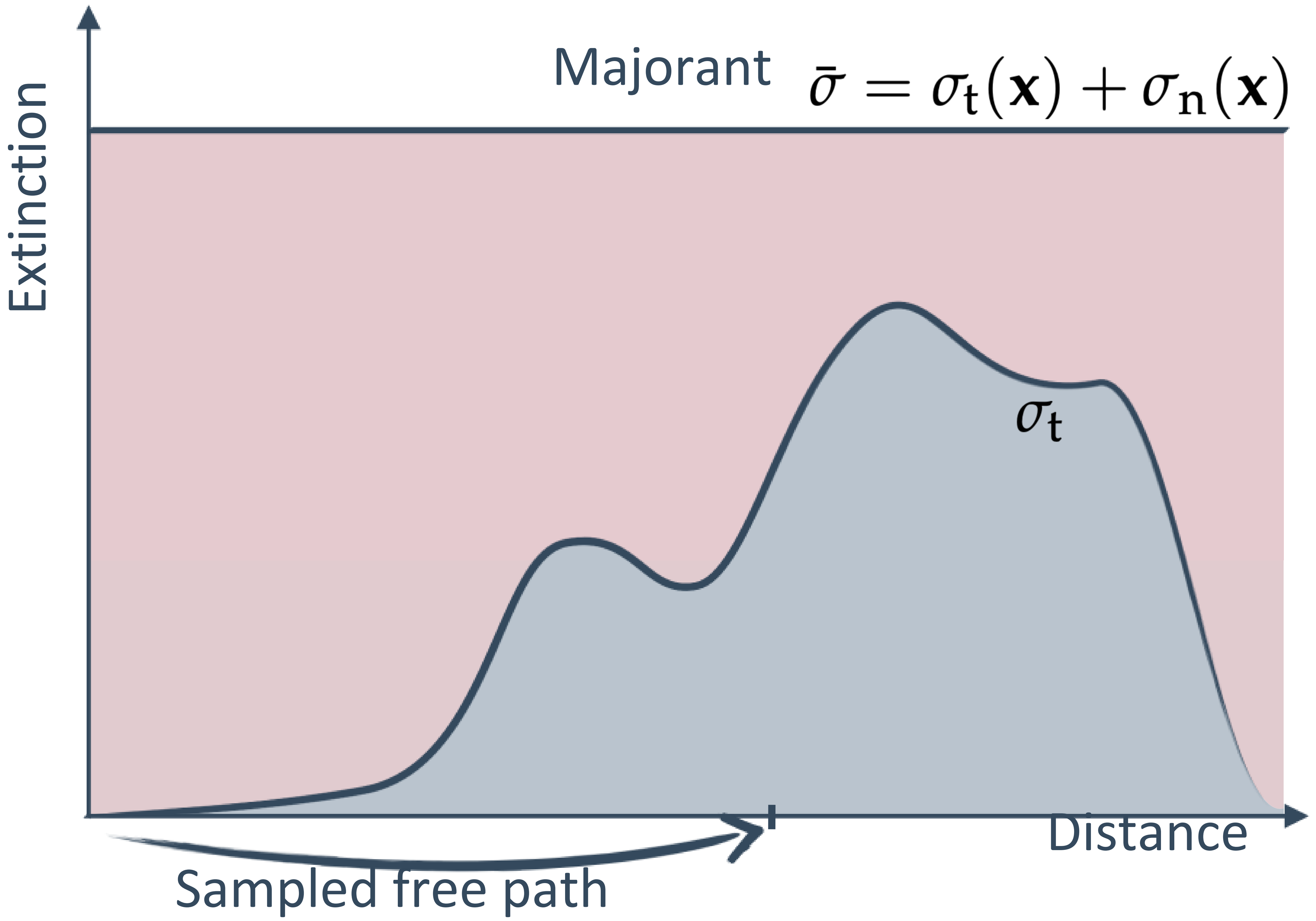
Stochastic Sampling

$$P_r(\mathbf{x}) = \frac{\sigma_t(\mathbf{x})}{\bar{\sigma}} \qquad P_n(\mathbf{x}) = \frac{\sigma_n(\mathbf{x})}{\bar{\sigma}}$$



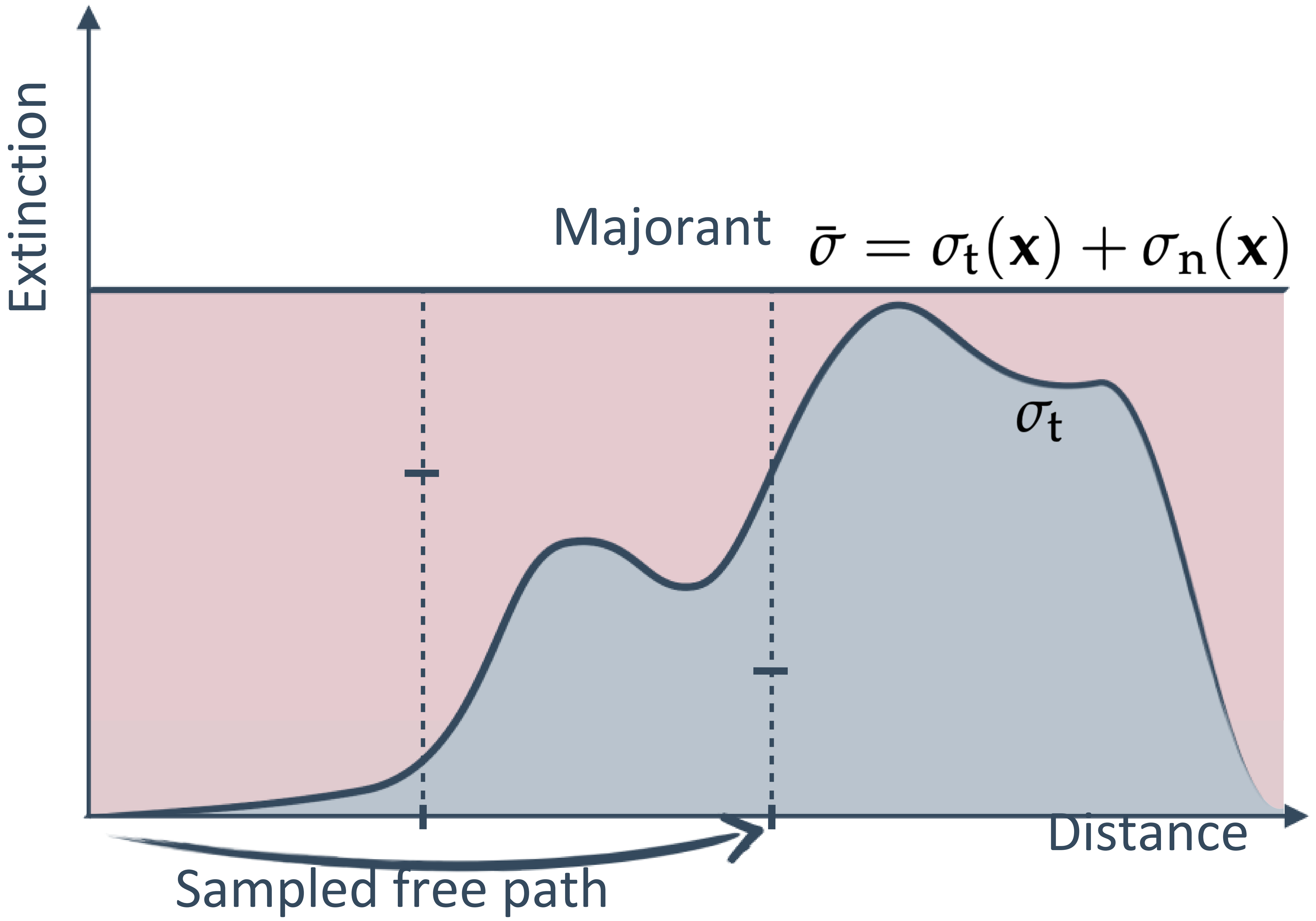


Impact of Majorant



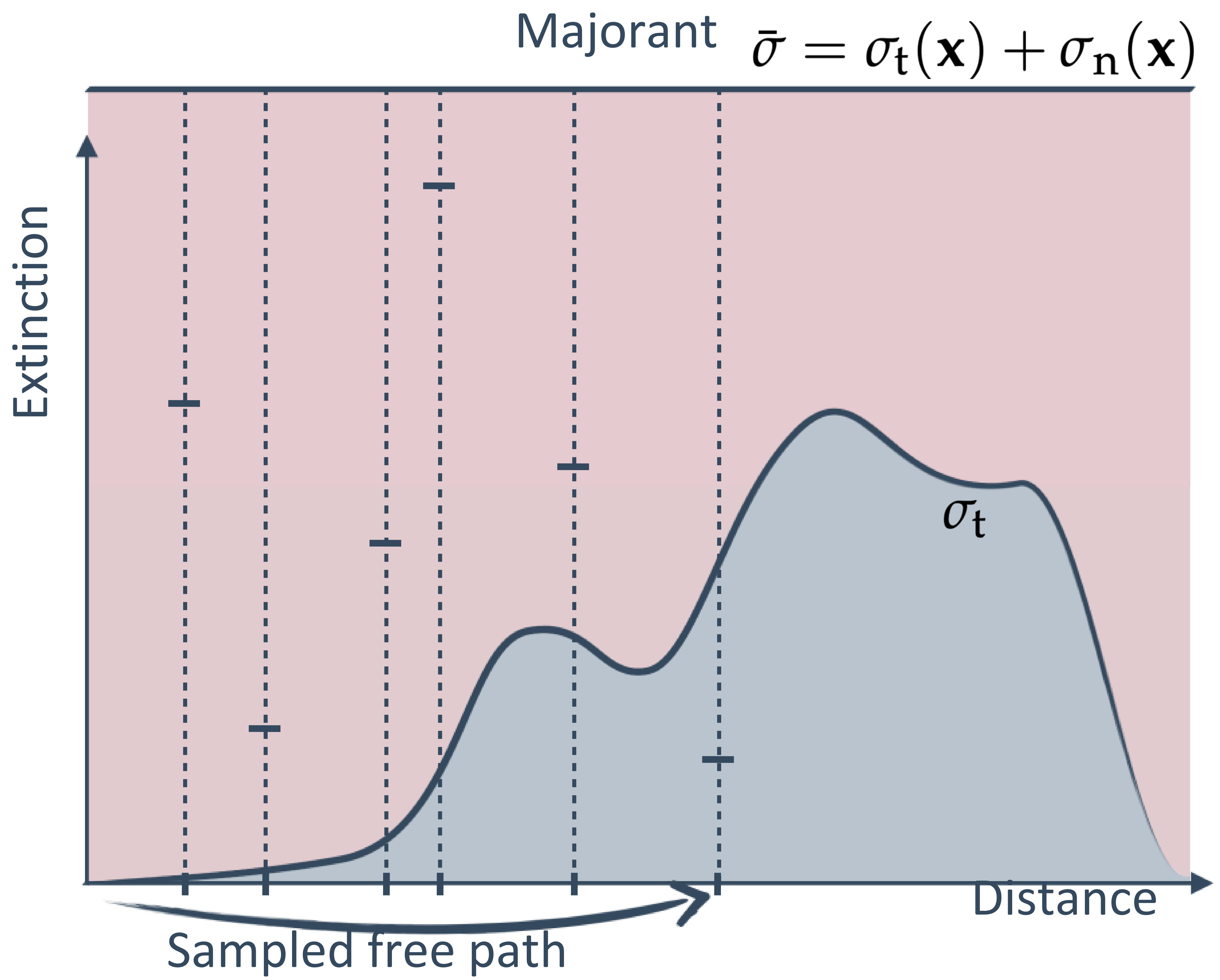
Impact of Majorant

Tight majorant = GOOD
(few rejected collisions)



Impact of Majorant

Loose majorant = BAD
(many expensive rejected collisions)



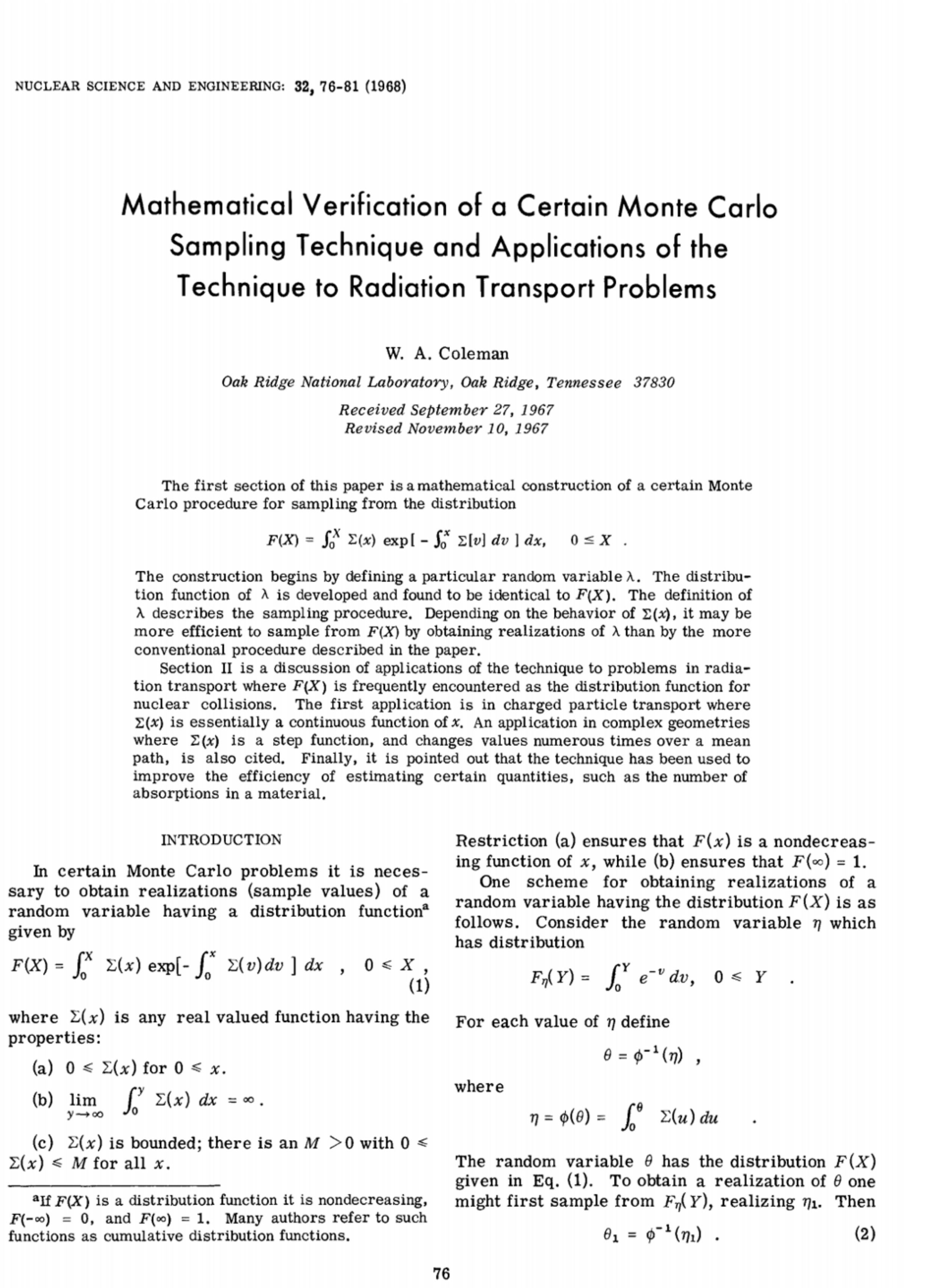
Delta Tracking

```
void preprocess()
majorant = findMaximumExtinction()

void sampleFreePath(x, ω)
t = 0
do:
// Sample distance to next tentative collision
t += -ln(1 - randf()) / majorant
// Compute probability of a real collision
Pr = getExtinction(x + t*ω) / majorant
while Pr < randf()
return t
```

Delta Tracking Summary

Unbiased, see [Coleman 68] for a proof



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Mathematical Verification of a Certain Monte Carlo Sampling Technique and Applications of the Technique to Radiation Transport Problems

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The first section of this paper is a mathematical construction of a certain Monte Carlo procedure for sampling from the distribution

$$F(X) = \int_0^X \Sigma(x) \exp[-\int_0^x \Sigma[v]\, dv]\, dx, \quad 0 \le X .$$

The construction begins by defining a particular random variable λ. The distribution function of λ is developed and found to be identical to $F(X)$. The definition of λ describes the sampling procedure. Depending on the behavior of $\Sigma(x)$, it may be more efficient to sample from $F(X)$ by obtaining realizations of λ than by the more conventional procedure described in the paper.

Section II is a discussion of applications of the technique to problems in radiation transport where $F(X)$ is frequently encountered as the distribution function for nuclear collisions. The first application is in charged particle transport where $\Sigma(x)$ is essentially a continuous function of x. An application in complex geometries where $\Sigma(x)$ is a step function, and changes values numerous times over a mean path, is also cited. Finally, it is pointed out that the technique has been used to improve the efficiency of estimating certain quantities, such as the number of absorptions in a material.

INTRODUCTION

In certain Monte Carlo problems it is necessary to obtain realizations (sample values) of a random variable having a distribution function[a] given by

$$F(X) = \int_0^X \Sigma(x) \exp[-\int_0^x \Sigma(v) dv]\, dx, \quad 0 \le X, \quad (1)$$

where $\Sigma(x)$ is any real valued function having the properties:

(a) $0 \le \Sigma(x)$ for $0 \le x$.

(b) $\lim_{y \to \infty} \int_0^y \Sigma(x)\, dx = \infty$.

(c) $\Sigma(x)$ is bounded; there is an $M > 0$ with $0 \le \Sigma(x) \le M$ for all x.

Restriction (a) ensures that $F(x)$ is a nondecreasing function of x, while (b) ensures that $F(\infty) = 1$.

One scheme for obtaining realizations of a random variable having the distribution $F(X)$ is as follows. Consider the random variable η which has distribution

$$F_\eta(Y) = \int_0^Y e^{-v} dv, \quad 0 \le Y .$$

For each value of η define

$$\theta = \phi^{-1}(\eta) ,$$

where

$$\eta = \phi(\theta) = \int_0^\theta \Sigma(u)\, du .$$

The random variable θ has the distribution $F(X)$ given in Eq. (1). To obtain a realization of θ one might first sample from $F_\eta(Y)$, realizing η_1. Then

$$\theta_1 = \phi^{-1}(\eta_1) . \quad (2)$$

[a] If $F(X)$ is a distribution function it is nondecreasing, $F(-\infty) = 0$, and $F(\infty) = 1$. Many authors refer to such functions as cumulative distribution functions.

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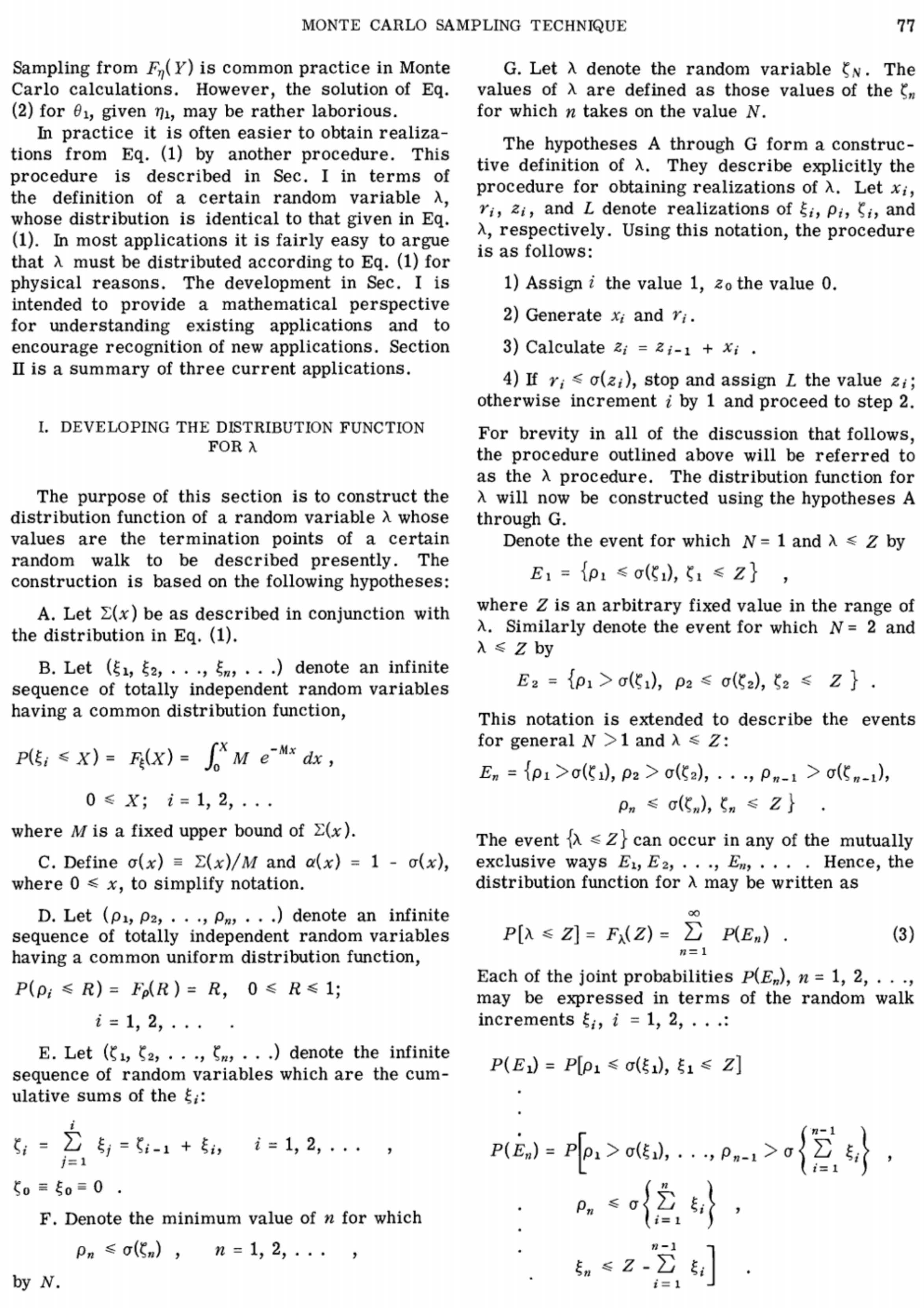

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Sampling from $F_\eta(Y)$ is common practice in Monte Carlo calculations. However, the solution of Eq. (2) for θ_1, given η_1, may be rather laborious.

In practice it is often easier to obtain realizations from Eq. (1) by another procedure. This procedure is described in Sec. I in terms of the definition of a certain random variable λ, whose distribution is identical to that given in Eq. (1). In most applications it is fairly easy to argue that λ must be distributed according to Eq. (1) for physical reasons. The development in Sec. I is intended to provide a mathematical perspective for understanding existing applications and to encourage recognition of new applications. Section II is a summary of three current applications.

I. DEVELOPING THE DISTRIBUTION FUNCTION FOR λ

The purpose of this section is to construct the distribution function of a random variable λ whose values are the termination points of a certain random walk to be described presently. The construction is based on the following hypotheses:

A. Let $\Sigma(x)$ be as described in conjunction with the distribution in Eq. (1).

B. Let $(\xi_1, \xi_2, \ldots, \xi_n, \ldots)$ denote an infinite sequence of totally independent random variables having a common distribution function,

$$P(\xi_i \le X) = F_\xi(X) = \int_0^X M\, e^{-Mx}\, dx ,$$
$$0 \le X; \quad i = 1, 2, \ldots$$

where M is a fixed upper bound of $\Sigma(x)$.

C. Define $\sigma(x) \equiv \Sigma(x)/M$ and $\alpha(x) = 1 - \sigma(x)$, where $0 \le x$, to simplify notation.

D. Let $(\rho_1, \rho_2, \ldots, \rho_n, \ldots)$ denote an infinite sequence of totally independent random variables having a common uniform distribution function,

$$P(\rho_i \le R) = F_\rho(R) = R, \quad 0 \le R \le 1;$$
$$i = 1, 2, \ldots .$$

E. Let $(\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots)$ denote the infinite sequence of random variables which are the cumulative sums of the ξ_i:

$$\zeta_i = \sum_{j=1}^{i} \xi_j = \zeta_{i-1} + \xi_i, \quad i = 1, 2, \ldots ,$$
$$\zeta_0 \equiv \xi_0 \equiv 0 .$$

F. Denote the minimum value of n for which

$$\rho_n \le \sigma(\zeta_n) , \quad n = 1, 2, \ldots ,$$

by N.

G. Let λ denote the random variable ζ_N. The values of λ are defined as those values of the ζ_n for which n takes on the value N.

The hypotheses A through G form a constructive definition of λ. They describe explicitly the procedure for obtaining realizations of λ. Let x_i, r_i, z_i, and L denote realizations of ξ_i, ρ_i, ζ_i, and λ, respectively. Using this notation, the procedure is as follows:

1) Assign i the value 1, z_0 the value 0.

2) Generate x_i and r_i.

3) Calculate $z_i = z_{i-1} + x_i$.

4) If $r_i \le \sigma(z_i)$, stop and assign L the value z_i; otherwise increment i by 1 and proceed to step 2.

For brevity in all of the discussion that follows, the procedure outlined above will be referred to as the λ procedure. The distribution function for λ will now be constructed using the hypotheses A through G.

Denote the event for which $N = 1$ and $\lambda \le Z$ by

$$E_1 = \{\rho_1 \le \sigma(\zeta_1), \zeta_1 \le Z\} ,$$

where Z is an arbitrary fixed value in the range of λ. Similarly denote the event for which $N = 2$ and $\lambda \le Z$ by

$$E_2 = \{\rho_1 > \sigma(\zeta_1), \rho_2 \le \sigma(\zeta_2), \zeta_2 \le Z\} .$$

This notation is extended to describe the events for general $N > 1$ and $\lambda \le Z$:

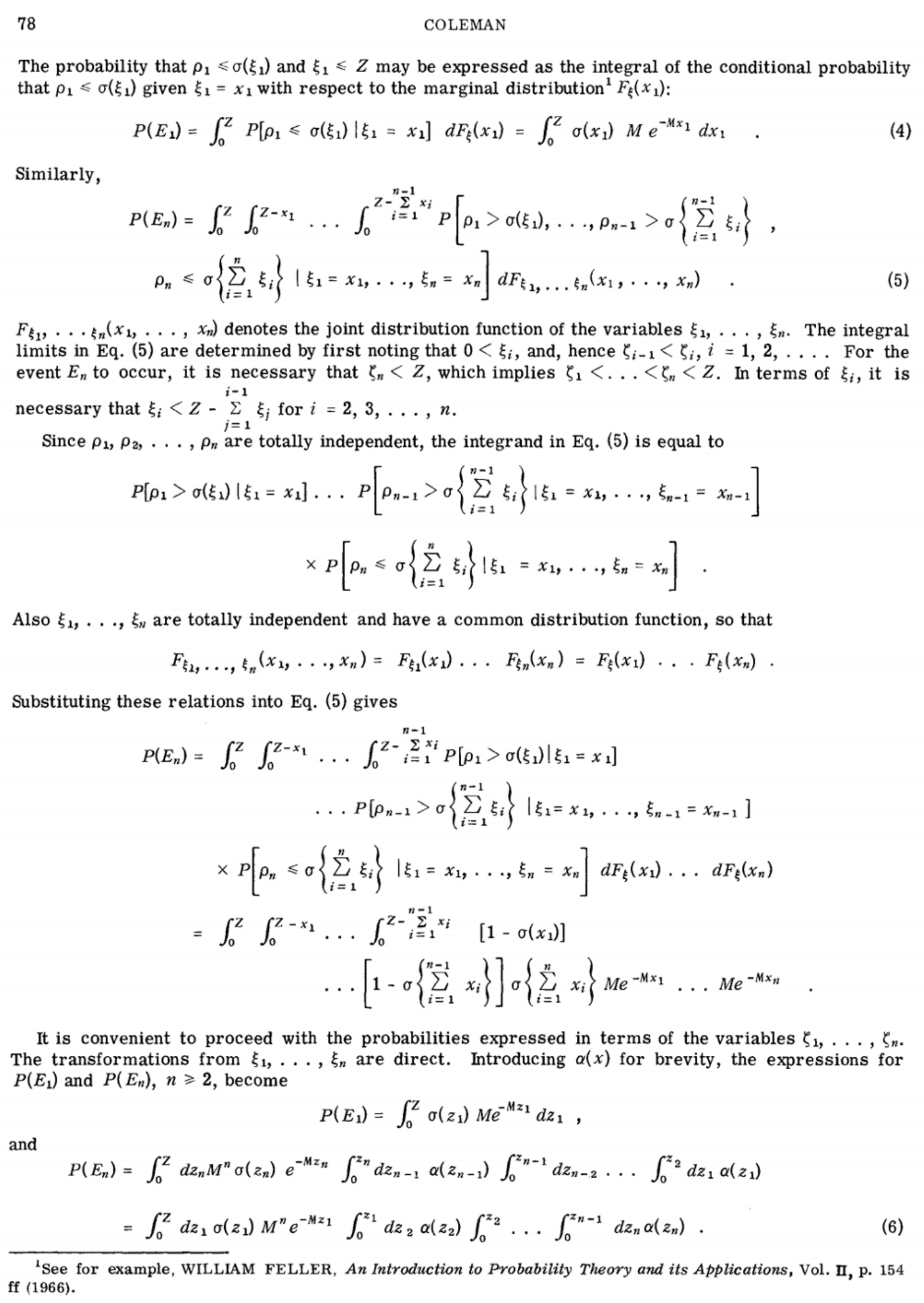
$$E_n = \{\rho_1 > \sigma(\zeta_1), \rho_2 > \sigma(\zeta_2), \ldots, \rho_{n-1} > \sigma(\zeta_{n-1}), \rho_n \le \sigma(\zeta_n), \zeta_n \le Z\} .$$

The event $\{\lambda \le Z\}$ can occur in any of the mutually exclusive ways $E_1, E_2, \ldots, E_n, \ldots$. Hence, the distribution function for λ may be written as

$$P[\lambda \le Z] = F_\lambda(Z) = \sum_{n=1}^{\infty} P(E_n) . \quad (3)$$

Each of the joint probabilities $P(E_n)$, $n = 1, 2, \ldots$, may be expressed in terms of the random walk increments ξ_i, $i = 1, 2, \ldots$:

$$P(E_1) = P[\rho_1 \le \sigma(\xi_1), \xi_1 \le Z]$$
$$\vdots$$
$$P(E_n) = P\left[\rho_1 > \sigma(\xi_1), \ldots, \rho_{n-1} > \sigma\left\{\sum_{i=1}^{n-1} \xi_i\right\}, \rho_n \le \sigma\left\{\sum_{i=1}^{n} \xi_i\right\}, \xi_n \le Z - \sum_{i=1}^{n-1} \xi_i\right] .$$



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The probability that $\rho_1 \le \sigma(\xi_1)$ and $\xi_1 \le Z$ may be expressed as the integral of the conditional probability that $\rho_1 \le \sigma(\xi_1)$ given $\xi_1 = x_1$ with respect to the marginal distribution[1] $F_\xi(x_1)$:

$$P(E_1) = \int_0^Z P[\rho_1 \le \sigma(\xi_1) \mid \xi_1 = x_1]\, dF_\xi(x_1) = \int_0^Z \sigma(x_1)\, M\, e^{-Mx_1}\, dx_1 . \quad (4)$$

Similarly,

$$P(E_n) = \int_0^Z \int_0^{Z-x_1} \cdots \int_0^{Z-\sum_{i=1}^{n-1} x_i} P\left[\rho_1 > \sigma(\xi_1), \ldots, \rho_{n-1} > \sigma\left\{\sum_{i=1}^{n-1} \xi_i\right\}, \rho_n \le \sigma\left\{\sum_{i=1}^{n} \xi_i\right\} \mid \xi_1 = x_1, \ldots, \xi_n = x_n\right] dF_{\xi_1, \ldots \xi_n}(x_1, \ldots, x_n) . \quad (5)$$

$F_{\xi_1, \ldots \xi_n}(x_1, \ldots, x_n)$ denotes the joint distribution function of the variables $\xi_1, \ldots, \xi_n$. The integral limits in Eq. (5) are determined by first noting that $0 < \xi_i$, and, hence $\zeta_{i-1} < \zeta_i$, $i = 1, 2, \ldots$. For the event E_n to occur, it is necessary that $\zeta_n < Z$, which implies $\zeta_1 < \ldots < \zeta_n < Z$. In terms of ξ_i, it is necessary that $\xi_i < Z - \sum_{j=1}^{i-1} \xi_j$ for $i = 2, 3, \ldots, n$.

Since $\rho_1, \rho_2, \ldots, \rho_n$ are totally independent, the integrand in Eq. (5) is equal to

$$P[\rho_1 > \sigma(\xi_1) \mid \xi_1 = x_1] \ldots P\left[\rho_{n-1} > \sigma\left\{\sum_{i=1}^{n-1} \xi_i\right\} \mid \xi_1 = x_1, \ldots, \xi_{n-1} = x_{n-1}\right]$$
$$\times P\left[\rho_n \le \sigma\left\{\sum_{i=1}^{n} \xi_i\right\} \mid \xi_1 = x_1, \ldots, \xi_n = x_n\right] .$$

Also $\xi_1, \ldots, \xi_n$ are totally independent and have a common distribution function, so that

$$F_{\xi_1, \ldots, \xi_n}(x_1, \ldots, x_n) = F_{\xi_1}(x_1) \ldots F_{\xi_n}(x_n) = F_\xi(x_1) \ldots F_\xi(x_n) .$$

Substituting these relations into Eq. (5) gives

$$P(E_n) = \int_0^Z \int_0^{Z-x_1} \cdots \int_0^{Z-\sum_{i=1}^{n-1} x_i} P[\rho_1 > \sigma(\xi_1) \mid \xi_1 = x_1]$$
$$\ldots P\left[\rho_{n-1} > \sigma\left\{\sum_{i=1}^{n-1} \xi_i\right\} \mid \xi_1 = x_1, \ldots, \xi_{n-1} = x_{n-1}\right]$$
$$\times P\left[\rho_n \le \sigma\left\{\sum_{i=1}^{n} \xi_i\right\} \mid \xi_1 = x_1, \ldots, \xi_n = x_n\right] dF_\xi(x_1) \ldots dF_\xi(x_n)$$
$$= \int_0^Z \int_0^{Z-x_1} \cdots \int_0^{Z-\sum_{i=1}^{n-1} x_i} [1 - \sigma(x_1)] \ldots \left[1 - \sigma\left\{\sum_{i=1}^{n-1} x_i\right\}\right] \sigma\left\{\sum_{i=1}^{n} x_i\right\} Me^{-Mx_1} \ldots Me^{-Mx_n} .$$

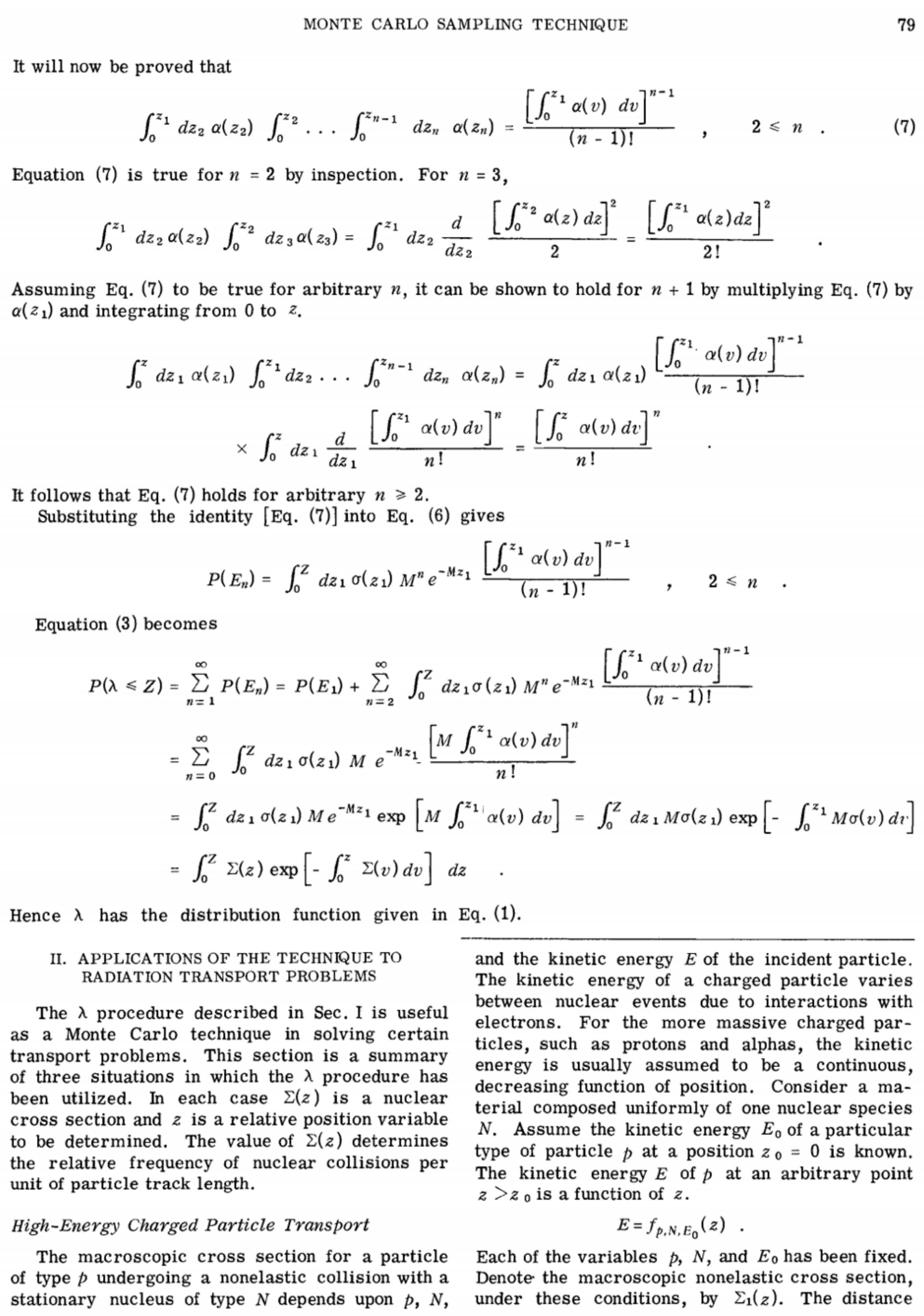
It is convenient to proceed with the probabilities expressed in terms of the variables $\zeta_1, \ldots, \zeta_n$. The transformations from $\xi_1, \ldots, \xi_n$ are direct. Introducing $\alpha(x)$ for brevity, the expressions for $P(E_1)$ and $P(E_n)$, $n \ge 2$, become

$$P(E_1) = \int_0^Z \sigma(z_1)\, Me^{-Mz_1}\, dz_1 ,$$

and

$$P(E_n) = \int_0^Z dz_n M^n \sigma(z_n)\, e^{-Mz_n} \int_0^{z_n} dz_{n-1}\, \alpha(z_{n-1}) \int_0^{z_{n-1}} dz_{n-2} \ldots \int_0^{z_2} dz_1\, \alpha(z_1)$$
$$= \int_0^Z dz_1\, \sigma(z_1)\, M^n e^{-Mz_1} \int_0^{z_1} dz_2\, \alpha(z_2) \int_0^{z_2} \ldots \int_0^{z_{n-1}} dz_n\, \alpha(z_n) . \quad (6)$$

[1] See for example, WILLIAM FELLER, *An Introduction to Probability Theory and its Applications*, Vol. **II**, p. 154 ff (1966).



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It will now be proved that

$$\int_0^{z_1} dz_2\, \alpha(z_2) \int_0^{z_2} \ldots \int_0^{z_{n-1}} dz_n\, \alpha(z_n) = \frac{\left[\int_0^{z_1} \alpha(v)\, dv\right]^{n-1}}{(n-1)!} , \quad 2 \le n . \quad (7)$$

Equation (7) is true for $n = 2$ by inspection. For $n = 3$,

$$\int_0^{z_1} dz_2\, \alpha(z_2) \int_0^{z_2} dz_3\, \alpha(z_3) = \int_0^{z_1} dz_2 \frac{d}{dz_2} \frac{\left[\int_0^{z_2} \alpha(z)\, dz\right]^2}{2} = \frac{\left[\int_0^{z_1} \alpha(z) dz\right]^2}{2!} .$$

Assuming Eq. (7) to be true for arbitrary n, it can be shown to hold for $n + 1$ by multiplying Eq. (7) by $\alpha(z_1)$ and integrating from 0 to z.

$$\int_0^z dz_1\, \alpha(z_1) \int_0^{z_1} dz_2 \ldots \int_0^{z_{n-1}} dz_n\, \alpha(z_n) = \int_0^z dz_1\, \alpha(z_1) \frac{\left[\int_0^{z_1} \alpha(v)\, dv\right]^{n-1}}{(n-1)!}$$
$$\times \int_0^z dz_1 \frac{d}{dz_1} \frac{\left[\int_0^{z_1} \alpha(v)\, dv\right]^n}{n!} = \frac{\left[\int_0^z \alpha(v)\, dv\right]^n}{n!} .$$

It follows that Eq. (7) holds for arbitrary $n \ge 2$.

Substituting the identity [Eq. (7)] into Eq. (6) gives

$$P(E_n) = \int_0^Z dz_1\, \sigma(z_1)\, M^n e^{-Mz_1} \frac{\left[\int_0^{z_1} \alpha(v)\, dv\right]^{n-1}}{(n-1)!} , \quad 2 \le n .$$

Equation (3) becomes

$$P(\lambda \le Z) = \sum_{n=1}^{\infty} P(E_n) = P(E_1) + \sum_{n=2}^{\infty} \int_0^Z dz_1 \sigma(z_1)\, M^n e^{-Mz_1} \frac{\left[\int_0^{z_1} \alpha(v)\, dv\right]^{n-1}}{(n-1)!}$$
$$= \sum_{n=0}^{\infty} \int_0^Z dz_1\, \sigma(z_1)\, M\, e^{-Mz_1} \frac{\left[M \int_0^{z_1} \alpha(v)\, dv\right]^n}{n!}$$
$$= \int_0^Z dz_1\, \sigma(z_1)\, M e^{-Mz_1} \exp\left[M \int_0^{z_1} \alpha(v)\, dv\right] = \int_0^Z dz_1\, M\sigma(z_1) \exp\left[-\int_0^{z_1} M\sigma(v)\, dv\right]$$
$$= \int_0^Z \Sigma(z) \exp\left[-\int_0^z \Sigma(v)\, dv\right] dz .$$

Hence λ has the distribution function given in Eq. (1).

II. APPLICATIONS OF THE TECHNIQUE TO RADIATION TRANSPORT PROBLEMS

The λ procedure described in Sec. I is useful as a Monte Carlo technique in solving certain transport problems. This section is a summary of three situations in which the λ procedure has been utilized. In each case $\Sigma(z)$ is a nuclear cross section and z is a relative position variable to be determined. The value of $\Sigma(z)$ determines the relative frequency of nuclear collisions per unit of particle track length.

High-Energy Charged Particle Transport

The macroscopic cross section for a particle of type p undergoing a nonelastic collision with a stationary nucleus of type N depends upon p, N, and the kinetic energy E of the incident particle. The kinetic energy of a charged particle varies between nuclear events due to interactions with electrons. For the more massive charged particles, such as protons and alphas, the kinetic energy is usually assumed to be a continuous, decreasing function of position. Consider a material composed uniformly of one nuclear species N. Assume the kinetic energy E_0 of a particular type of particle p at a position $z_0 = 0$ is known. The kinetic energy E of p at an arbitrary point $z > z_0$ is a function of z.

$$E = f_{p,N,E_0}(z) .$$

Each of the variables p, N, and E_0 has been fixed. Denote the macroscopic nonelastic cross section, under these conditions, by $\Sigma_1(z)$. The distance

Delta Tracking Summary

Unbiased, see [Coleman 68] for a proof

Majorant extinction

- defines the combined homogeneous volume
- must bound the real extinction
- loose majorants lead to many fictitious collisions