Ray tracing and geometric representations
Course announcements

- Programming assignment 0 is available on Canvas.
- Programming assignment 1 will be posted on Friday 1/28 and will be due two weeks later.
- Take-home quiz 1 will be posted on Tuesday 1/25 and will be due a week later.
- Office hours for this week only (will finalize starting next week based on survey results):
  - Yannis—Thursday 2-4 pm.
  - Zoom details on Piazza and Canvas.
Course announcements

• Is anyone not on Piazza?
  
  https://piazza.com/class/ky96bnus9u54ul

• Is anyone not on Canvas?
  
  https://canvas.cmu.edu/courses/27795

• Is anyone not on Slack?
Overview of today’s lecture

- Introduction to ray tracing.
- Intersections with geometric primitives.
- Triangular meshes.
Most of these slides were directly adapted from:

- Wojciech Jarosz (Dartmouth).
Two forms of 3D rendering

Rasterization: object point to image plane
- start with a 3D object point
- apply transforms
- determine the 2D image plane point it projects to

Ray tracing: image plane to object point
- start with a 2D image point
- generate a ray
- determine the visible 3D object point

Inverse processes
Two forms of 3D rendering

**Rasterization**
- for (each triangle)
- for (each pixel)
  - if (triangle covers pixel)
    - keep closest hit

**Ray-centric**
- for (each pixel or ray)
  - for (each triangle)
    - if (ray hits triangle)
      - keep closest hit

After a slide by Frédéric Durand
Rasterization advantages

Modern scenes are more complicated than images
- A 1920x1080 frame (1080p) at 64-bit color and 32-bit depth per pixel is 24 MB (not that much)
  • of course, if we have more than one sample per pixel this gets larger, but e.g. 4x supersampling is still a relatively comfortable ~100 MB
- Our scenes are routinely larger than this
  • This wasn’t always true

A rasterization-based renderer can stream over the triangles, no need to keep entire dataset around
- Allows parallelism and optimizations of memory systems
Rasterization limitations

Restricted to scan-convertible primitives
- Pretty much: triangles

Faceting, shading artifacts
- This is largely going away with programmable per-pixel shading, though

No unified handling of shadows, reflection, transparency
Ray/path tracing

Advantages
- Generality: can render anything that can be intersected with a ray
- Easily allows recursion (shadows, reflections, etc.)

Disadvantages
- Hard to implement in hardware (lacks computation coherence, must fit entire scene in memory, bad memory behavior)
  - Not such a big point anymore given general purpose GPUs
- Has traditionally been too slow for interactive applications
- Both of the above are changing rather rapidly right now!
A ray-traced image
Ray tracing today
Rapid change in film industry

2008:
- Most CGI in films rendered using micro-polygon rasterization.
- “You’d be crazy to render a full-feature film with ray/path tracing.”
- Ray/path tracing mostly interesting to academics

2018:
- Most major films now rendered using ray/path tracing.
- “You’d be crazy \textit{not} to render a full-feature film using path tracing.”
Albrecht Dürer (1525)
René Descartes (1650)
Isaac Newton (1670)
Ray casting
- Generate an image by sending one ray per pixel
- Check for shadows by sending a ray towards the light
Whitted (1979)

recursive ray tracing (reflection & refraction)
Light Transport - Assumptions

Geometric optics:
- no diffraction, no polarization, no interference

Light travels in a straight line in a vacuum
- no atmospheric scattering or refraction
- no gravity effects

Color can be represented as three numbers: (R,G,B)
Emission theory of vision

Supported by:
- Ancient greeks
- 50% of US college students*

Ray Tracing - Overview

“light tracing”

eye point

image plane

light source
Basic Ray Tracing Pipeline

Ray Generation
Basic Ray Tracing Pipeline

1. Ray Generation
2. Intersection
Basic Ray Tracing Pipeline

Ray Generation

Intersection

Shading
Basic Ray Tracing Pipeline

- Ray Generation
- Intersection
- Shading
Basic Ray Tracing Pipeline

- Ray Generation
- Intersection
- Shading
Basic Ray Tracing Pipeline

- Ray Generation
- Intersection
- Shading
Basic Ray Tracing Pipeline

Ray Generation

Intersection

Shading
Ray Tracing Pseudocode

```plaintext
def rayTraceImage()
{
    parse scene description

    for each pixel
        ray = generateCameraRay(pixel)
        pixelColor = trace(ray)
}
```
Ray Tracing Pseudocode

trace(ray) {
    hit = find first intersection with scene objects
    color = shade(hit)
    return color
}

might trace more rays (recursive)
Ray Tracing Pseudocode

rayTraceImage()
{
    parse scene description

    for each pixel
    {
        ray = generateCameraRay(pixel)
        pixelColor = trace(ray)
    }

    what is a ray? how do we generate a camera ray?
Ray: a half line

Standard representation: origin (point) $\mathbf{0}$ and direction $\mathbf{d}$

$$\mathbf{r}(t) = \mathbf{0} + t\mathbf{d}$$

- this is a parametric equation for the line
- lets us directly generate the points on the line
- if we restrict to $t > 0$ then we have a ray
- note replacing $\mathbf{d}$ with $a\mathbf{d}$ does not change ray (for $a > 0$)
Generating eye rays

Orthographic

Perspective

view rect

pixel position

viewing ray

view rect

viewpoint

pixel position

viewing ray

After a slide by Steve Marschner
Pinhole Camera (Camera Obscura)
Pinhole Camera

Pinhole Camera

film / physical image plane

virtual image plane

viewing volume

pinhole
Pinhole Camera
Generating eye rays—perspective

Establish view rectangle in X–Y plane, specified by, e.g.
- l, r, t, b

Place rectangle at $z = -d$

$s = [u, v, -d]^T$

$d = s$

$r(t) = o + td$

Does distance $d$ matter?
Placing the camera in the scene
Generating eye rays—orthographic

How do you generate a ray for an orthographic camera?
Ray-Surface Intersections

Ray Generation

Intersection

Shading
Ray-Surface Intersections

Surface primitives
- spheres
- planes
- triangles
- general implicits
- etc.
Ray-Sphere Intersection

Algebraic approach:

- Condition 1: point is on ray: \( r(t) = o + td \)

- Condition 2: point is on sphere: \( \|x - c\|^2 - r^2 = 0 \)

- substitute and solve for \( t \):

\[ \|o + td - c\|^2 - r^2 = 0 \]
Ray-Sphere Intersection

substitute and solve for $t$

$$\|\mathbf{o} + t\mathbf{d} - \mathbf{c}\|^2 - r^2 = 0 \rightarrow (o_x + td_x - c_x)^2 +$$

$$ (o_y + td_y - c_y)^2 +$$

$$ (o_z + td_z - c_z)^2 - r^2 = 0$$

which reduces to: $At^2 + Bt + C = 0$

Solve for $t$ using quadratic equation:

$$t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

What happens when square root is zero or negative?
Ray-Surface Intersections

Surface primitives

- spheres
- planes
- triangles
- general implicit
- etc.
Ray-Plane Intersection

Plane equation (implicit)

Algebraic form:

$$ax + by + cz + d = 0$$
Ray-Plane Intersection

Plane equation (implicit)

\[(x - p) \cdot n = 0\]

point of interest  point on plane  plane normal

substitute ray equation for \(x\) and solve for \(t\)

\[(o + td - p) \cdot n = 0\]
\[td \cdot n + (o - p) \cdot n = 0\]
\[t = -\frac{(o - p) \cdot n}{d \cdot n}\]
Ray-Surface Intersections

Surface primitives

- spheres
- planes
- triangles
- general implicits
- etc.
Ray-Triangle intersection

Condition 1: point is on ray: \( r(t) = o + td \)

Condition 2: point is on plane: \( (x - p) \cdot n = 0 \)

Condition 3: point is on the inside of all three edges

First solve 1&2 (ray–plane intersection) for \( t \):

\[
(o + td - p) \cdot n = 0
\]

\[
t = -\frac{(o - p) \cdot n}{d \cdot n}
\]

Several options for 3
Ray-Triangle intersection (Approach 1)

In plane, triangle is the intersection of 3 half spaces
Ray-Triangle intersection (Approach 1)

\[ \mathbf{n} = (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) \]

Which way does \( \mathbf{n} \) point?
Ray-Triangle intersection (Approach 1)

\[ \mathbf{n} = (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) \]
\[ \mathbf{n}_{x13} = (\mathbf{x} - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) \]

Which way does \( \mathbf{n} \) point?
What about \( \mathbf{n}_{x13} \)?
Ray-Triangle intersection (Approach 1)

\[ \mathbf{n} = (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) \]
\[ \mathbf{n}_{x13} = (\mathbf{x} - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) \]

Which way does \( \mathbf{n} \) point?
What about \( \mathbf{n}_{x13} \)?
- How about now?
Ray-Triangle intersection (Approach 1)

\[ \mathbf{n} = (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) \]
\[ \mathbf{n}_{x13} = (\mathbf{x} - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) \]

Which way does \( \mathbf{n} \) point?
What about \( \mathbf{n}_{x13} \)?

- How about now?
- Edge test: \( (\mathbf{n}_{x13} \cdot \mathbf{n}) < 0 \)
Ray-Triangle intersection (Approach 1)

\[ \mathbf{n} = (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) \]

\[ \mathbf{n}_{x13} = (\mathbf{x} - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1) \]

Which way does \( \mathbf{n} \) point?

What about \( \mathbf{n}_{x13} \)?

- How about now?
- Edge test: \( (\mathbf{n}_{x13} \cdot \mathbf{n}) < 0 \)
Ray-Triangle Intersection (Approach 2)

Intersect ray with triangle’s plane

Test whether hit-point is within triangle

- compute sub-triangle areas $\alpha, \beta, \gamma$

- test inside triangle conditions
Barycentric coordinates

Barycentric coordinates: \[ x(\alpha, \beta, \gamma) = \alpha p_1 + \beta p_2 + \gamma p_3 \]

Inside triangle conditions:

\[ \alpha + \beta + \gamma = 1 \quad 0 \leq \alpha \leq 1 \]
\[ \gamma = 1 - \alpha - \beta \quad 0 \leq \beta \leq 1 \]
\[ 0 \leq \gamma \leq 1 \]
Interpretations of barycentric coords

Sub-triangle areas

\[ \alpha = \frac{\Delta p_2p_3x}{\Delta p_1p_2p_3} \]
\[ \beta = \frac{\Delta p_1p_3x}{\Delta p_1p_2p_3} \]
\[ \gamma = \frac{\Delta p_1p_2x}{\Delta p_1p_2p_3} \]

\[ x = \alpha p_1 + \beta p_2 + \gamma p_3 \]
Ray-Triangle Intersection (Approach 3)

Insert ray equation:
\[ \alpha p_1 + \beta p_2 + (1 - \alpha - \beta) p_3 = o + td \]
\[ \alpha (p_1 - p_3) + \beta (p_2 - p_3) + p_3 = o + td \]
\[ \alpha (p_1 - p_3) + \beta (p_2 - p_3) - td = o - p_3 \]
\[ \alpha a + \beta b - td = e \]

Solve directly
\[ \begin{bmatrix} -d & a & b \end{bmatrix} \begin{bmatrix} t \\ \alpha \\ \beta \end{bmatrix} = e \]

Can be much faster!
Ray-Surface Intersections

Other primitives

- cylinder
- cone, paraboloid, hyperboloid
- torus
- disk
- general polygons, meshes
- etc.
Intersecting transformed primitive?

Option 1: Transform the primitive
- simple for triangles, since they transform to triangles
- other primitives get more complicated (e.g. sphere $\rightarrow$ ellipsoid)

Option 2: Transform the ray (by the inverse transform)
- more elegant; works on any primitive
- allows simpler intersection tests
  (e.g., just use existing sphere-intersection routine)
Intersection and coordinate systems

World space

Local space
Intersection and coordinate systems

World space

Local space
Intersection and coordinate systems

We have a sphere now
But with a different ray
Transformations in homogeneous coords

A 3D transformation matrix:

\[ M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{24} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \]

A 3D homogenous vector:

\[ \mathbf{v} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \]

A position has \( w \neq 0 \), and a direction has \( w = 0 \)
Transformations

Matrix-vector multiplication, $Mv$, transforms the vector

A translation matrix:

$$M_t = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A scaling matrix:

$$M_s = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
Intersection and coordinate systems

Have a transform $M$, a ray $r(t)$, and a surface $S$

To intersect:
1. Transform ray to local coords (by inverse of $M$)
2. Call surface intersection
3. Transform hit data back to global coords (by $M$)

How to transform a ray $r(t) = p + td$ by $M^{-1}$?
- $r'(t) = M^{-1}p + tM^{-1}d$
- Remember: $p$ forms as a point, $d$ as a direction!
Ray-Surface Intersections

Other primitives

- cylinder
- cone, paraboloid, hyperboloid
- torus
- disk
- general polygons, meshes
- etc.
Image so far

With eye ray generation and sphere intersection

parse scene description

for each pixel:
    ray = camera.getRay(pixel);
    hit = s.intersect(ray, 0, +inf);
    if hit:
        image.set(pixel, white);
Intersecting many shapes

Intersect each primitive

Pick closest intersection

- Only within considered range \([t_{\text{min}}, t_{\text{max}}]\)
- After each valid intersection, update \(t_{\text{max}}\)

Essentially a line search
Intersection against many shapes

The basic idea is:

```
Surfaces::intersect(ray, tMin, tMax):
    tBest = +inf; firstHit = null;
    for s in surfaces:
        hit = s.intersect(ray, tMin, tBest);
        if hit:
            tBest = hit.t;
            firstHit = hit;
    return firstHit;
```

- this is linear in number of surfaces but there are sublinear methods (acceleration structures)
Image so far

With eye ray generation and scene intersection

for each pixel:
    ray = camera.getRay(pixel);
    c = scene.trace(ray, 0, +inf);
    image.set(pixel, c);

Scene::trace(ray, tMin, tMax):
    hit = surfaces.intersect(ray, tMin, tMax);
    if (hit)
        return hit.color();
    else
        return backgroundColor;
Ray-Surface Intersections

Other primitives

- cylinder
- cone, paraboloid, hyperboloid
- torus
- disk
- general polygons, meshes
- etc.
How should we represent complex geometry?

How are they obtained?
- modeled by hand
- scanned

What operations must we support?
- modeling/editing
- animating
- texturing
- rendering
## Surface representation zoo!

<table>
<thead>
<tr>
<th>Parametric</th>
<th>Implicit</th>
<th>Discrete/Sampled</th>
</tr>
</thead>
</table>
| • Splines, tensor-product surfaces  
• Subdivision surfaces | • Metaballs/blobs  
• Distance fields  
• Procedural, CSG  
• Neural nets | • Meshes  
• Point set surfaces |

After a slide by Olga Sorkine-Hornung
Polygonal Meshes

Boundary representations of objects

- Piecewise linear
A small triangle mesh

12 triangles, 8 vertices
A large mesh

10 million triangles from a high-resolution 3D scan
After a slide by Steve Marschner
spheres approximate sphere

After a slide by Steve Marschner
Meshes as Approx. of Smooth Surfaces

Piecewise linear approximation

- Error is $O(h^2)$
Meshes as Approx. of Smooth Surfaces

Piecewise linear approximation

- Error is $O(h^2)$

After a slide by Olga Sorkine-Hornung

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#faces vs. approximation error

- 0.4%
- 25%
- 6.5%
- 1.7%
- 0.4%
Polygonal Meshes

Polygonal meshes are a good representation

- approximation $O(h^2)$
- arbitrary topology
- piecewise smooth surfaces
- adaptive refinement
- efficient rendering
Data Structures: What should be stored?

Geometry: 3D coordinates

Attributes
- Normal, color, texture coordinates
- Per vertex, face, edge

Connectivity
- Adjacency relationships
Separate Triangle List or Face Set (STL)

Face: 3 vertex positions

Storage:
- 4 Bytes/coordinate (using 32-bit floats)
- 36 Bytes/face

Wastes space

<table>
<thead>
<tr>
<th>Triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>5</td>
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<tr>
<td>6</td>
</tr>
<tr>
<td>...</td>
</tr>
</tbody>
</table>
Indexed Face Set (OBJ, OFF, WRL)

Vertex: position

Face: vertex indices

Storage:
- 12 Bytes/vertex
- 12 Bytes/face

Reduces wasted space

Even better with per-vertex attributes
Data on meshes

Often need to store additional information besides just the geometry

Can store additional data at faces, vertices, or edges

Examples

- colors stored on faces, for faceted objects
- information about sharp creases stored at edges
- any quantity that varies *continuously* (without sudden changes, or *discontinuities*) gets stored at vertices
Key types of vertex data

Surface normals
- when a mesh is approximating a curved surface, store normals at vertices

Texture coordinates
- 2D coordinates that tell you how to paste images on the surface

Positions
- at some level this is just another piece of data
Defining normals

Face normals: same normal for all points in face
- geometrically correct, but faceted look
Problems with face normals

Piecewise planar approximation converges pretty quickly to the smooth geometry as the number of triangles increases
- error is $O(h^2)$

But the surface normals don’t converge so well
- normal is constant over each triangle, with discontinuous jumps across edges
- error is only $O(h)$
Problems with face normals—2D example

Approximating circle with increasingly many segments

Max error in position error drops by factor of 4 each step

Max error in normal only drops by factor of 2
Problems with face normals—solution

Piecewise planar approximation converges pretty quickly to the smooth geometry as the number of triangles increases
- for mathematicians: error is $O(h^2)$

But the surface normals don’t converge so well
- normal is constant over each triangle, with discontinuous jumps across edges
- for mathematicians: error is only $O(h)$

Better: store the “real” normal at each vertex, and interpolate to get normals that vary gradually across triangles
Defining normals

Vertex normals: store normal at vertices, interpolate in face
- geometrically “inconsistent”, but smooth look
Barycentric coordinates

Barycentric interpolation:

\[ p(\alpha, \beta, \gamma) = \alpha p_1 + \beta p_2 + \gamma p_3 \]

Can use this eqn. to interpolate any vertex quantity across triangle!
Barycentric coordinates

Barycentric interpolation:

\[ p(\alpha, \beta, \gamma) = \alpha p_1 + \beta p_2 + \gamma p_3 \]

\[ c(\alpha, \beta, \gamma) = \alpha c_1 + \beta c_2 + \gamma c_3 \]

Can use this eqn. to interpolate any vertex quantity across triangle!
Barycentric coordinates

Barycentric interpolation:

\[ p(\alpha, \beta, \gamma) = \alpha p_1 + \beta p_2 + \gamma p_3 \]
\[ c(\alpha, \beta, \gamma) = \alpha c_1 + \beta c_2 + \gamma c_3 \]
\[ n(\alpha, \beta, \gamma) = \alpha n_1 + \beta n_2 + \gamma n_3 \]

not guaranteed to be unit length

Can use this eqn. to interpolate any vertex quantity across triangle!
Realism through geometric complexity
Ray Tracing Acceleration

Ray-surface intersection is at the core of every ray tracing algorithm

Brute force approach:
- intersect every ray with every primitive
- many unnecessary ray-surface intersection tests
Ray Tracing Cost

“the time required to compute the intersections of rays and surfaces is over 95 percent” [Whitted 1980]

Cost = $O(n_x \cdot n_y \cdot n_o)$
- (number of pixels) \cdot (number of objects)
- Assumes 1 ray per pixel

Example: 1024 x 1024 image of a scene with 1000 triangles
- Cost is (at least) $10^9$ ray-triangle intersections

Typically measured per ray:
- Naive: $O(n_o)$ - linear with number of objects
$O(n^2)$ Ray Tracing (The Problem)

8 primitives $\rightarrow$ 3 seconds

50K trees each with 1M polygons $=$ 50B polygons

$\rightarrow$ 594 years!
Sub-linear Ray Tracing

50K trees each with 1M polygons = 50B polygons → 11 minutes
300,000,000x speedup!

Andreas Byström
The solution

Improve efficiency of ray-surface intersections by constructing acceleration structures.
- A spatial organization of objects in a scene to minimize the necessary number of ray-object intersection tests.

Spatial sorting/subdivision (e.g. grid, kd-tree, ochre)
- Decompose space into disjoint regions & assign objects to regions

Object sorting/subdivision (bounding volume hierarchy)
- Decompose objects into disjoint sets & bound using simple volumes for fast rejection
Bounding Volumes

Spheres

70 K
Bounding Volumes

Axis-aligned bounding boxes (most common)
Bounding Volumes Hierarchies

Now do this hierarchically!
BVH Traversal

```c++
void BVHNode::interactBVH(ray, &hit):
    if (bound.hit(ray)):
        if (leaf):
            leaf.intersect(ray, hit);
        else:
            leftChild.intersectBVH(ray, hit);
            rightChild.intersectBVH(ray, hit);
```
Constructing BVHs

Top-down:
- partition objects along an axis and create two sub-sets

Bottom-up:
- recursively group nearby objects together
Divisive (top-down) BBH construction

1. Choose split axis
2. Choose split plane location
3. Choose whether to create leaf or split + repeat

Many strategies for each of these steps
Choosing axis based on centroid extents
Object-median splitting

1. Sort bbox centroids along split axis
2. Take first half as left child, second half as right