Bidirectional path tracing

15-468, 15-668, 15-868
Physics-based Rendering
Spring 2022, Lecture 13

http://graphics.cs.cmu.edu/courses/15-468

Course announcements

- Take-home quiz 6 posted, due Tuesday 3/15 at 23:59.
- Programming assignment 3 posted, due Friday 3/18 at 23:59.
  - How many of you have looked at/started/finished it?
  - Any questions?
- Suggest topics for fourth reading group this Friday, 3/18.
- No lecture on Thursday 3/24! Vote on Piazza to reschedule: https://piazza.com/class/ky96bnus9u54ul?cid=103

Overview of today’s lecture

- Types of light paths.
- Light tracing.
- Bidirectional path tracing.

Slide credits

Most of these slides were directly adapted from:

- Wojciech Jarosz (Dartmouth).

Light Paths

Light Paths

Express light paths in terms of the surface interactions that have occurred

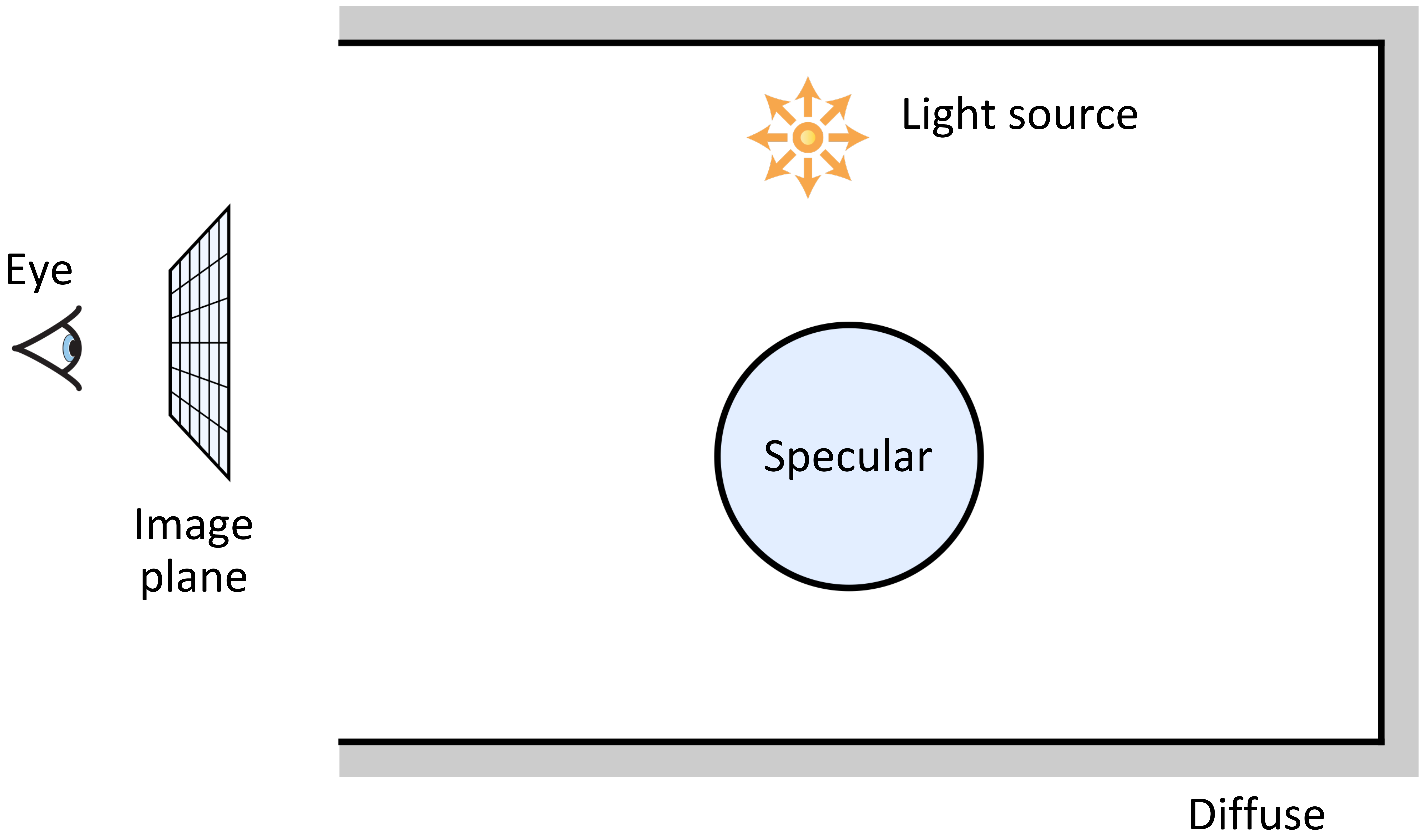
A light path is a chain of linear segments joined at event "vertices"

Heckbert's Classification

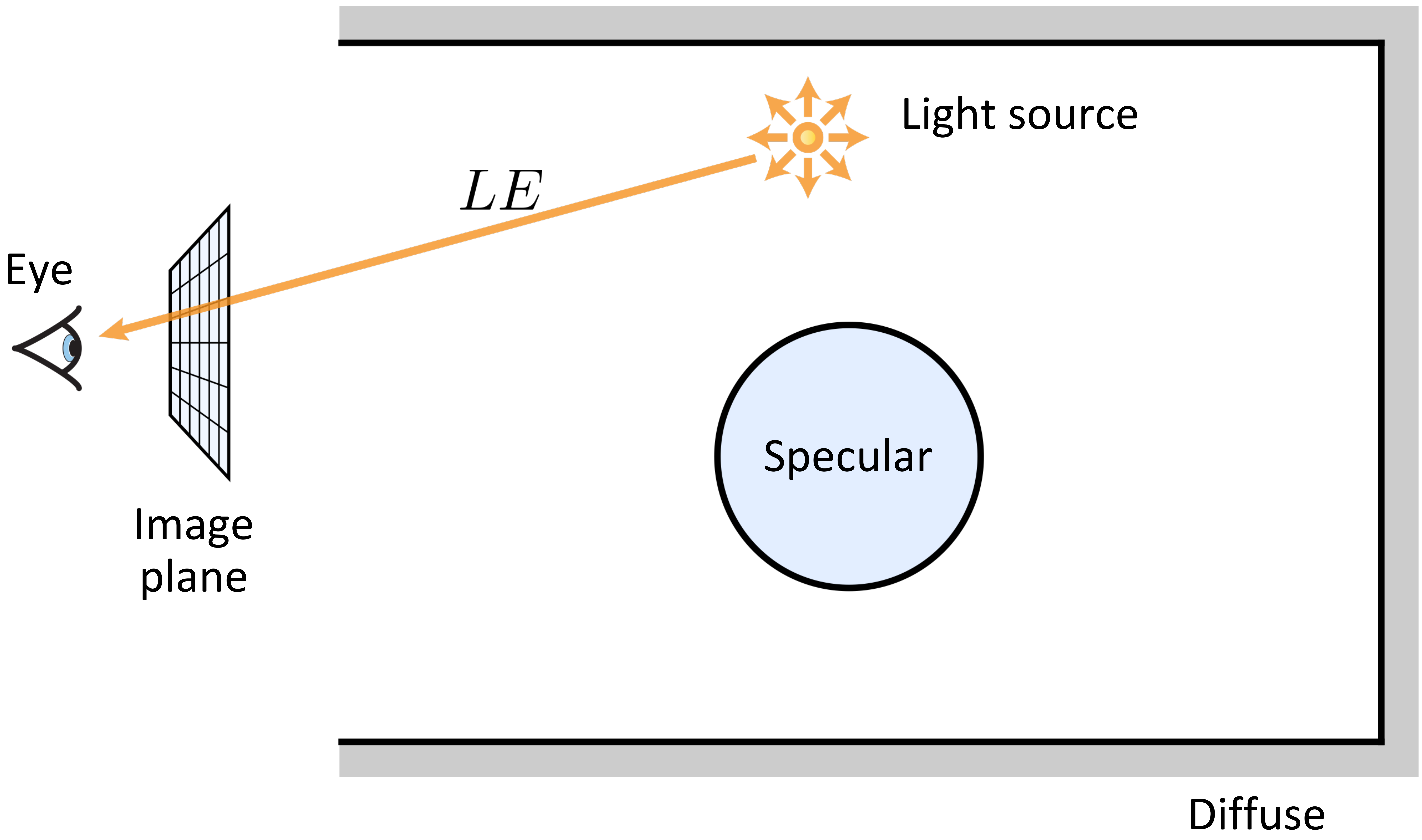
Classification of "vertices":

- L : a light source
- E : the eye
- S : a specular reflection
- D : a diffuse reflection

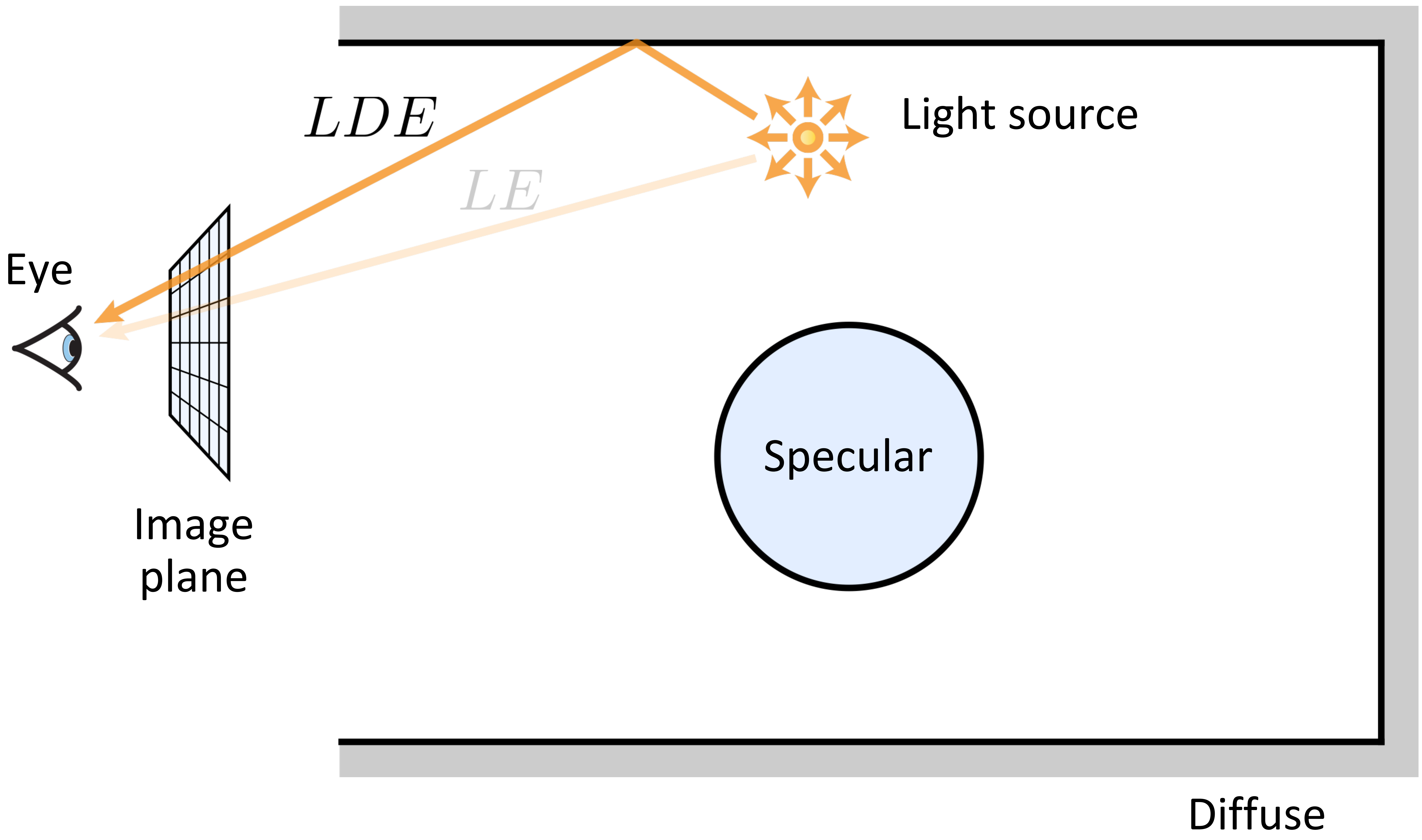
Heckbert's Classification



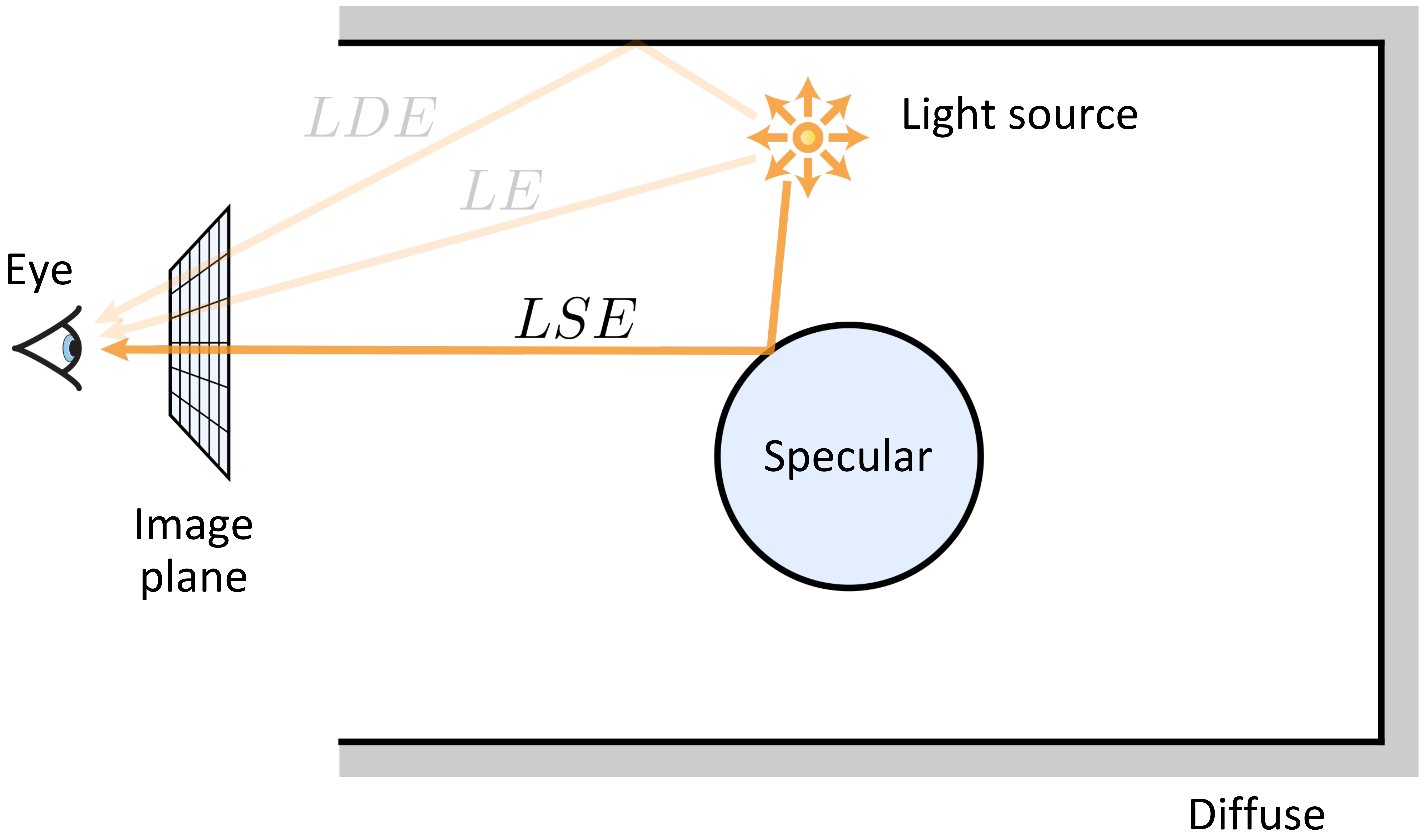
Heckbert's Classification



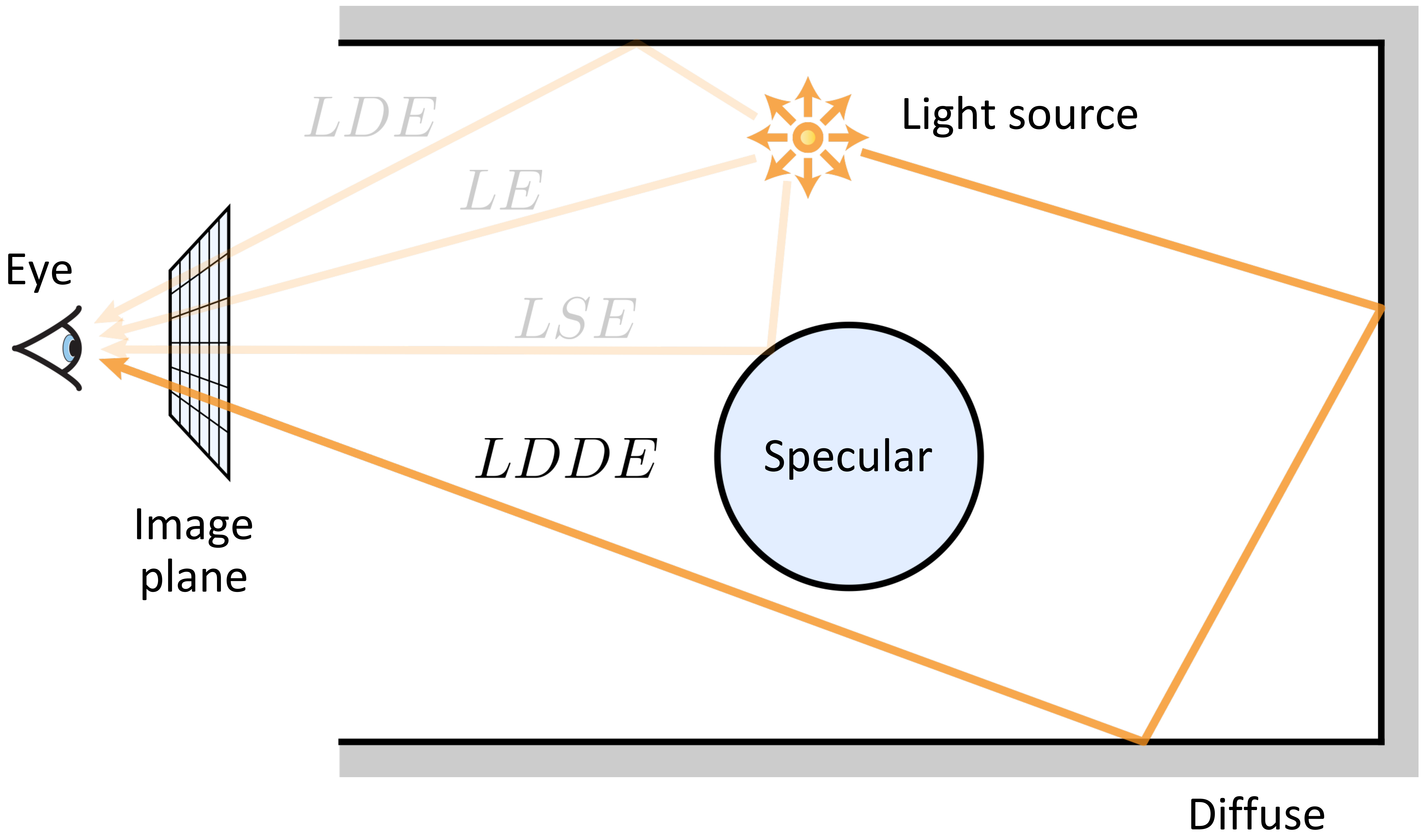
Heckbert's Classification



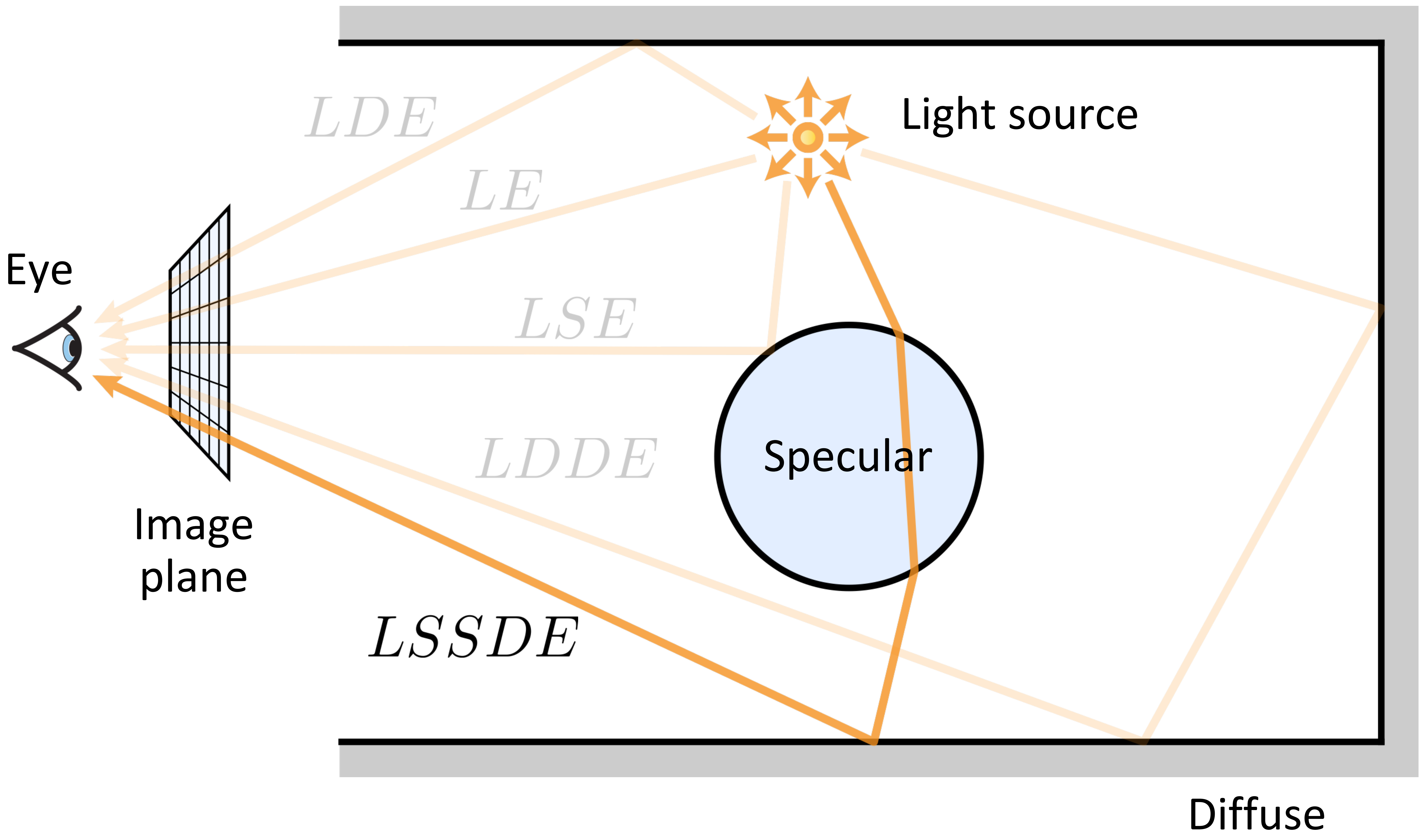
Heckbert's Classification



Heckbert's Classification



Heckbert’s Classification



Heckbert's Classification

Can express arbitrary classes of paths using a regular expression type syntax:

- k^+ : one or more of event k
- k^* : zero or more of event k
- $k?$: zero or one k events
- $(k|h)$: a k or h event

Heckbert's Classification

Direct illumination: $L(D|S)E$

Indirect illumination: $L(D|S)(D|S)^{+}E$

Heckbert's Classification

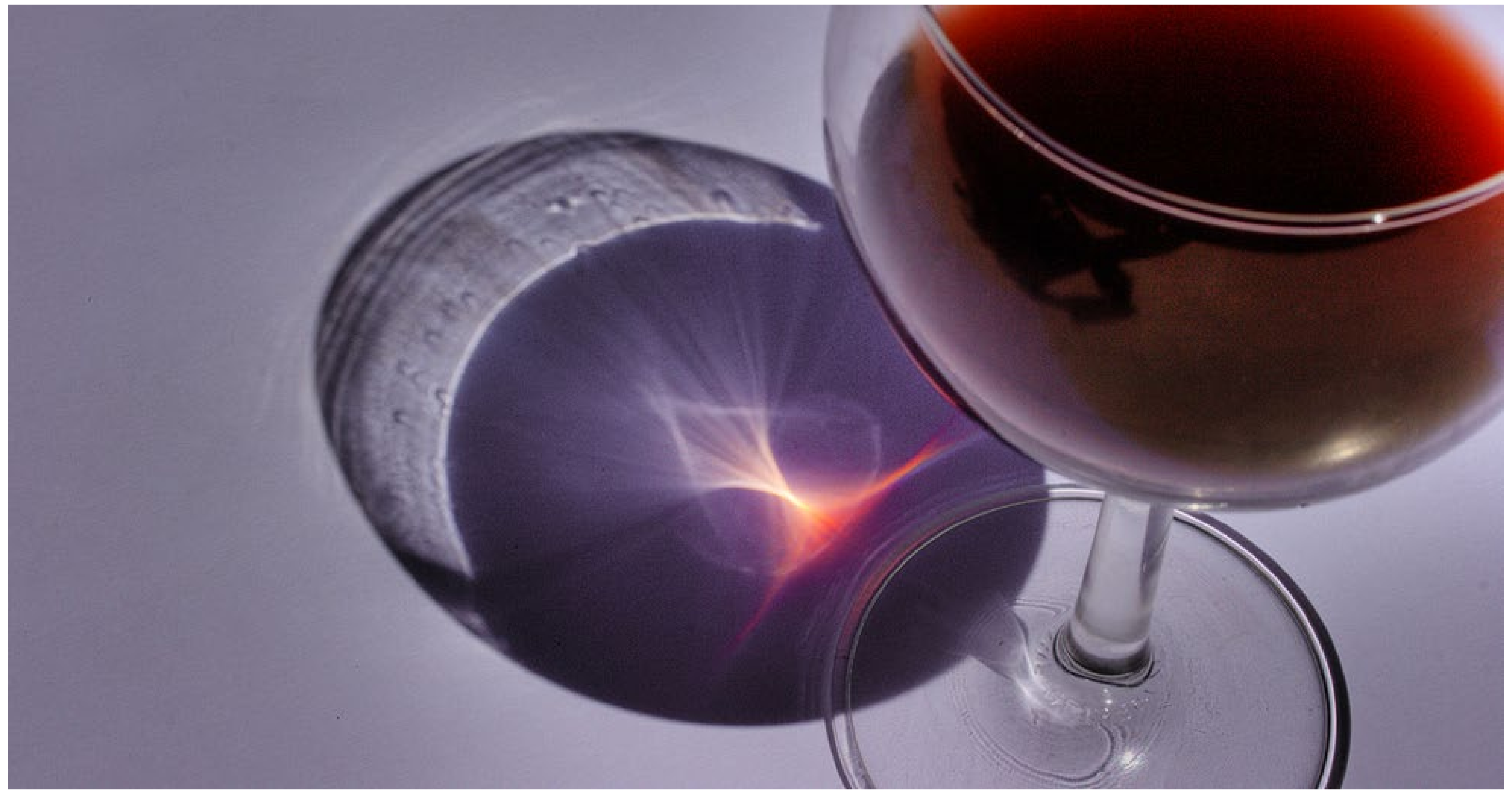
Direct illumination: $L(D|S)E$

Indirect illumination: $L(D|S)(D|S)^{+}E$

Full global illumination: $L(D|S)^{*}E$

Diffuse inter-reflections: $LDD^{+}E$

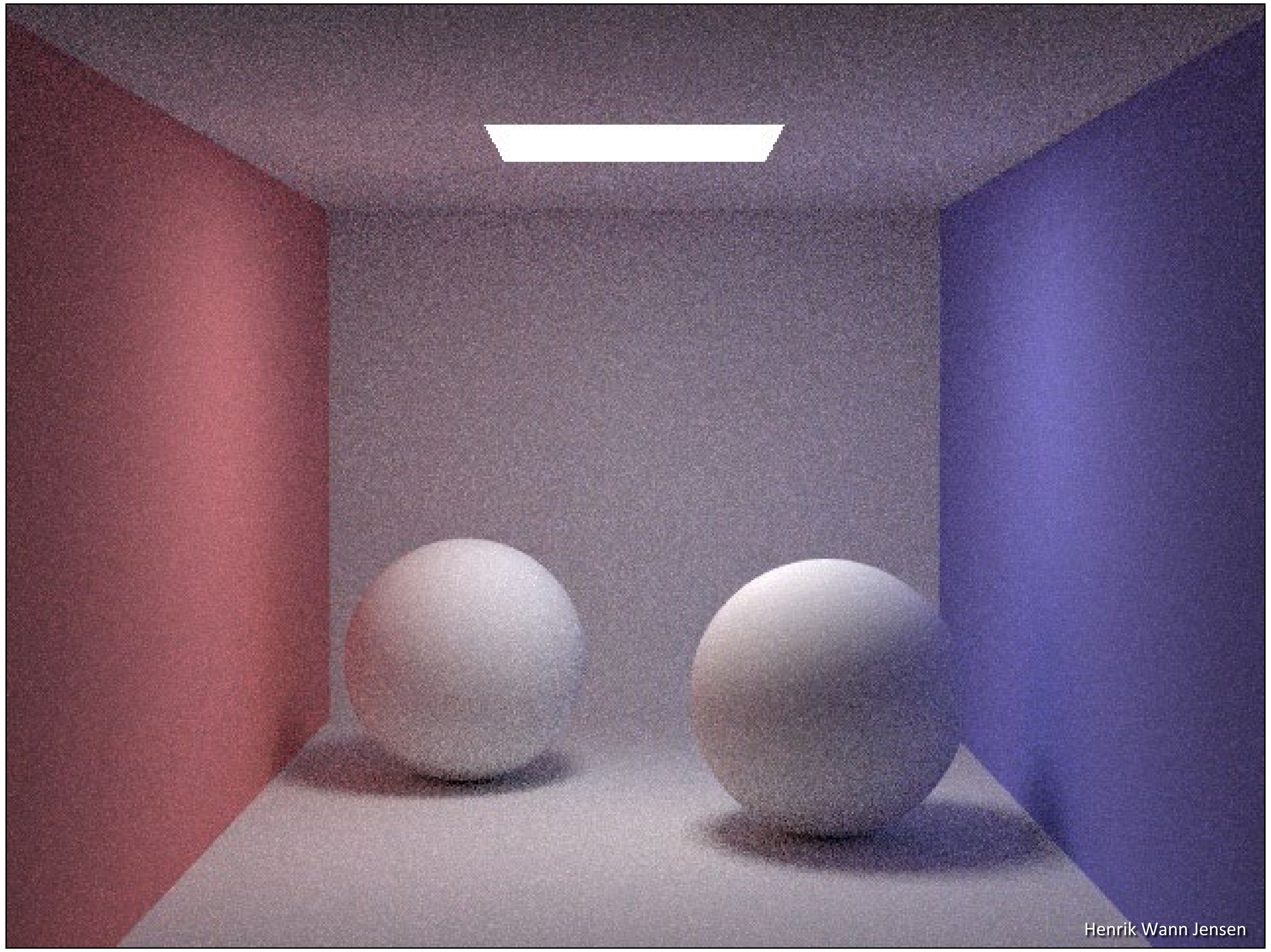
Caustics: $LS^{+}DE$




source: Flickr

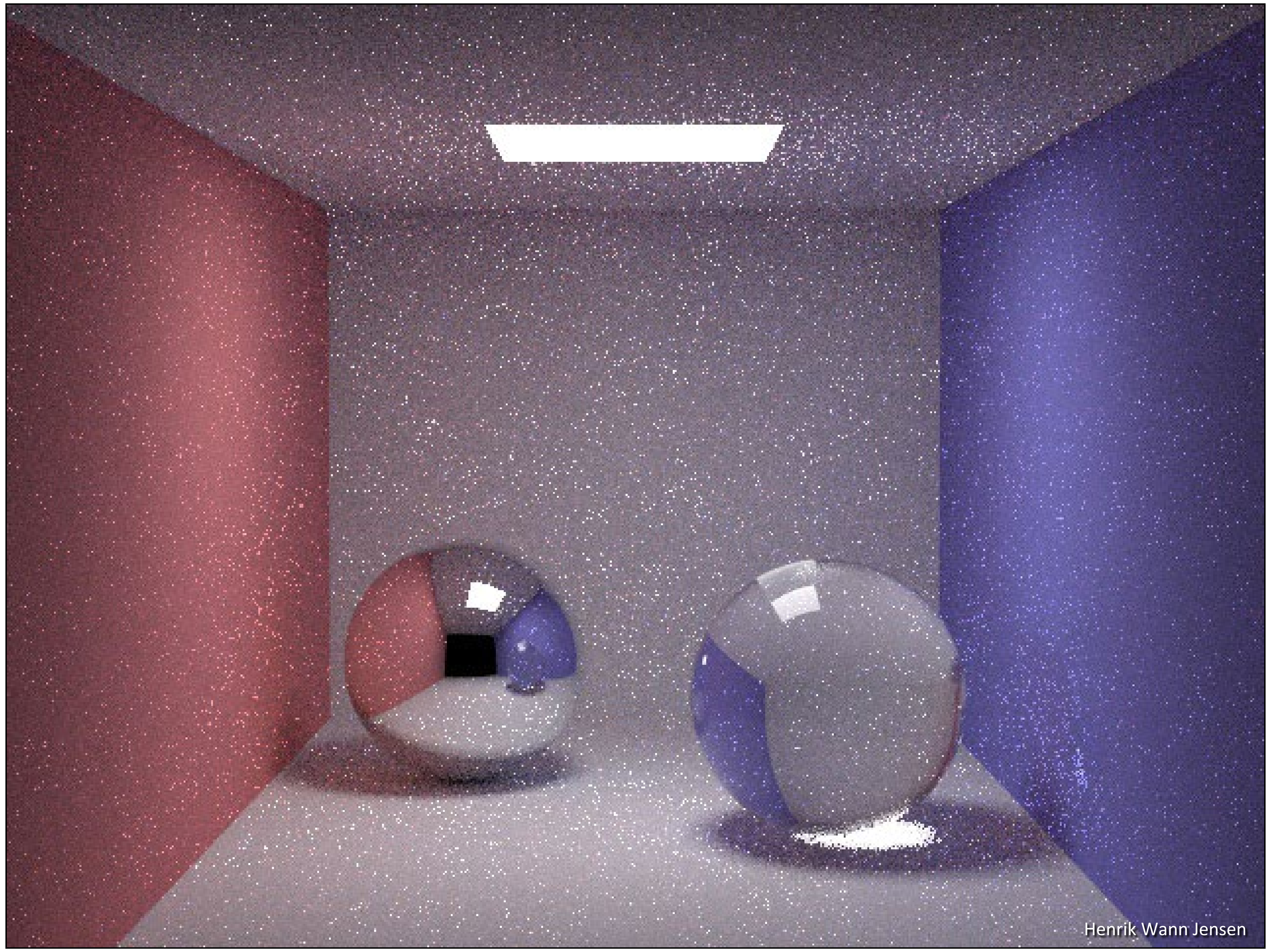
Subsurface Scattering

A Simple Scene



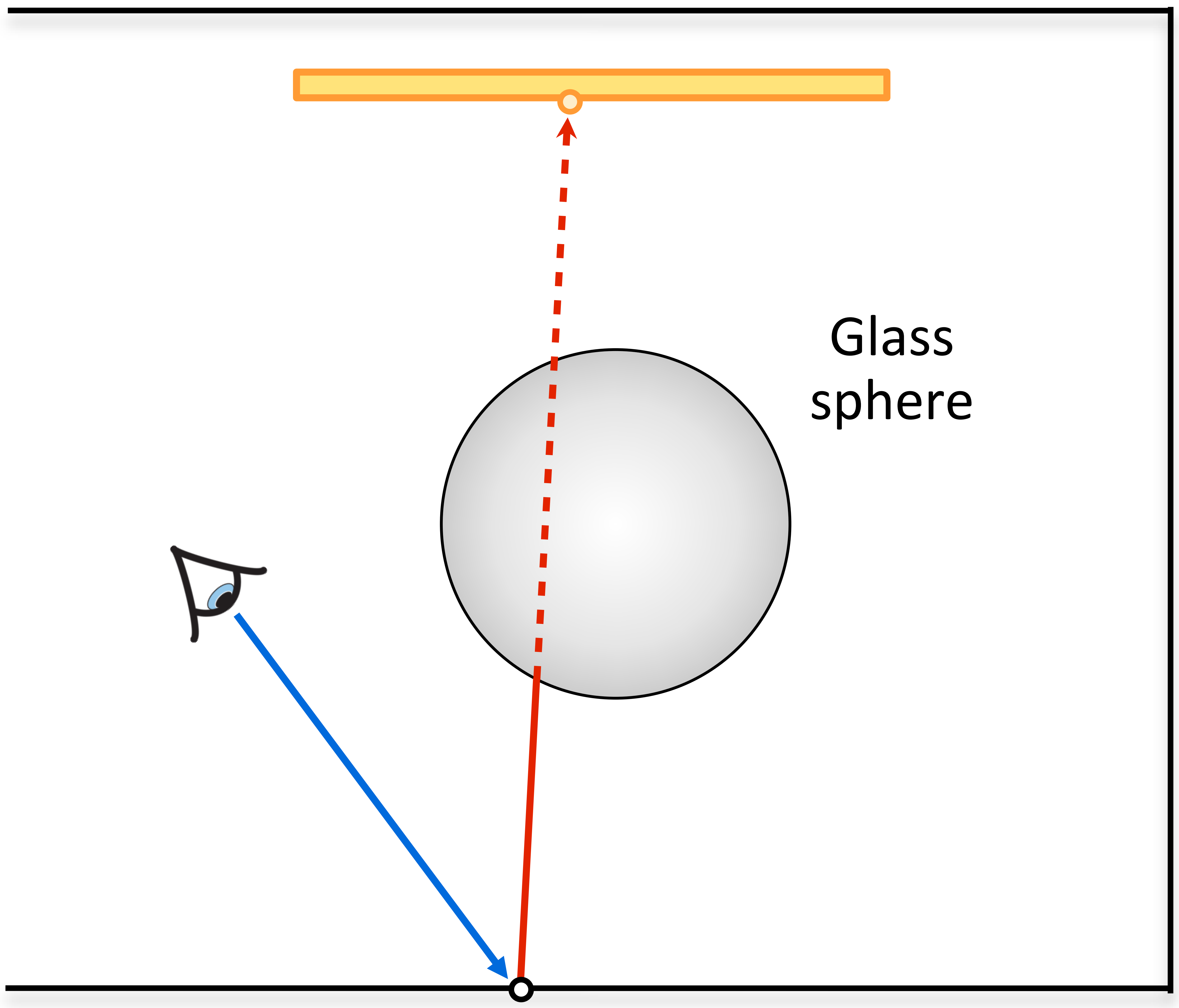
10 paths/pixel

+ Glass/Mirror Material

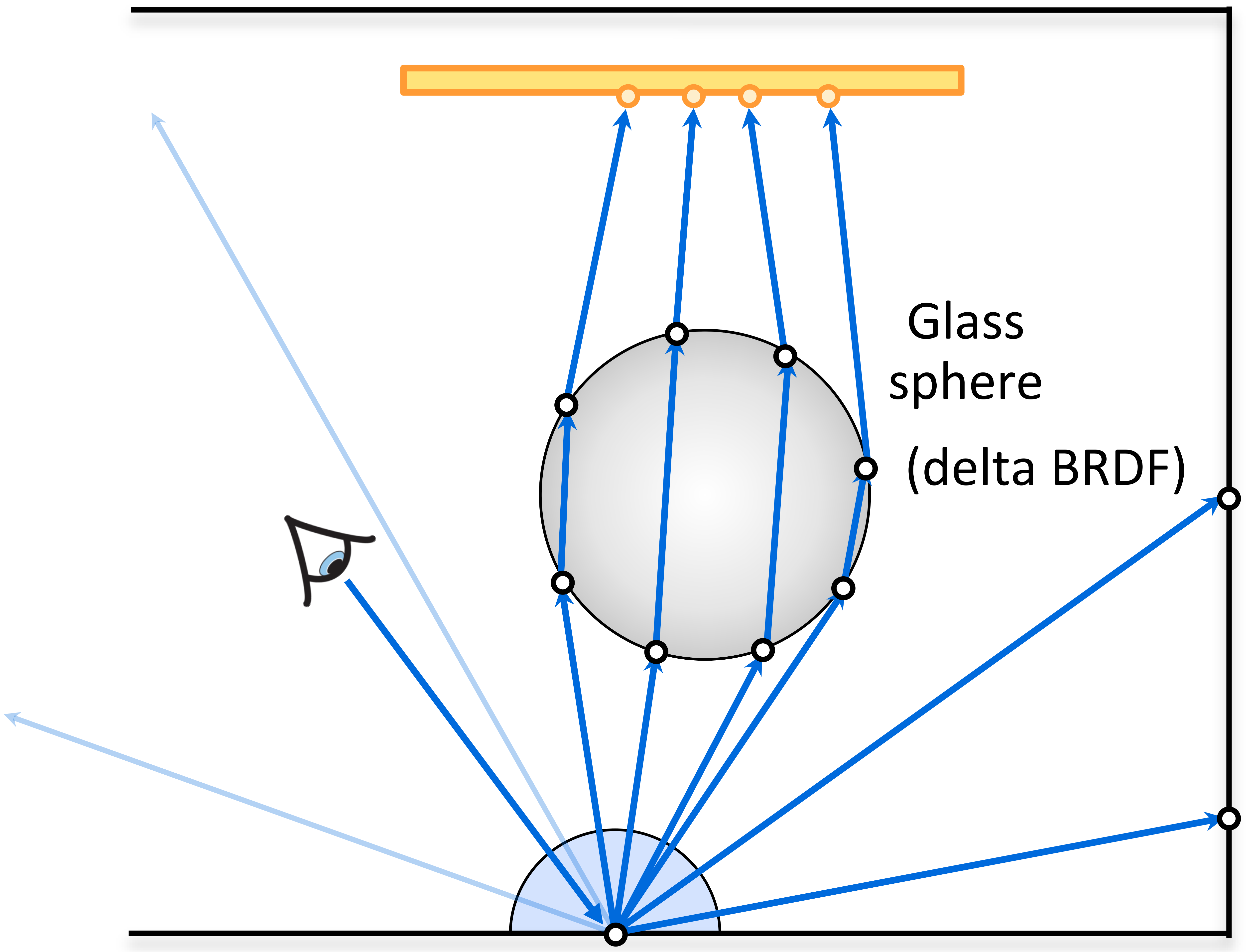


10 paths/pixel

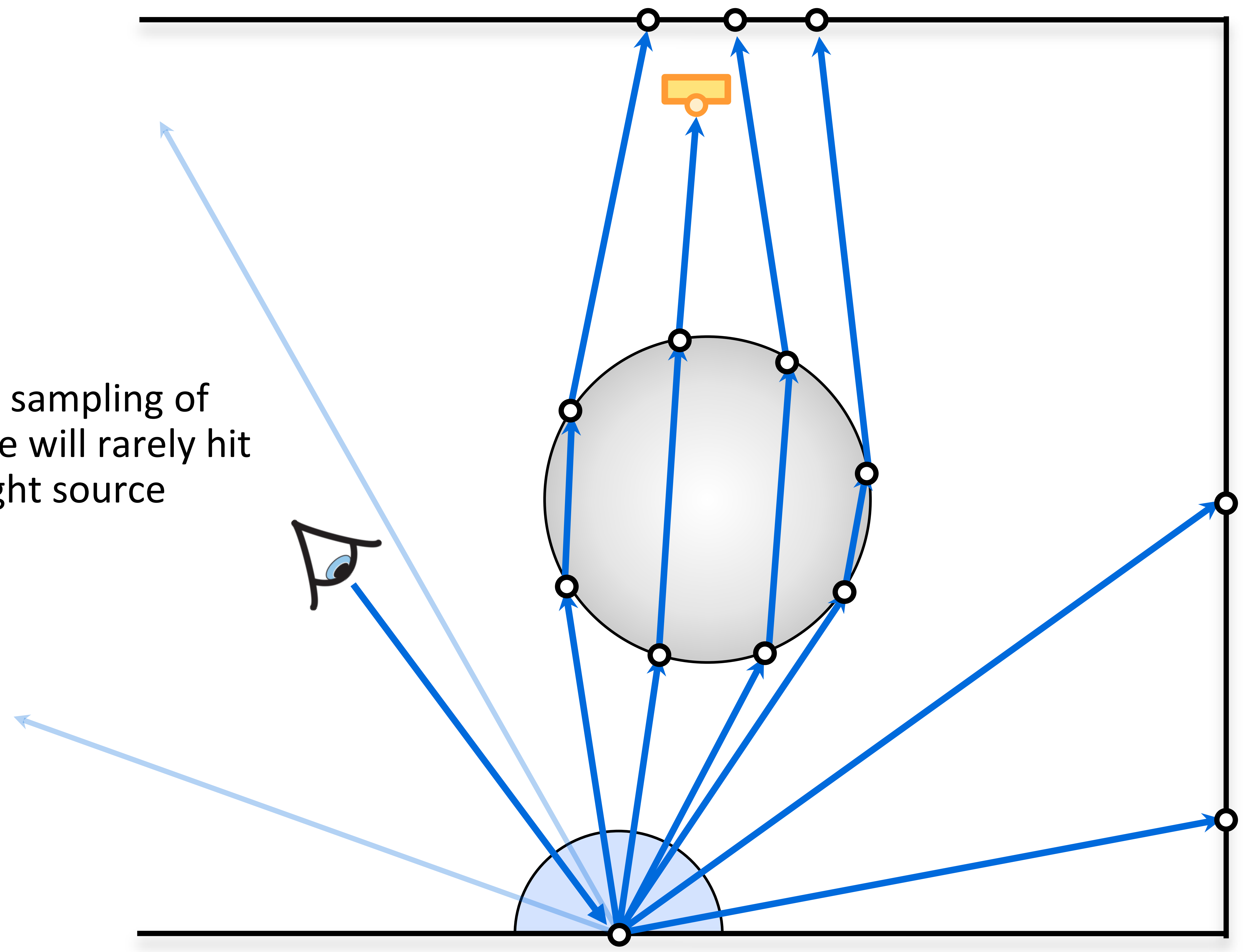
Path Tracing Caustics



Path Tracing Caustics



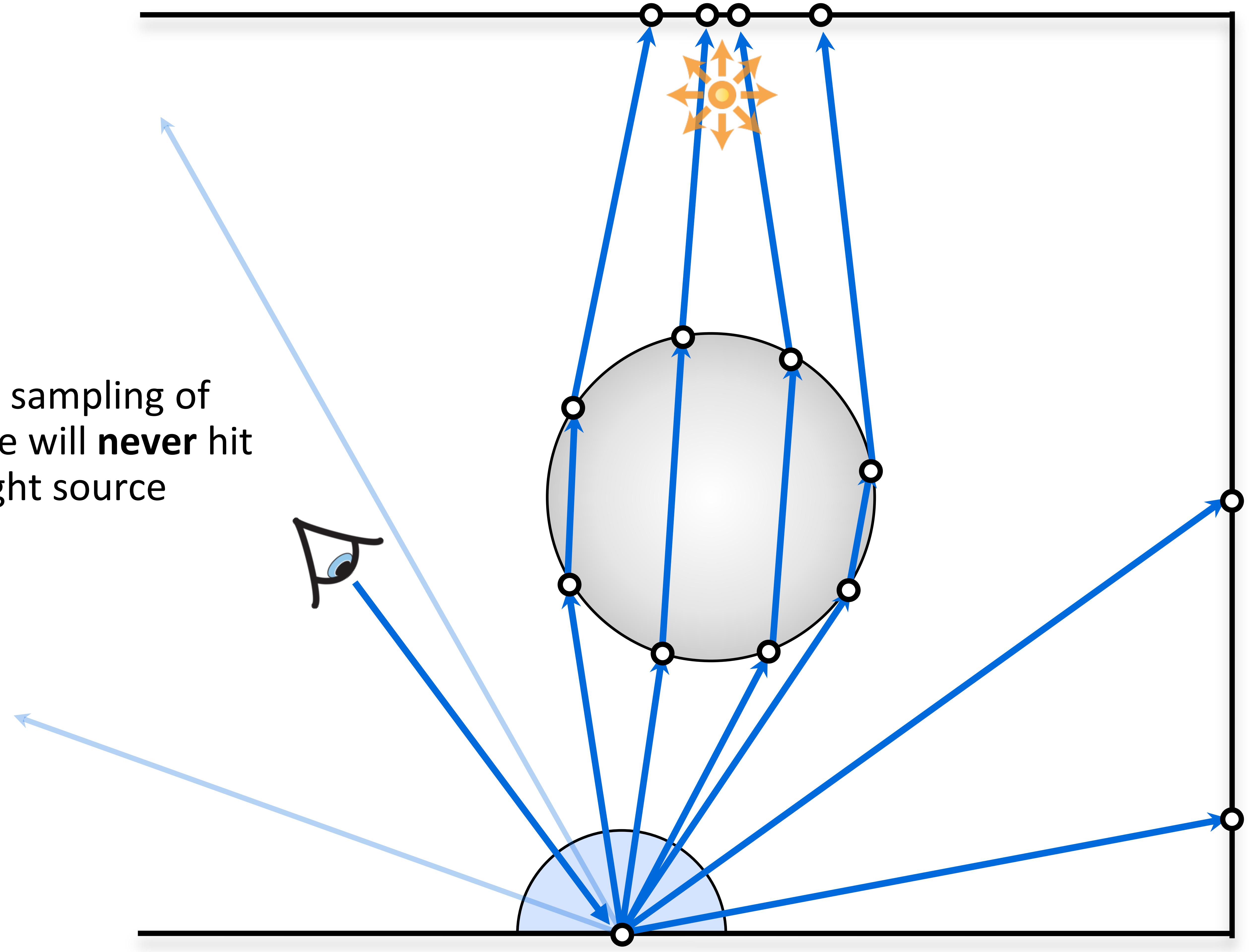
Path Tracing Caustics



Random sampling of
hemisphere will rarely hit
the light source

Path Tracing Caustics

Random sampling of
hemisphere will **never** hit
the light source



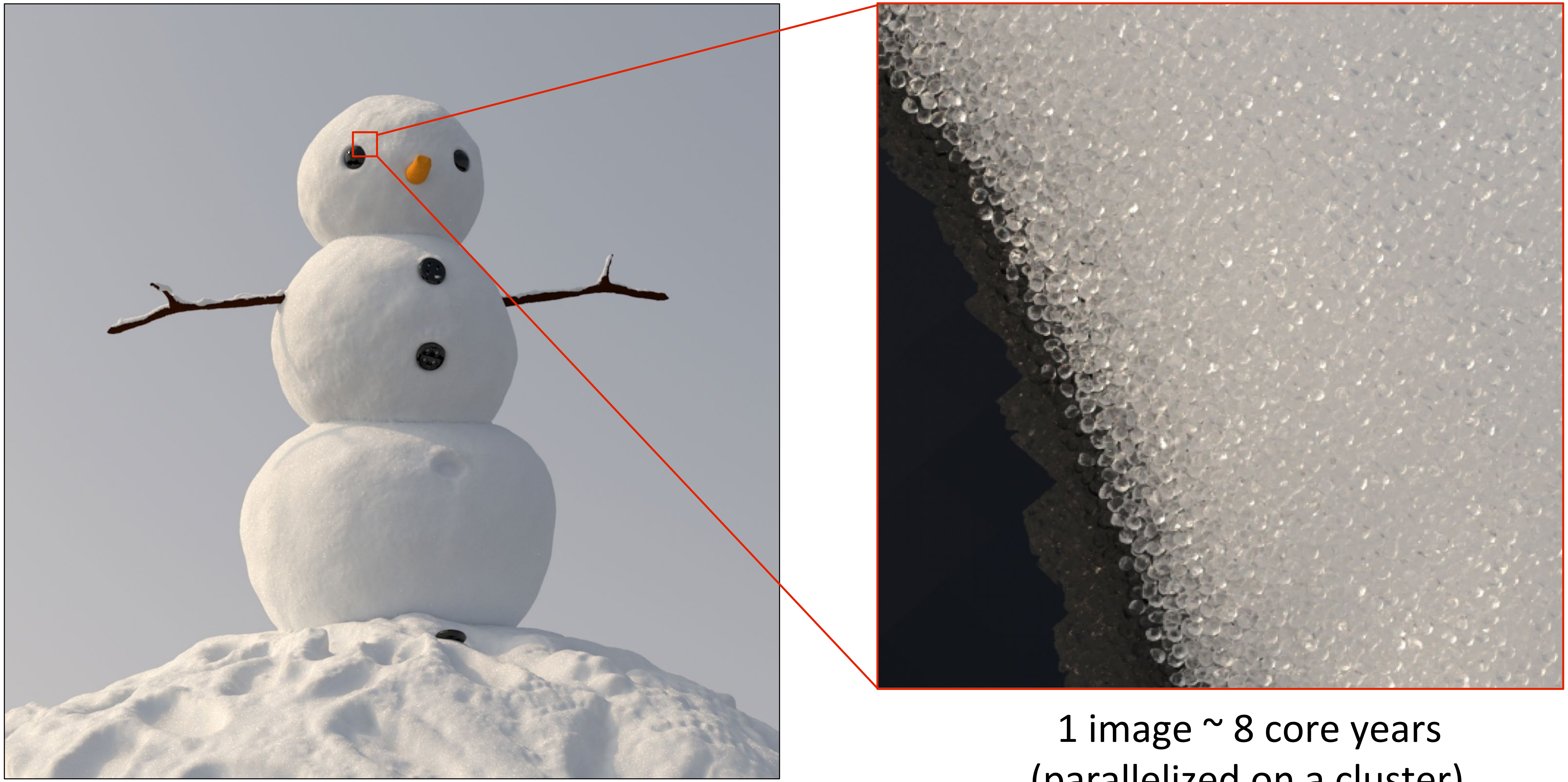
Let's just give it more time...

Nature ~ 2×10^{33} / second

Fastest GPU ray tracer ~ 2×10^{8} / second

Tim Webber, Gravity VFX supervisor

Let’s just give it more time...



1 image ~ 8 core years
(parallelized on a cluster)

Path Tracing - Summary

✓ Full solution to the rendering equation

✓ Simple to implement

✗ Slow convergence

– requires 4x more samples to half the error

✗ Robustness issues

– does not handle some light paths well (or not at all), e.g. caustics ($LS^{+}DE$)

✗ No reuse or caching of computation

✗ General sampling issue

– makes only locally good decisions

Today’s agenda

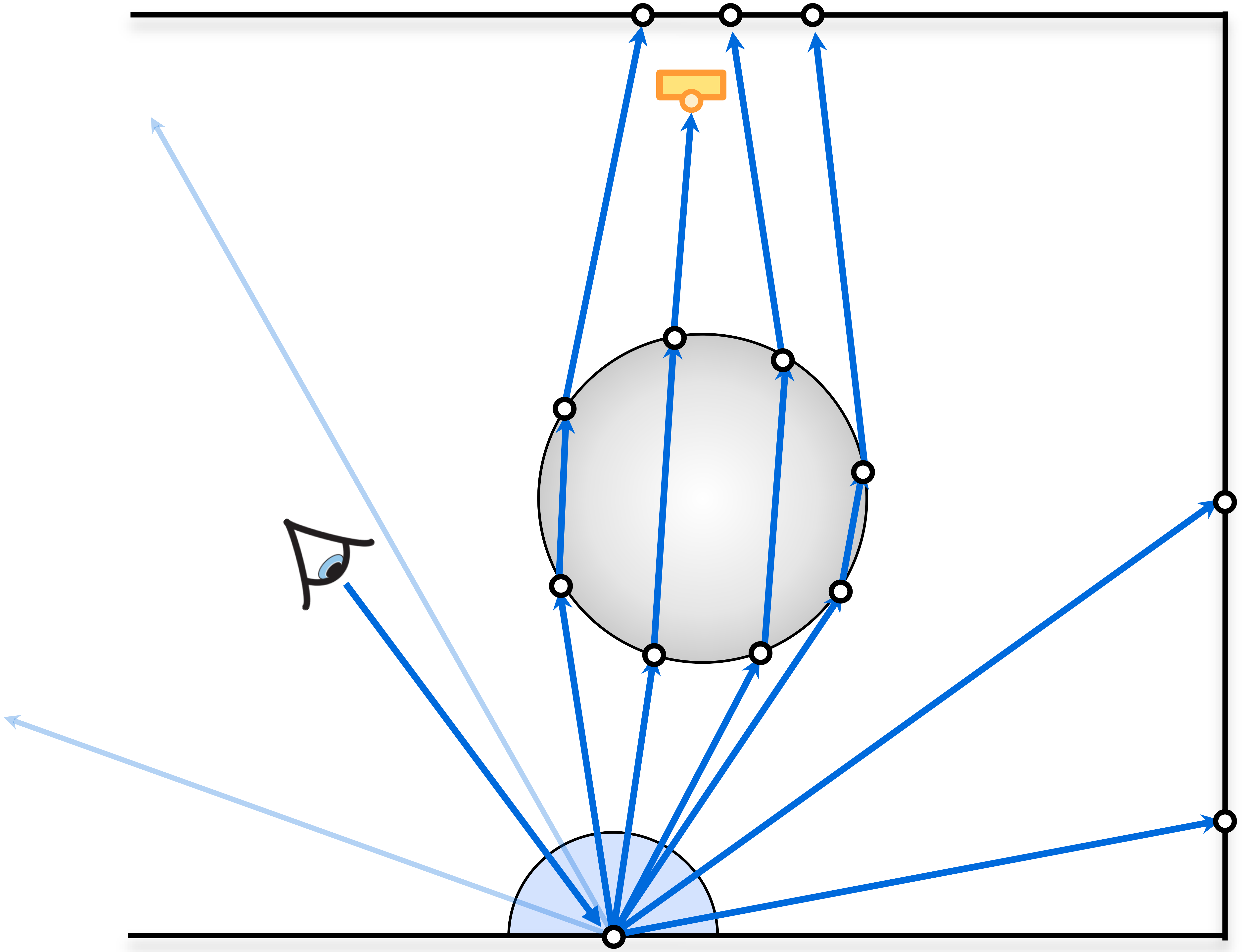
Measurement Equation

Path Integral Framework

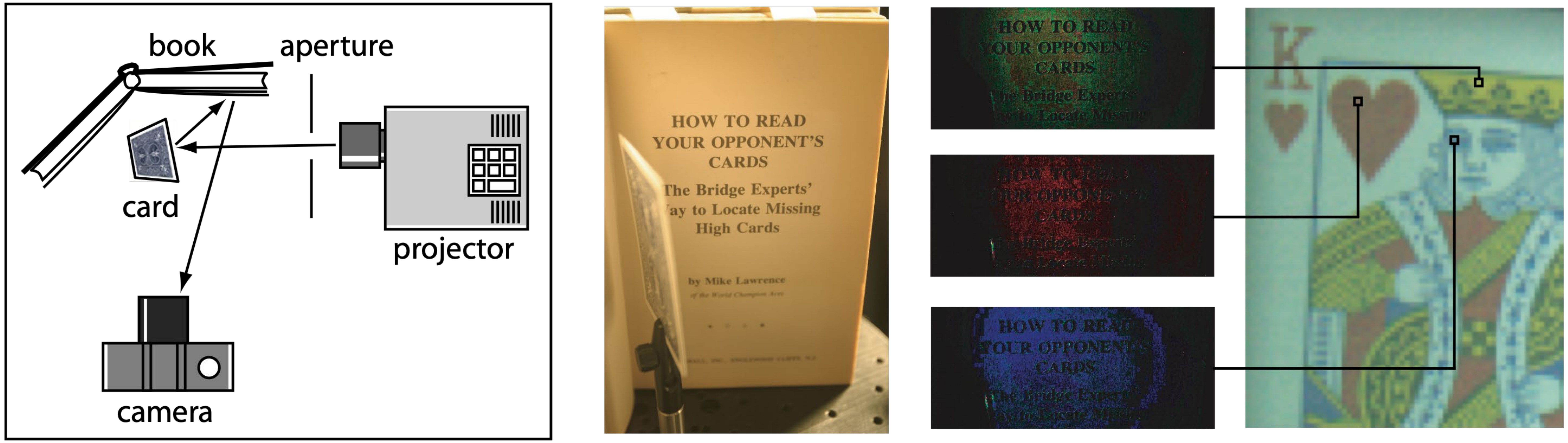
Solving the Rendering Equation

- Light tracing
- Bidirectional path tracing

Can we simulate this better?



Light transport is symmetric



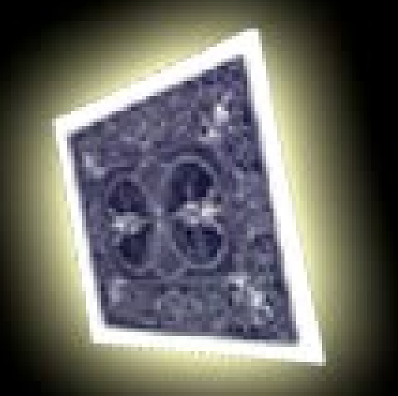
Dual Photography [Sen et al. 2005]

Dual Photography

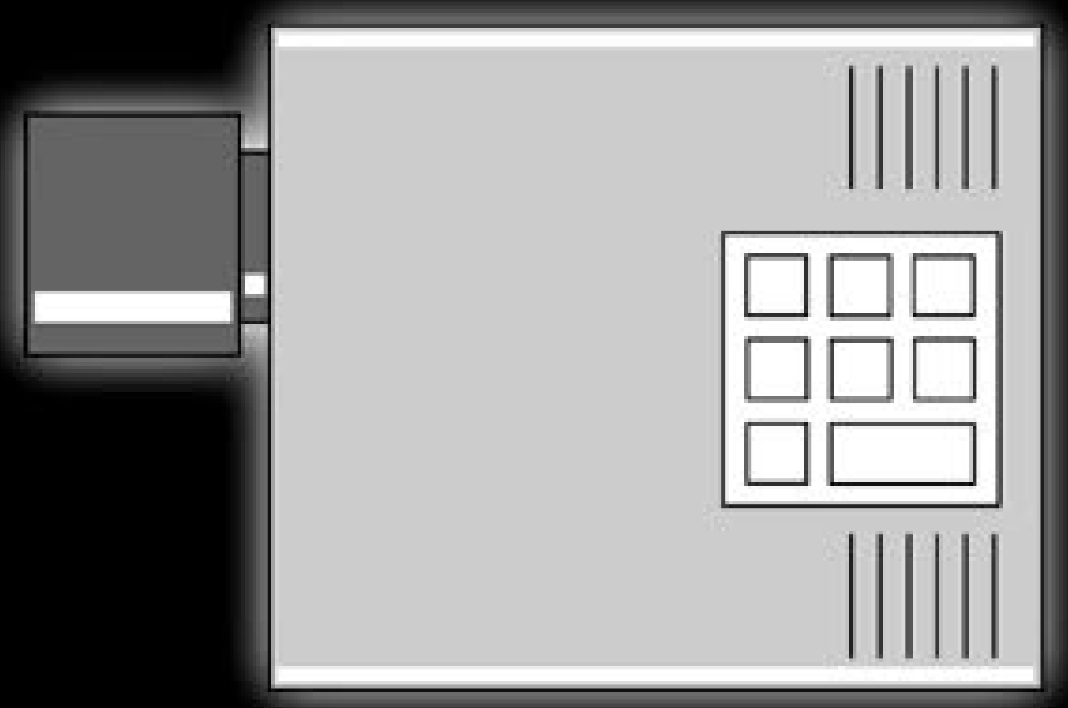
Pradeep Sen* Billy Chen* Gaurav Garg* Stephen R. Marschner†
Mark Horowitz* Marc Levoy* Hendrik P.A. Lensch*

*Stanford University †Cornell University

SIGGRAPH2005



card



projector

Duality of Radiance and Importance

Measurement Equation

Rendering equation describes radiative equilibrium at point $\mathbf{x}$:

$$L_o(\mathbf{x}, \vec{\omega}_o) = L_e(\mathbf{x}, \vec{\omega}_o) + \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_o) L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

We are interested in the total radiance contributing to pixel j:

$$I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x}$$

response of the sensor at film location $\mathbf{x}$
to radiance arriving from direction $\vec{\omega}$
(often referred to as *emitted importance*)

Radiometry as Measurements

Weighted integral of 5D radiance function

$$\int_V \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L(\mathbf{x}, \vec{\omega}) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\mathbf{x}$$

Other radiometric quantities are measurements

- expressing *irradiance* in terms of radiance:

$$\int_{H^2} L(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} = E(\mathbf{x})$$

Integrate radiance over hemisphere

- expressing *flux/power* in terms of radiance:

$$\int_A \int_{H^2} L(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} dA(\mathbf{x}) = \Phi(A)$$

Integrate radiance over hemisphere and area

Radiance vs. Importance

Radiance

- emitted from light sources
- describes *amount of light* traveling within a differential beam

Importance

- “emitted” from sensors
- describes the *response of the sensor* to radiance traveling within a differential beam

Duality of Radiance & Importance

$$I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x}$$

Duality of Radiance & Importance

$$I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x}$$

$$= \int_{A_{\text{film}}} \int_{A} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) L_o(\mathbf{y}, \mathbf{x}) \, d\mathbf{y} d\mathbf{x}$$

outgoing quantities

Let's expand L_o and consider direct illumination only

Duality of Radiance & Importance

$$\begin{aligned} I_j &= \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x} \\ &= \int_{A_{\text{film}}} \int_{A} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) L_o(\mathbf{y}, \mathbf{x}) \, d\mathbf{y} d\mathbf{x} \\ &= \int_{A_{\text{film}}} \int_{A} \int_{A_{\text{light}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}, \mathbf{z}, \mathbf{x}) G(\mathbf{y}, \mathbf{z}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{z} d\mathbf{y} d\mathbf{x} \end{aligned}$$

emitted quantities with
identical measure

Let's swap the inner
and outer integral

Duality of Radiance & Importance

$$\begin{aligned}
I_j &= \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) L_o(\mathbf{y}, \mathbf{x}) \, d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} \int_{A_{\text{light}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}, \mathbf{z}, \mathbf{x}) G(\mathbf{y}, \mathbf{z}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{z} d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{light}}} \int_{A} \int_{A_{\text{film}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}, \mathbf{z}, \mathbf{x}) G(\mathbf{y}, \mathbf{z}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{x} d\mathbf{y} d\mathbf{z}
\end{aligned}$$

symmetric functions

Duality of Radiance & Importance

$$
\begin{aligned}
I_j &= \int_{A_{\mathrm{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x} \\
&= \int_{A_{\mathrm{film}}} \int_{A} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) L_o(\mathbf{y}, \mathbf{x}) \, d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\mathrm{film}}} \int_{A} \int_{A_{\mathrm{light}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}, \mathbf{z}, \mathbf{x}) G(\mathbf{y}, \mathbf{z}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{z} d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\mathrm{light}}} \int_{A} \int_{A_{\mathrm{film}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{y}, \mathbf{x}) f(\mathbf{y}, \mathbf{x}, \mathbf{z}) G(\mathbf{z}, \mathbf{y}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{x} d\mathbf{y} d\mathbf{z}
\end{aligned}
$$

symmetric functions

Duality of Radiance & Importance

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\begin{aligned}
I_j &= \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) L_o(\mathbf{y}, \mathbf{x}) \, d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} \int_{A_{\text{light}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}, \mathbf{z}, \mathbf{x}) G(\mathbf{y}, \mathbf{z}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{z} d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{light}}} \int_{A} \int_{A_{\text{film}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{y}, \mathbf{x}) f(\mathbf{y}, \mathbf{x}, \mathbf{z}) G(\mathbf{z}, \mathbf{y}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{x} d\mathbf{y} d\mathbf{z} \\
&= \int_{A_{\text{light}}} \int_{A} W_o(\mathbf{y}, \mathbf{z}) G(\mathbf{z}, \mathbf{y}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{y} d\mathbf{z}
\end{aligned}
$$

Duality of Radiance & Importance

$$
\begin{aligned}
I_j &= \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) L_o(\mathbf{y}, \mathbf{x}) \, d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{film}}} \int_{A} \int_{A_{\text{light}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}, \mathbf{z}, \mathbf{x}) G(\mathbf{y}, \mathbf{z}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{z} d\mathbf{y} d\mathbf{x} \\
&= \int_{A_{\text{light}}} \int_{A} \int_{A_{\text{film}}} W_e(\mathbf{x}, \mathbf{y}) G(\mathbf{y}, \mathbf{x}) f(\mathbf{y}, \mathbf{x}, \mathbf{z}) G(\mathbf{z}, \mathbf{y}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{x} d\mathbf{y} d\mathbf{z} \\
&= \int_{A_{\text{light}}} \int_{A} W_o(\mathbf{y}, \mathbf{z}) G(\mathbf{z}, \mathbf{y}) L_e(\mathbf{z}, \mathbf{y}) \, d\mathbf{y} d\mathbf{z} \\
&= \int_{A_{\text{light}}} \int_{H^2} W_i(\mathbf{z}, \vec{\omega}) L_e(\mathbf{z}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{z}
\end{aligned}
$$

Duality of Radiance & Importance

emitted *importance*

incident *radiance*

$$I_j = \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x}$$

$$= \int_{A_{\text{light}}} \int_{H^2} W_i(\mathbf{z}, \vec{\omega}) L_e(\mathbf{z}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{z}$$

emitted *radiance*

incident *importance*

Duality of Radiance & Importance

Path tracing

start from *film*, search for *radiance* at light

$$
\begin{aligned}
I_j &= \int_{A_{\text{film}}} \int_{H^2} W_e(\mathbf{x}, \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{x} \\
&= \int_{A_{\text{light}}} \int_{H^2} W_i(\mathbf{z}, \vec{\omega}) L_e(\mathbf{z}, \vec{\omega}) \cos\theta \, d\vec{\omega} d\mathbf{z}
\end{aligned}
$$

Light tracing

start from *light*, search for *importance* at sensor

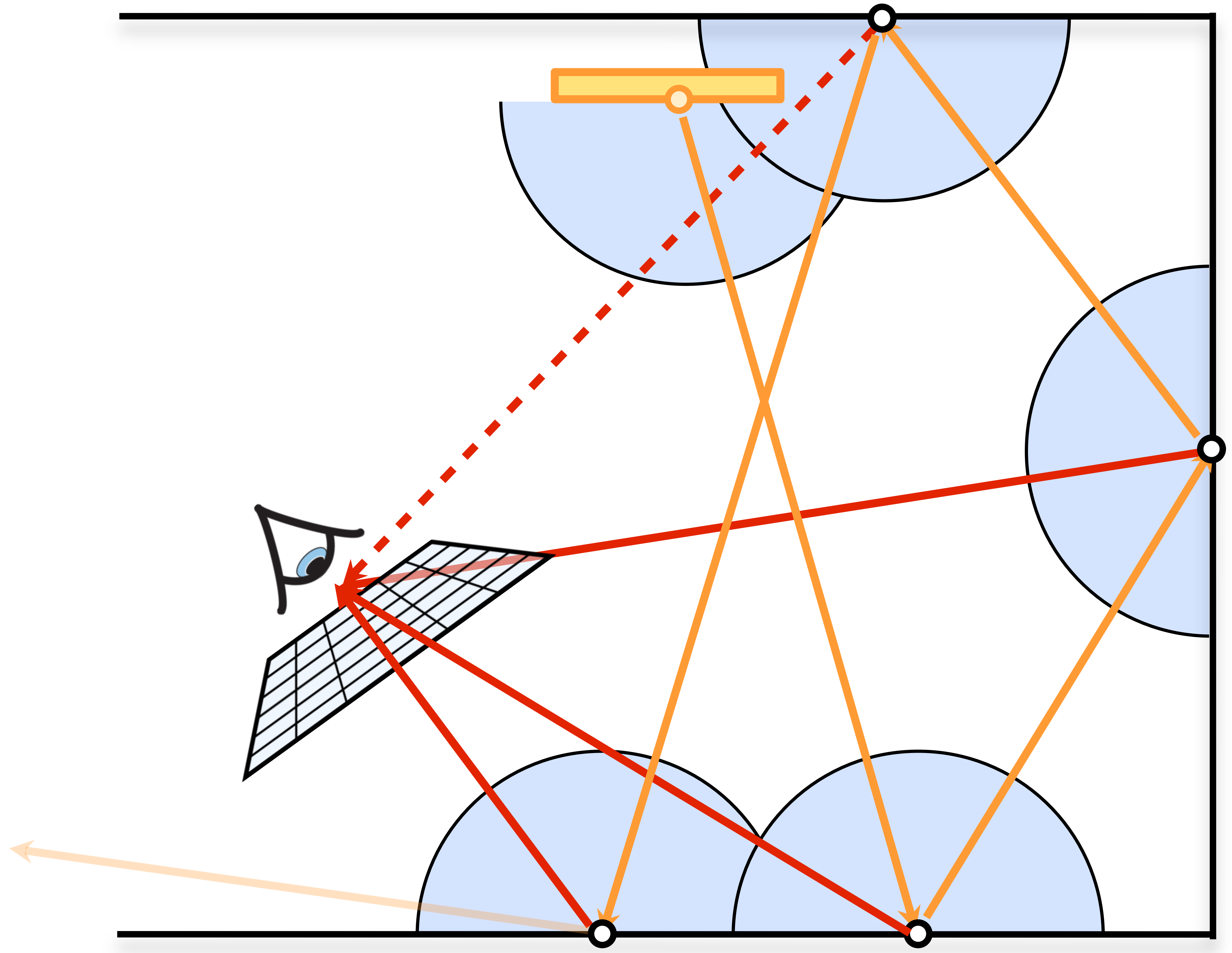
Light Tracing

Light Tracing

Shoot multiple paths from light sources hoping to randomly hit the sensor

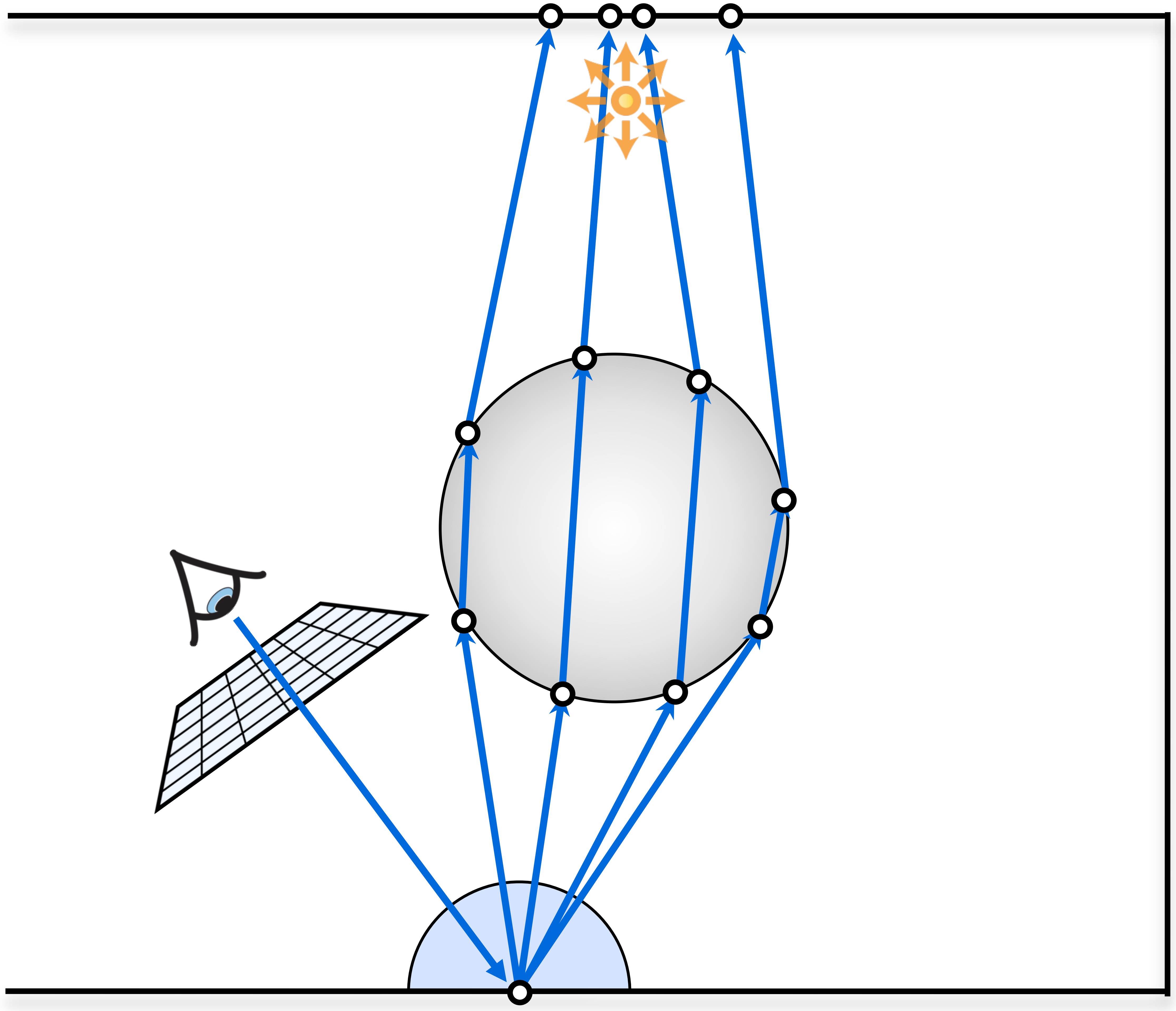
- Optionally: at each path vertex, connect to the image using next-event estimation (a.k.a. shadow rays in PT)

Light Tracing with NEE

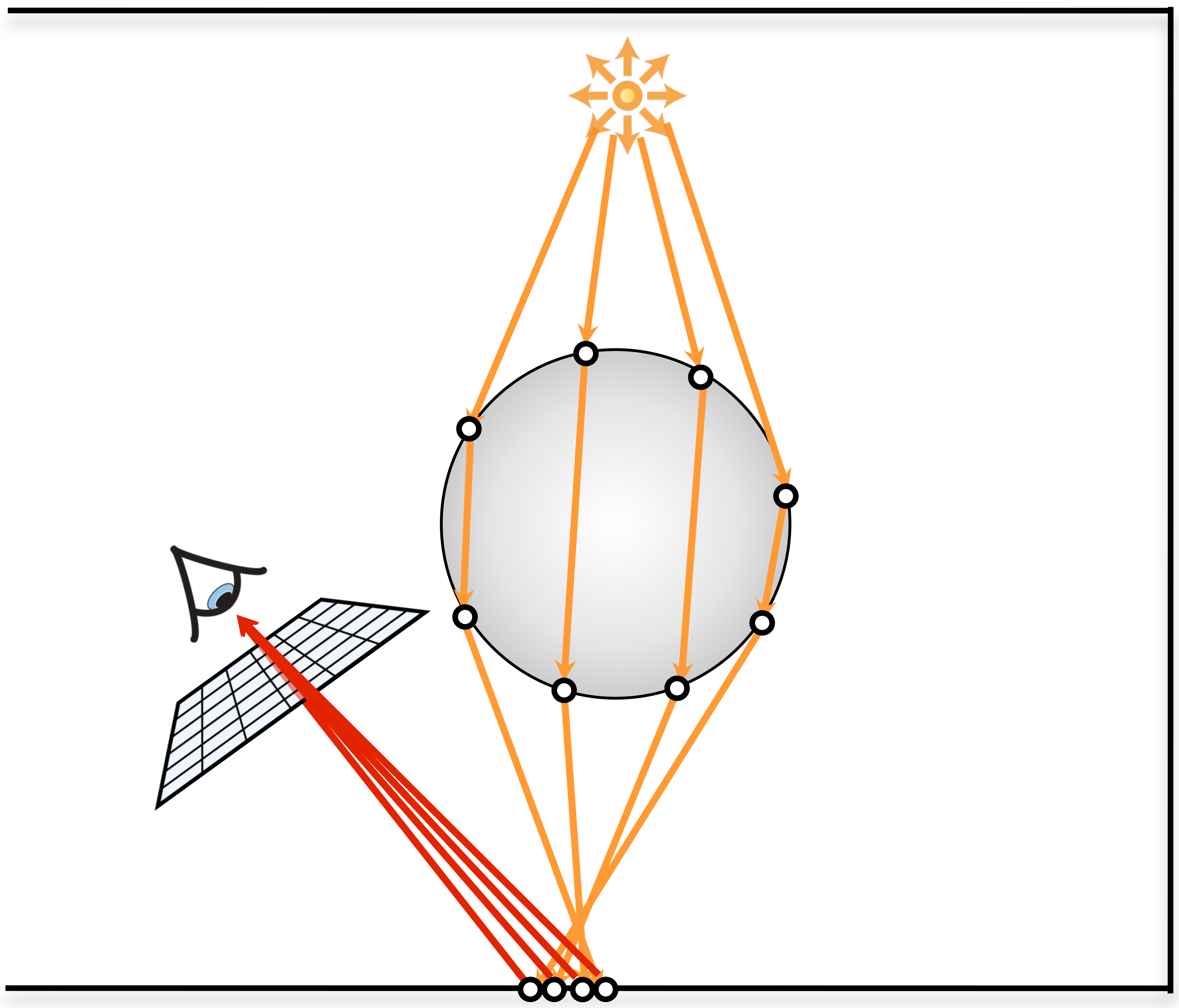


Splat to the image at each vertex

Path Tracing Caustics

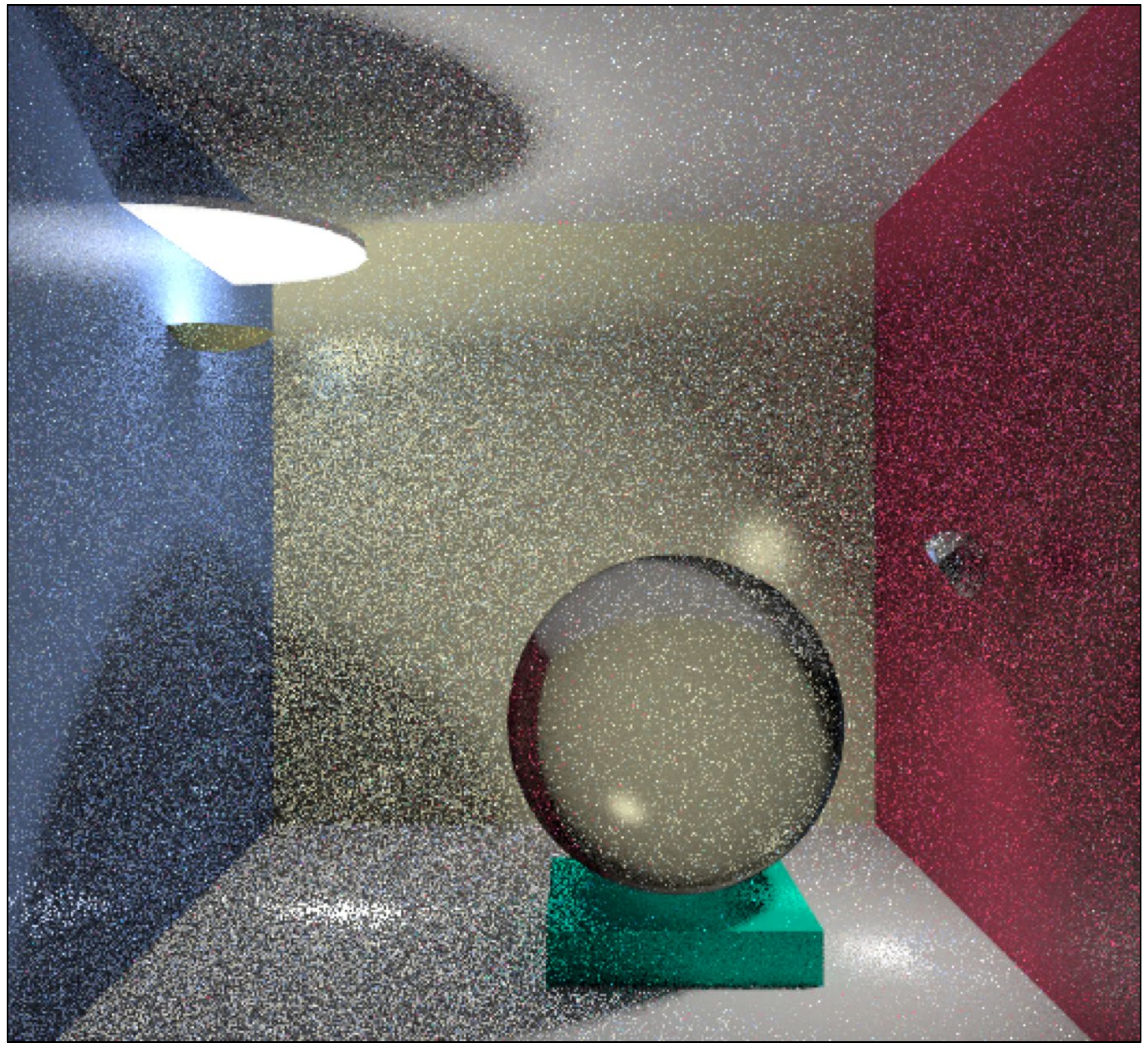


Light Tracing Caustics

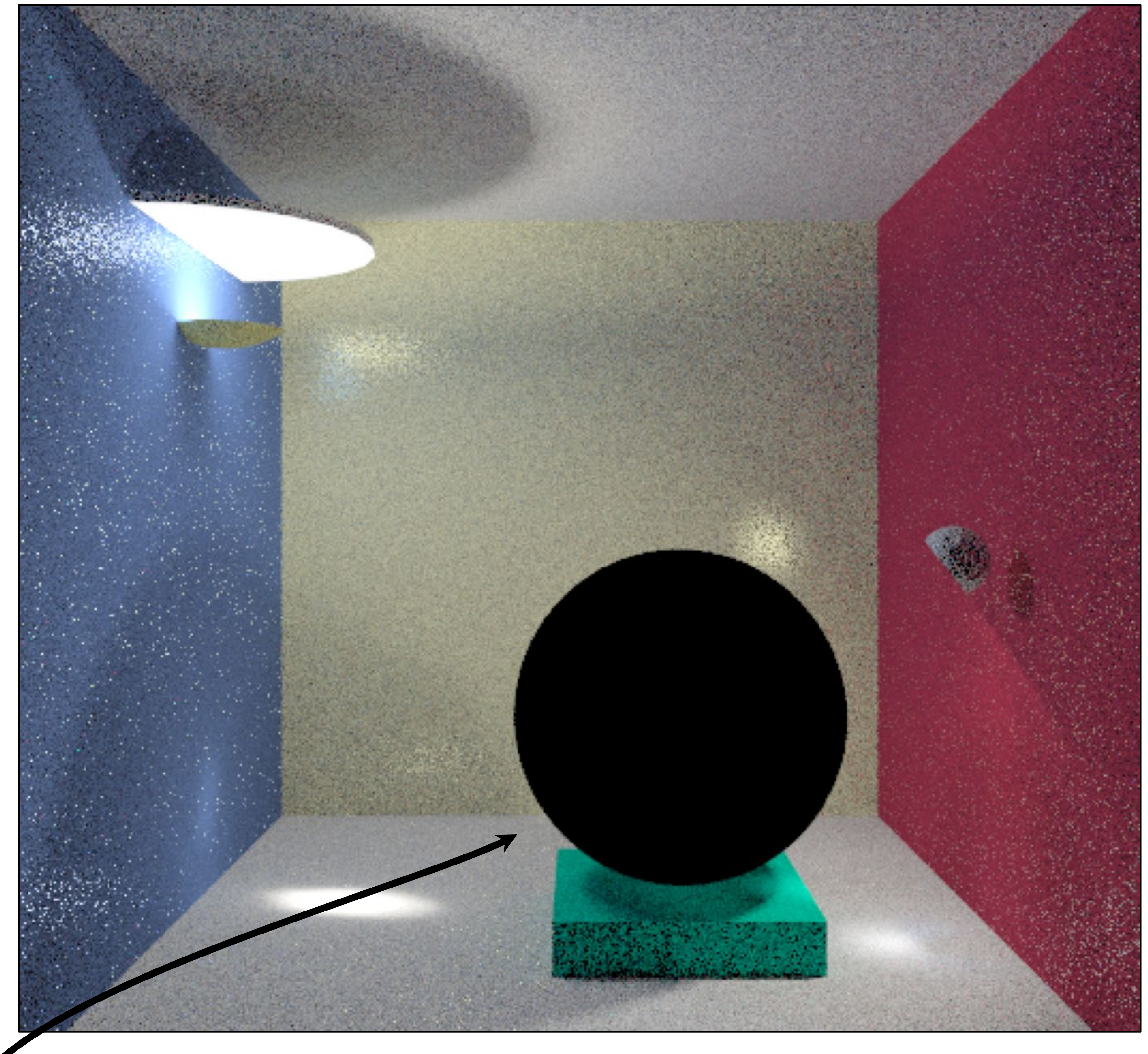


Path vs. Light Tracing

Path tracing



Light tracing

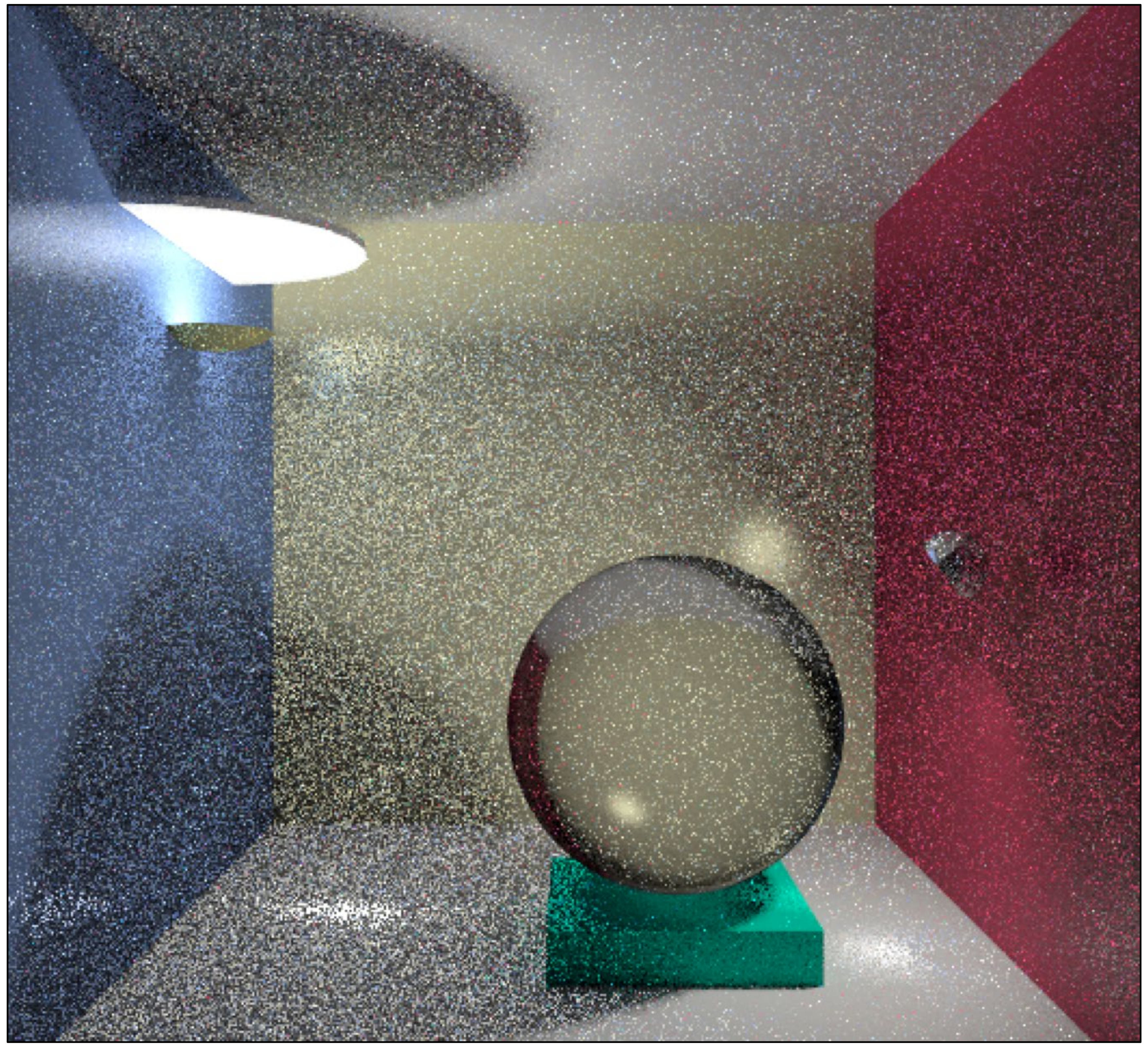


Images courtesy of F. Suykens

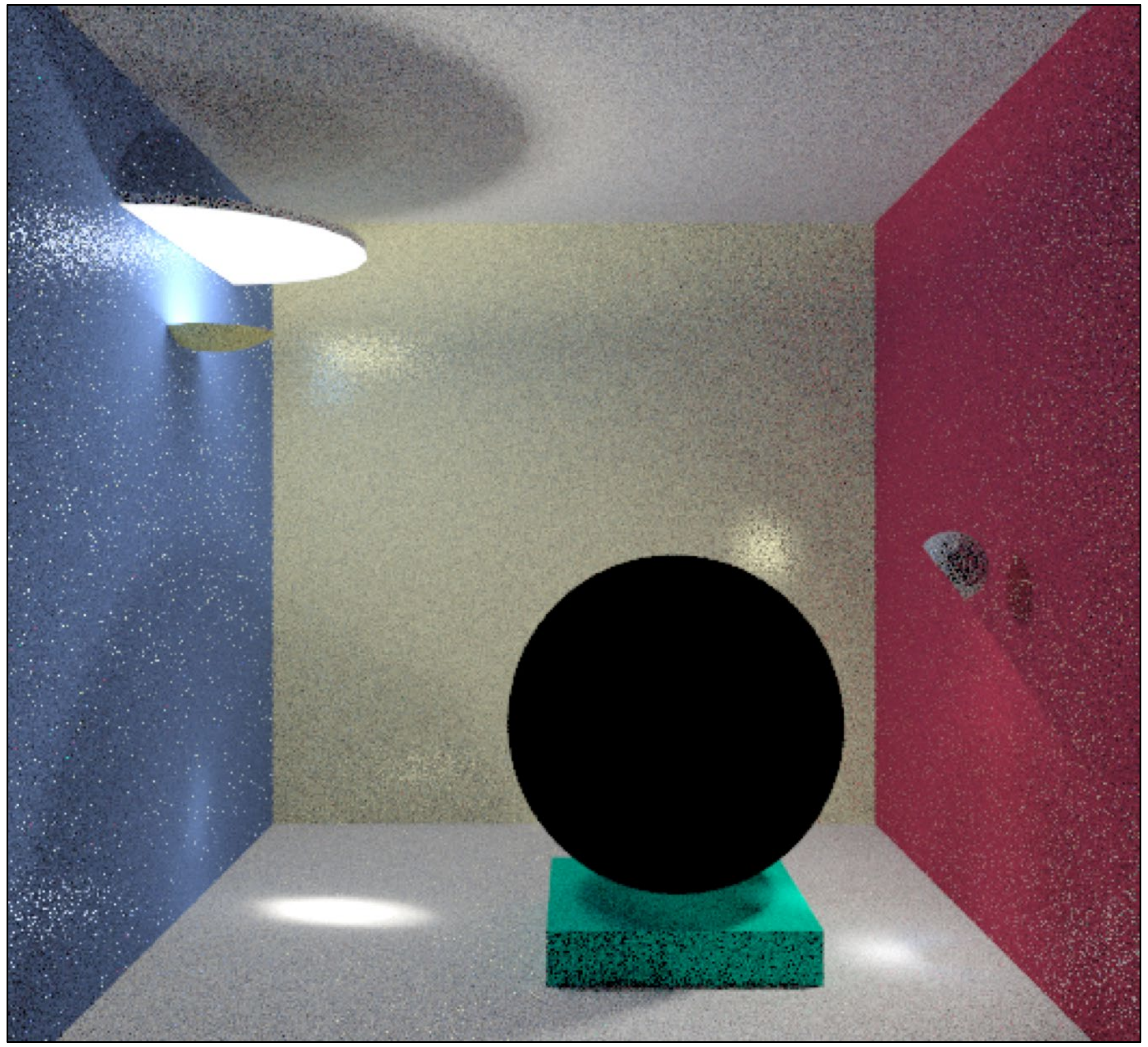
Why is this glass sphere black?

Path vs. Light Tracing

Path tracing



Light tracing



Images courtesy of F. Suykens

Can we combine them?

Path Integral Framework

Measurement Equation

$$I_j = \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_o(\mathbf{x}_1, \mathbf{x}_0) \, d\mathbf{x}_1 d\mathbf{x}_0$$

$$= \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_1, \mathbf{x}_0) + \int_A f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_0) G(\mathbf{x}_1, \mathbf{x}_2) L_o(\mathbf{x}_2, \mathbf{x}_1) d\mathbf{x}_2 d\mathbf{x}_1 d\mathbf{x}_0$$

$$= \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_1, \mathbf{x}_0) + \int_A f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_0) G(\mathbf{x}_1, \mathbf{x}_2) L_e(\mathbf{x}_2, \mathbf{x}_1) + \int_A f(\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_1) G(\mathbf{x}_2, \mathbf{x}_3) \, L_e(\mathbf{x}_3, \mathbf{x}_2) + \int_A \cdots \, d\mathbf{x}_4 d\mathbf{x}_3 d\mathbf{x}_2 d\mathbf{x}_1 d\mathbf{x}_0$$

Hard to concisely express arbitrary light transport with all the nested integrals

Let’s find a better way

Path Integral Form of Measurement Eq.

$$I_j = \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_o(\mathbf{x}_1, \mathbf{x}_0)\, d\mathbf{x}_1 d\mathbf{x}_0$$

Emission

$$= \iint_A W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_1, \mathbf{x}_0) G(\mathbf{x}_0, \mathbf{x}_1) d\mathbf{x}_1 d\mathbf{x}_0$$

Direct illumination (3 vertices)

$$+ \iiint_A W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_2, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_0) G(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_2 d\mathbf{x}_1 d\mathbf{x}_0 + \cdots$$

(k-2)-bounce illumination (k vertices)

$$+ \int \cdots \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) G(\mathbf{x}_0, \mathbf{x}_1) \prod_{j=1}^{k-1} f(\mathbf{x}_j, \mathbf{x}_{j+1}, \mathbf{x}_{j-1}) G(\mathbf{x}_j, \mathbf{x}_{j+1})\, d\mathbf{x}_k \cdots d\mathbf{x}_0 + \cdots$$

introduce: $\mathcal{P}_k = \{\bar{\mathbf{x}} = \mathbf{x}_0 \cdots \mathbf{x}_k;\ \mathbf{x}_0 \cdots \mathbf{x}_k \in A\}$

space of all paths with k segments

Path Integral Form of Measurement Eq.

$$I_j = \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_o(\mathbf{x}_1, \mathbf{x}_0)\, d\mathbf{x}_1 d\mathbf{x}_0$$

Emission

$$= \int_{\mathcal{P}_1} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_1, \mathbf{x}_0) G(\mathbf{x}_0, \mathbf{x}_1) d\bar{\mathbf{x}}_1$$

Direct illumination (3 vertices)

$$+ \int_{\mathcal{P}_2} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_2, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_0) G(\mathbf{x}_1, \mathbf{x}_2) d\bar{\mathbf{x}}_2 + \cdots$$

(k-2)-bounce illumination (k vertices)

$$+ \int_{\mathcal{P}_k} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) G(\mathbf{x}_0, \mathbf{x}_1) \prod_{j=1}^{k-1} f(\mathbf{x}_j, \mathbf{x}_{j+1}, \mathbf{x}_{j-1}) G(\mathbf{x}_j, \mathbf{x}_{j+1})\, d\bar{\mathbf{x}}_k + \cdots$$

introduce: $$T(\bar{\mathbf{x}}_k) = G(\mathbf{x}_0, \mathbf{x}_1) \prod_{j=1}^{k-1} f(\mathbf{x}_j, \mathbf{x}_{j+1}, \mathbf{x}_{j-1}) G(\mathbf{x}_j, \mathbf{x}_{j+1})$$

throughput of path $\bar{\mathbf{x}}_k$

Path Integral Form of Measurement Eq.

$$I_j = \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_o(\mathbf{x}_1, \mathbf{x}_0)\, d\mathbf{x}_1 d\mathbf{x}_0$$

Emission

$$= \int_{\mathcal{P}_1} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_1, \mathbf{x}_0) T(\bar{\mathbf{x}}_1) d\bar{\mathbf{x}}_1$$

Direct illumination (3 vertices)

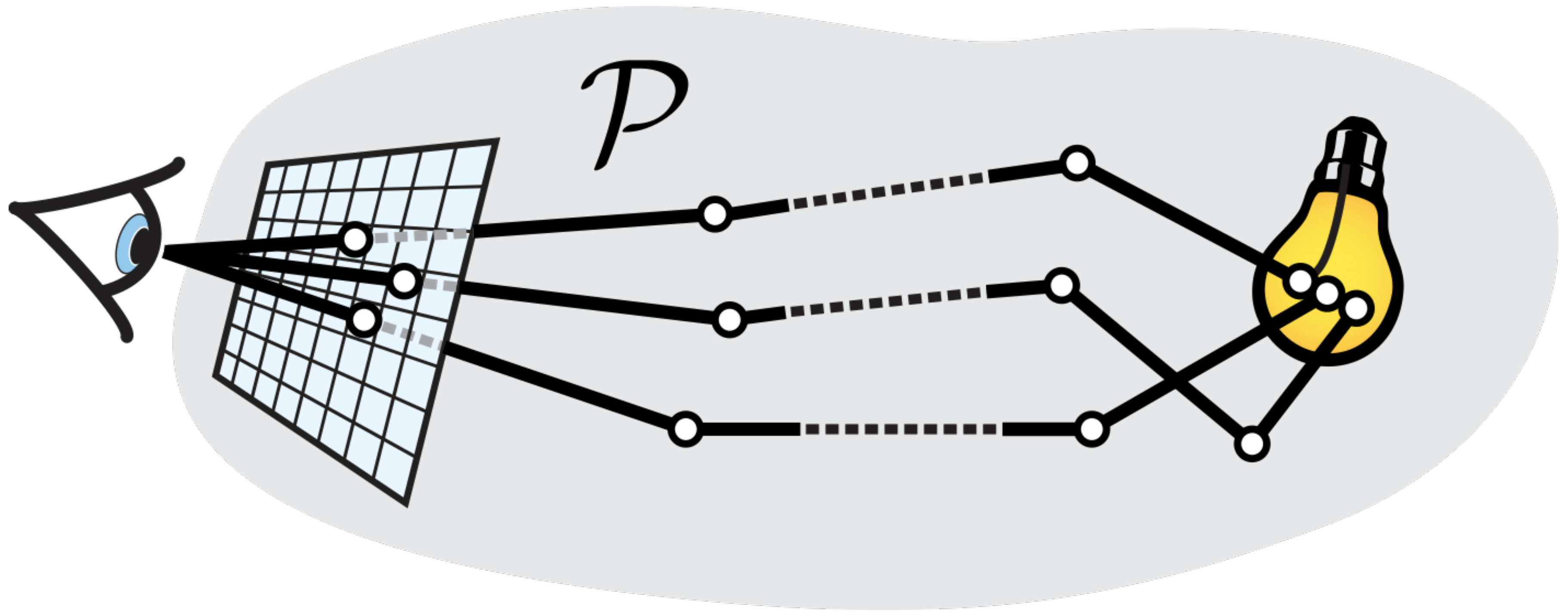
$$+ \int_{\mathcal{P}_2} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_2, \mathbf{x}_1) T(\bar{\mathbf{x}}_2)\, d\bar{\mathbf{x}}_2 + \cdots$$

(k-2)-bounce illumination (k vertices)

$$+ \int_{\mathcal{P}_k} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) T(\bar{\mathbf{x}}_k)\, d\bar{\mathbf{x}}_k + \cdots$$

introduce: $\mathcal{P} = \bigcup_{k=1}^{\infty} \mathcal{P}_k$

the *path space*, i.e. the space of all paths of all lengths



Path Integral Form of Measurement Eq.

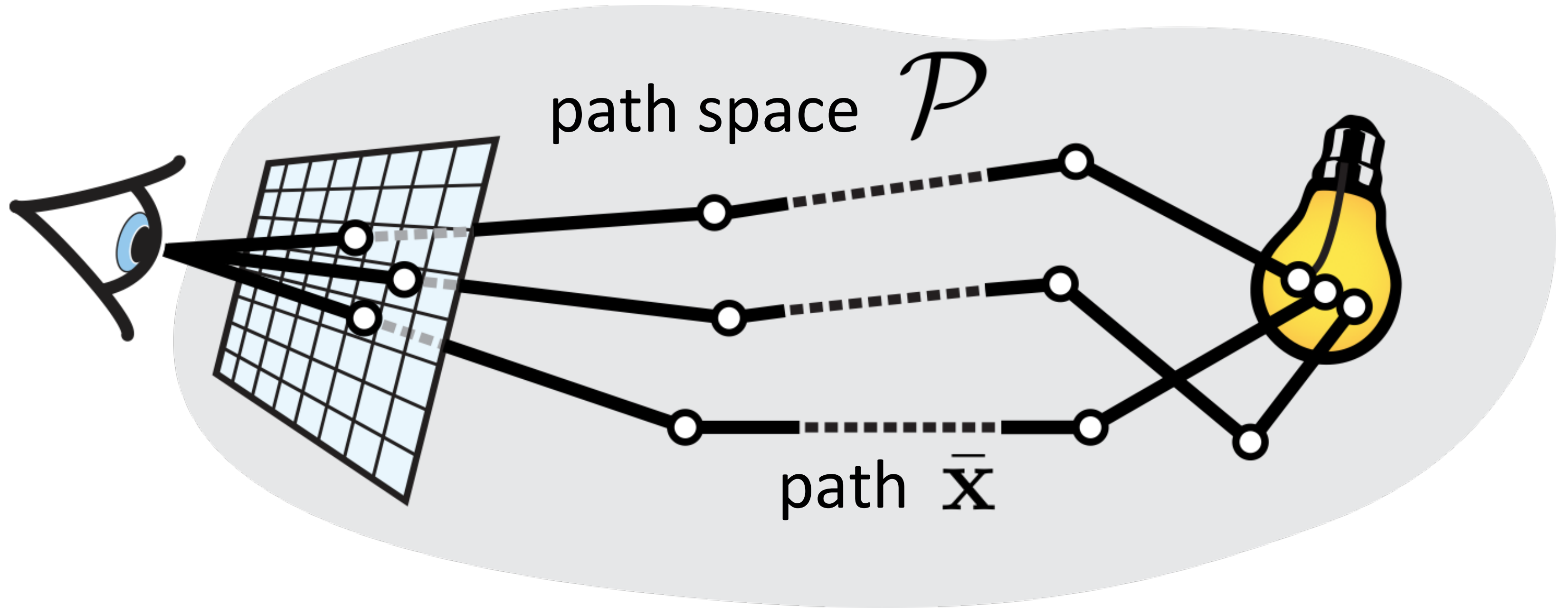
$$I_j = \int_A \int_A W_e(\mathbf{x}_0, \mathbf{x}_1) G(\mathbf{x}_0, \mathbf{x}_1) L_o(\mathbf{x}_1, \mathbf{x}_0)\, d\mathbf{x}_1 d\mathbf{x}_0$$

global illumination (all paths of all lengths)

$$= \int_{\mathcal{P}} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) T(\bar{\mathbf{x}})\, d\bar{\mathbf{x}}$$

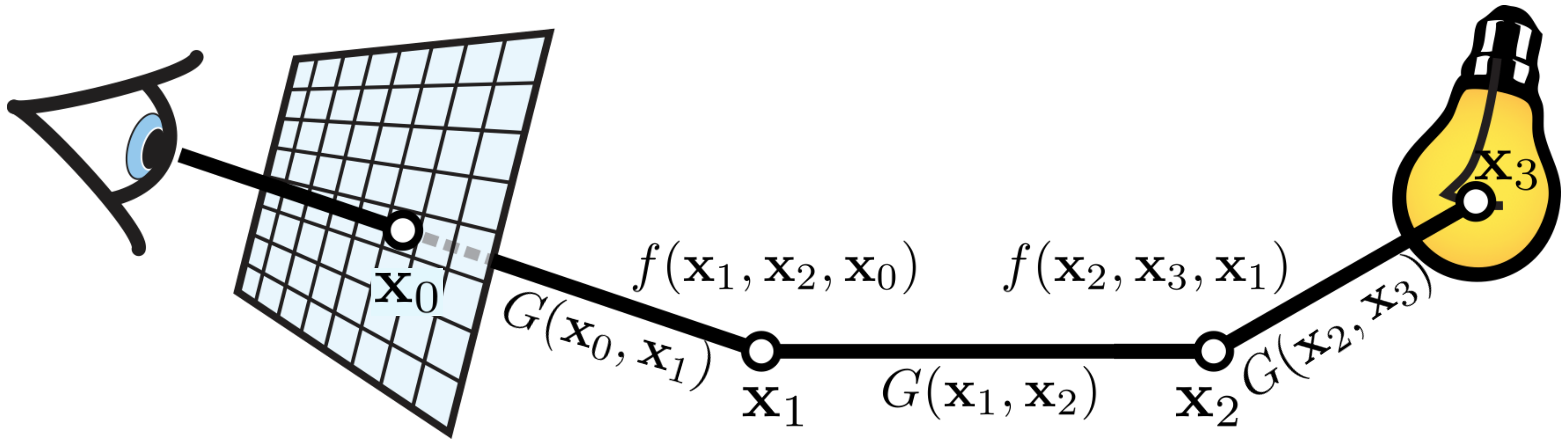
Path Integral Form of Measurement Eq.

$$I_j = \int_{\mathcal{P}} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) T(\bar{\mathbf{x}})\, d\bar{\mathbf{x}}$$



path throughput

$$T(\bar{\mathbf{x}}) = G(\mathbf{x}_0, \mathbf{x}_1) \prod_{j=1}^{k-1} f(\mathbf{x}_j, \mathbf{x}_{j+1}, \mathbf{x}_{j-1}) G(\mathbf{x}_j, \mathbf{x}_{j+1})$$



Path Integral Form of Measurement Eq.

$$I_j = \int_{\mathcal{P}} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) T(\bar{\mathbf{x}})\, d\bar{\mathbf{x}}$$

Advantages:

- no recursion, no “nasty” nested integrals
- emphasizes symmetry of light transport
- easy to relate different rendering algorithms
- focuses on path geometry, independent of strategy for constructing paths
- MC estimator on path space looks much simpler

Path Integral Form of Measurement Eq.

$$I_j = \int_{\mathcal{P}} W_e(\mathbf{x}_0, \mathbf{x}_1) L_e(\mathbf{x}_k, \mathbf{x}_{k-1}) T(\bar{\mathbf{x}}) \, d\bar{\mathbf{x}}$$

Monte Carlo estimator:

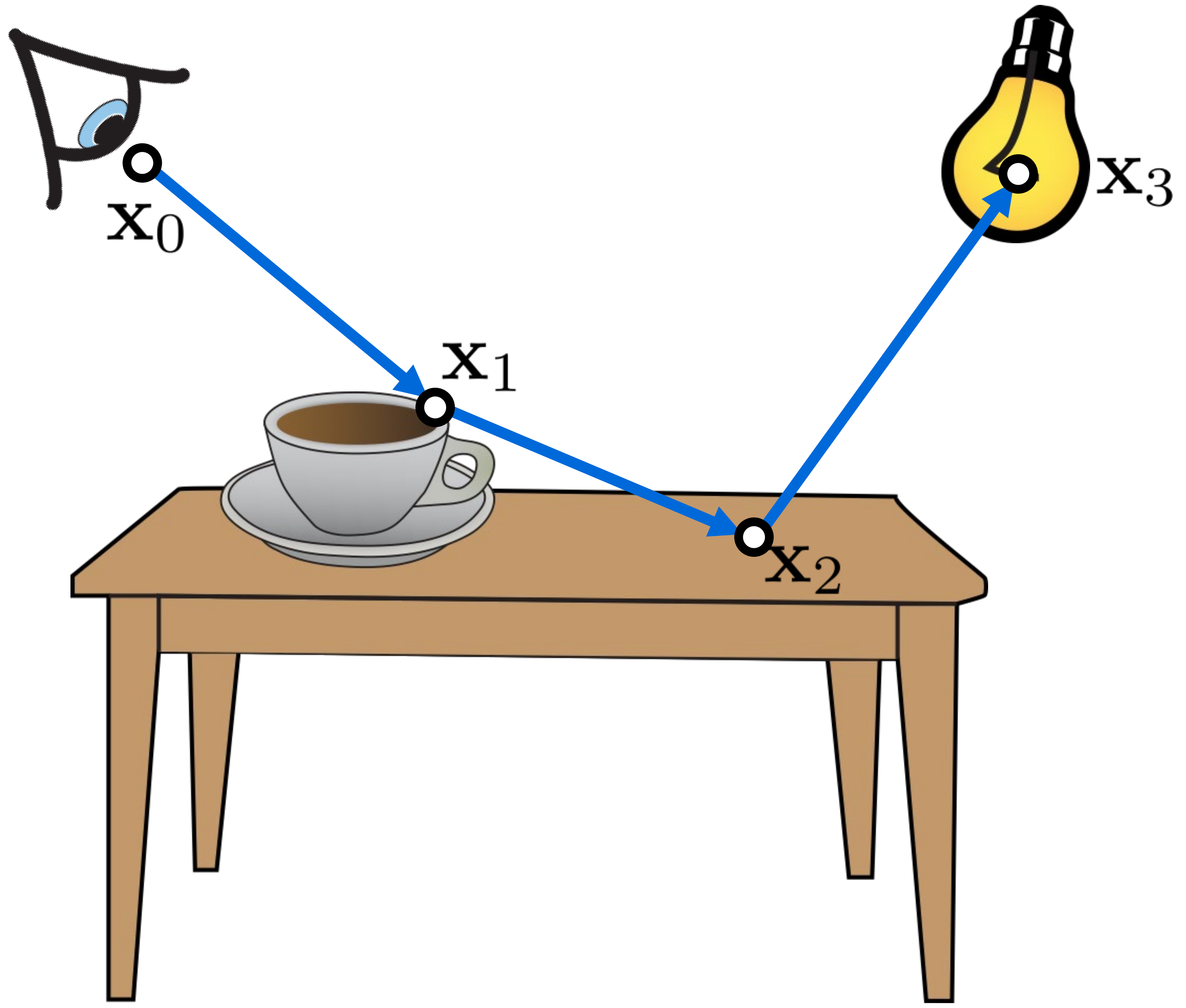
$$I_j \approx \frac{1}{N} \sum_{i=1}^{N} \frac{W_e(\mathbf{x}_{i,0}, \mathbf{x}_{i,1}) L_e(\mathbf{x}_{i,k}, \mathbf{x}_{i,k-1}) T(\bar{\mathbf{x}}_i)}{p(\bar{\mathbf{x}}_i)}$$

$$\underset{\text{path PDF}}{p(\bar{\mathbf{x}})} = \underbrace{p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k)}_{\text{joint PDF of path vertices}}$$

Path Construction

$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k)$$

Path tracing w/o NEE

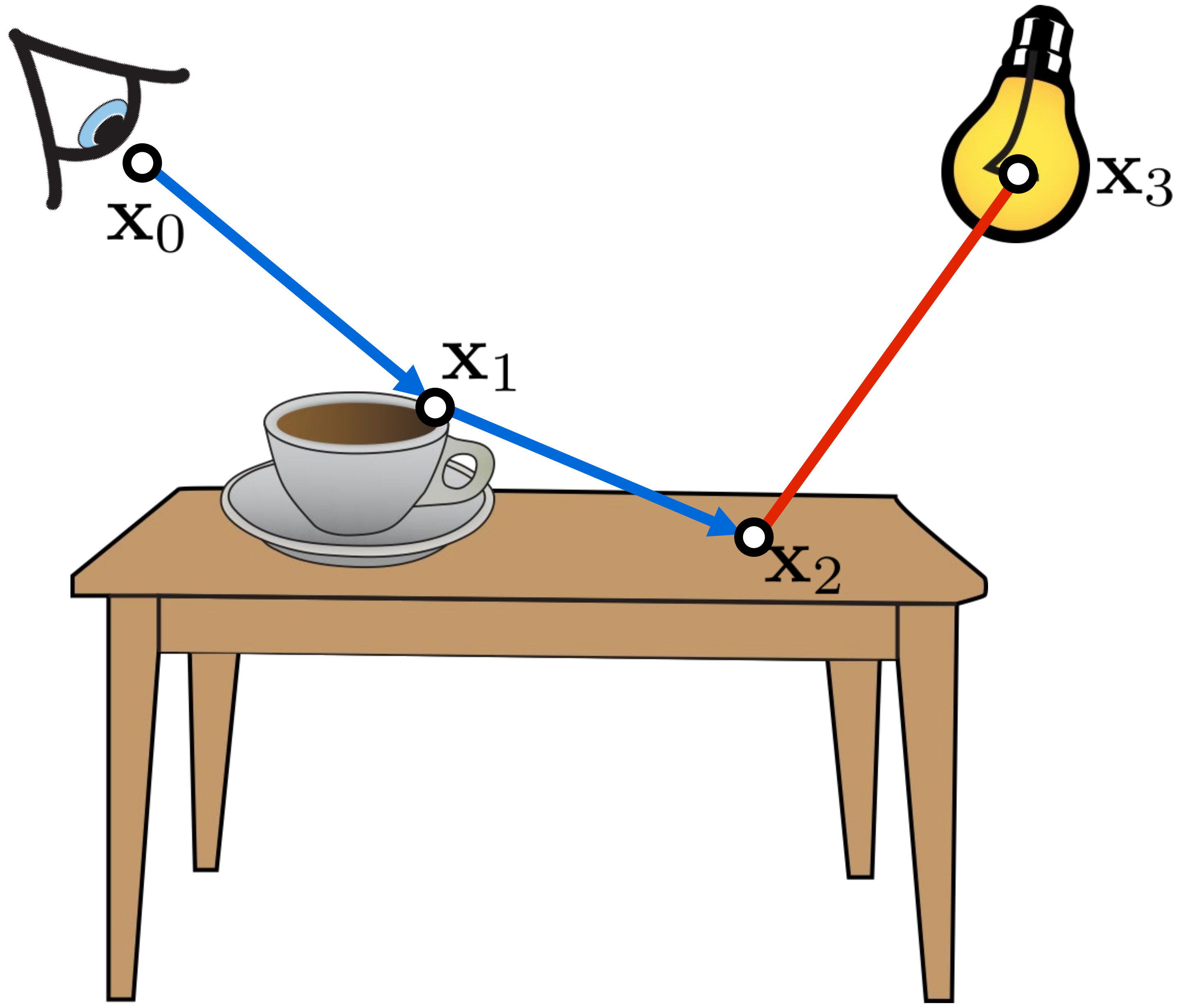


$$\begin{aligned} p(\bar{\mathbf{x}}) = & \, p(\mathbf{x}_0) \\ & \times p(\mathbf{x}_1 | \mathbf{x}_0) \\ & \times p(\mathbf{x}_2 | \mathbf{x}_0 \mathbf{x}_1) \\ & \times p(\mathbf{x}_3 | \mathbf{x}_0 \mathbf{x}_1 \mathbf{x}_2) \end{aligned}$$

Path Construction

$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k)$$

Path tracing with NEE



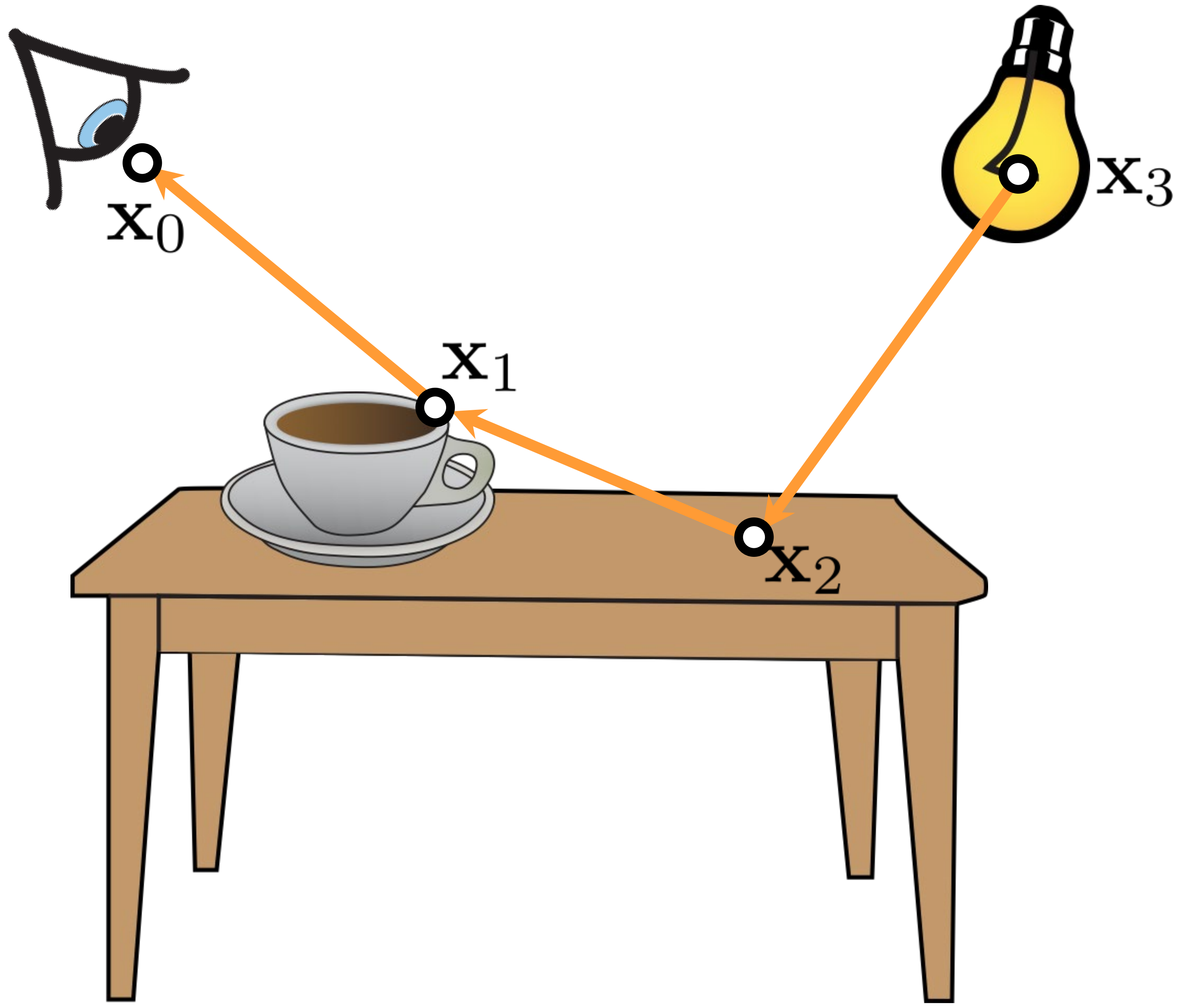
$$\begin{aligned} p(\bar{\mathbf{x}}) = &\, p(\mathbf{x}_0) \\ &\times p(\mathbf{x}_1 | \mathbf{x}_0) \\ &\times p(\mathbf{x}_2 | \mathbf{x}_0 \mathbf{x}_1) \\ &\times p(\mathbf{x}_3) \end{aligned}$$

assuming uniform
area sampling

Path Construction

$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k)$$

Light tracing

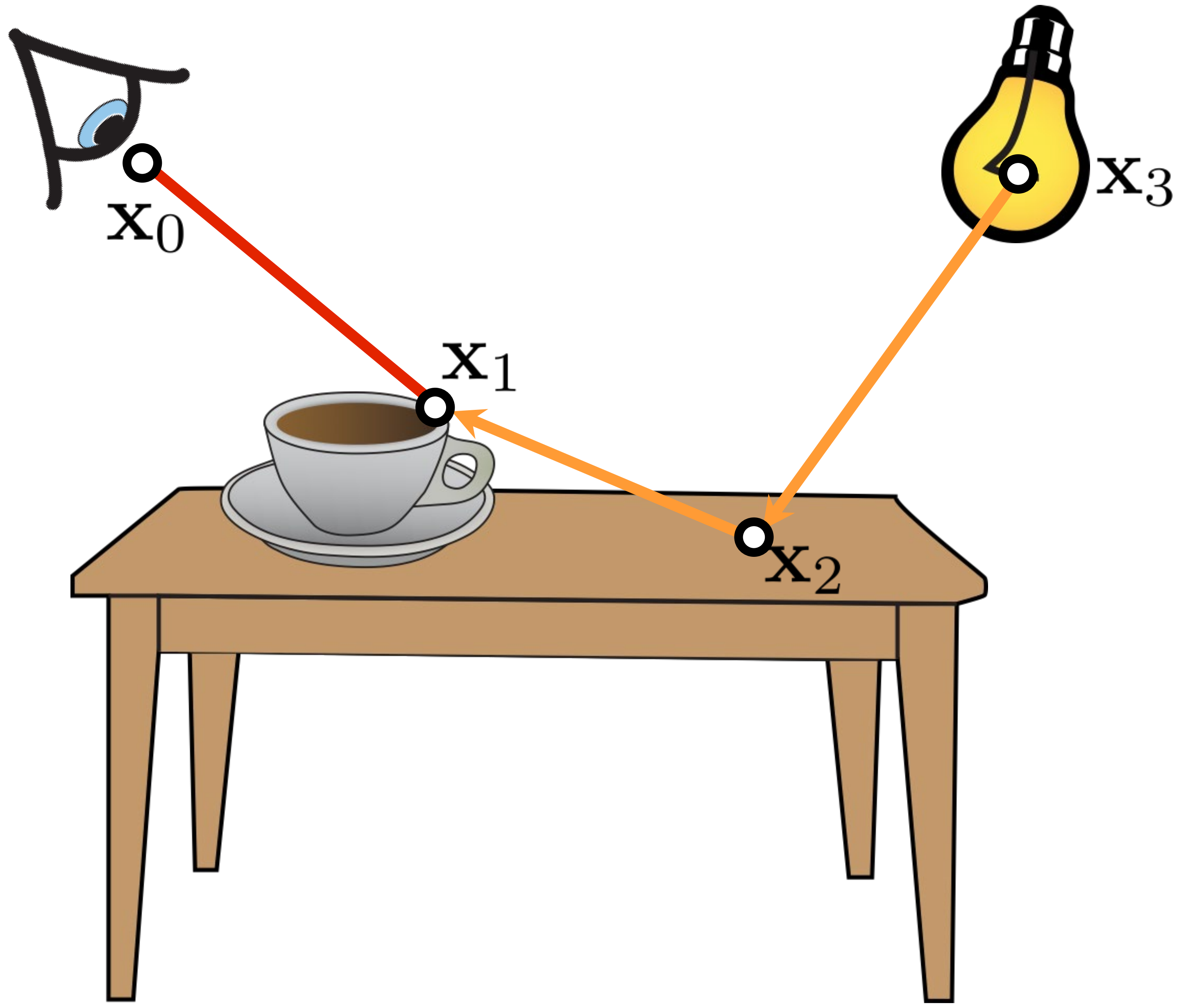


$$\begin{aligned} p(\bar{\mathbf{x}}) &= p(\mathbf{x}_0 | \mathbf{x}_3 \mathbf{x}_2 \mathbf{x}_1) \\ &\times p(\mathbf{x}_1 | \mathbf{x}_3 \mathbf{x}_2) \\ &\times p(\mathbf{x}_2 | \mathbf{x}_3) \\ &\times p(\mathbf{x}_3) \end{aligned}$$

Path Construction

$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k)$$

Light tracing with NEE



assuming uniform aperture sampling

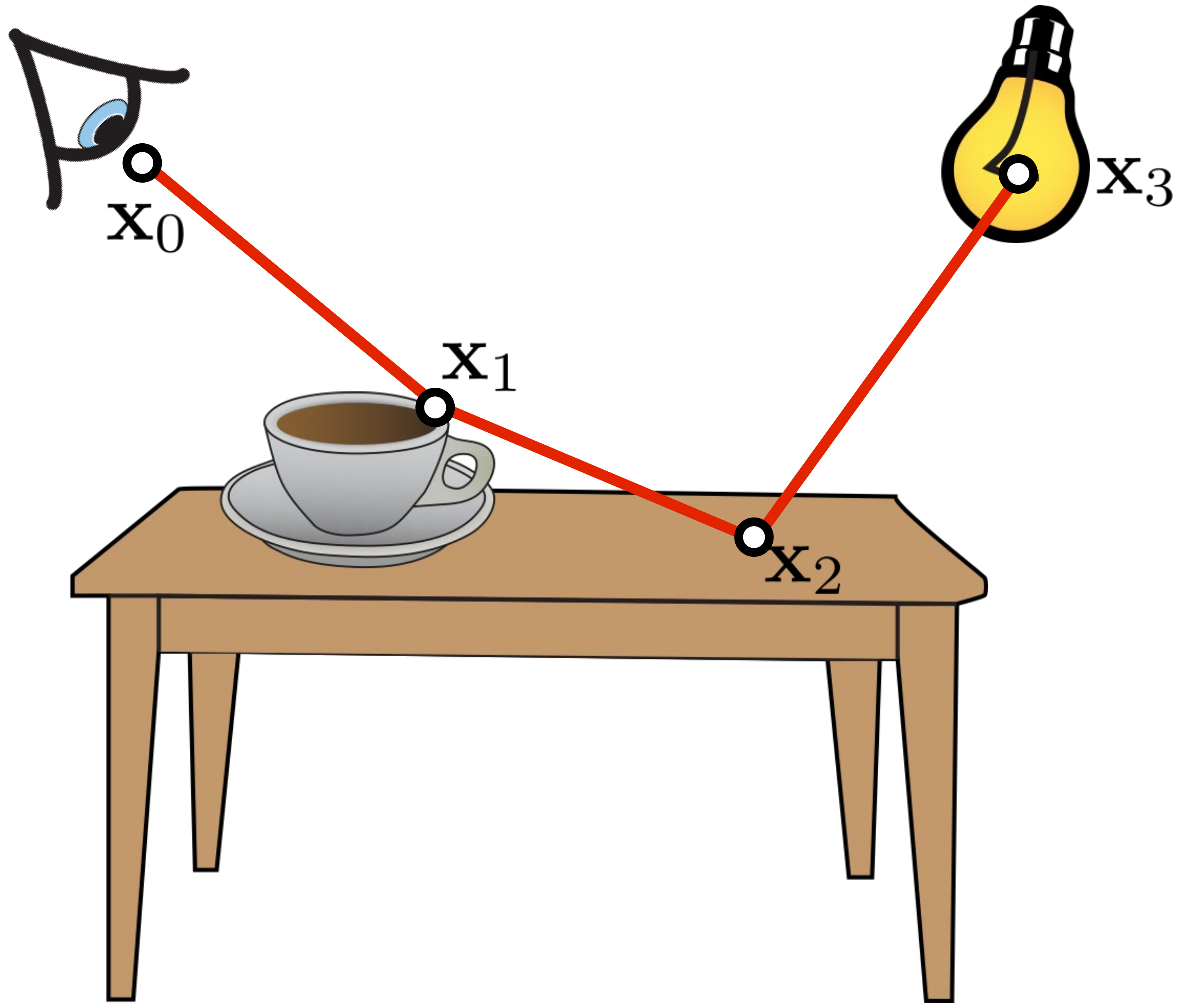
$$\begin{aligned} p(\bar{\mathbf{x}}) = & \, p(\mathbf{x}_0) \\ & \times p(\mathbf{x}_1 | \mathbf{x}_3 \mathbf{x}_2) \\ & \times p(\mathbf{x}_2 | \mathbf{x}_3) \\ & \times p(\mathbf{x}_3) \end{aligned}$$

Path Construction

$$p(\bar{\mathbf{x}}) = p(\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}, \mathbf{x}_k)$$

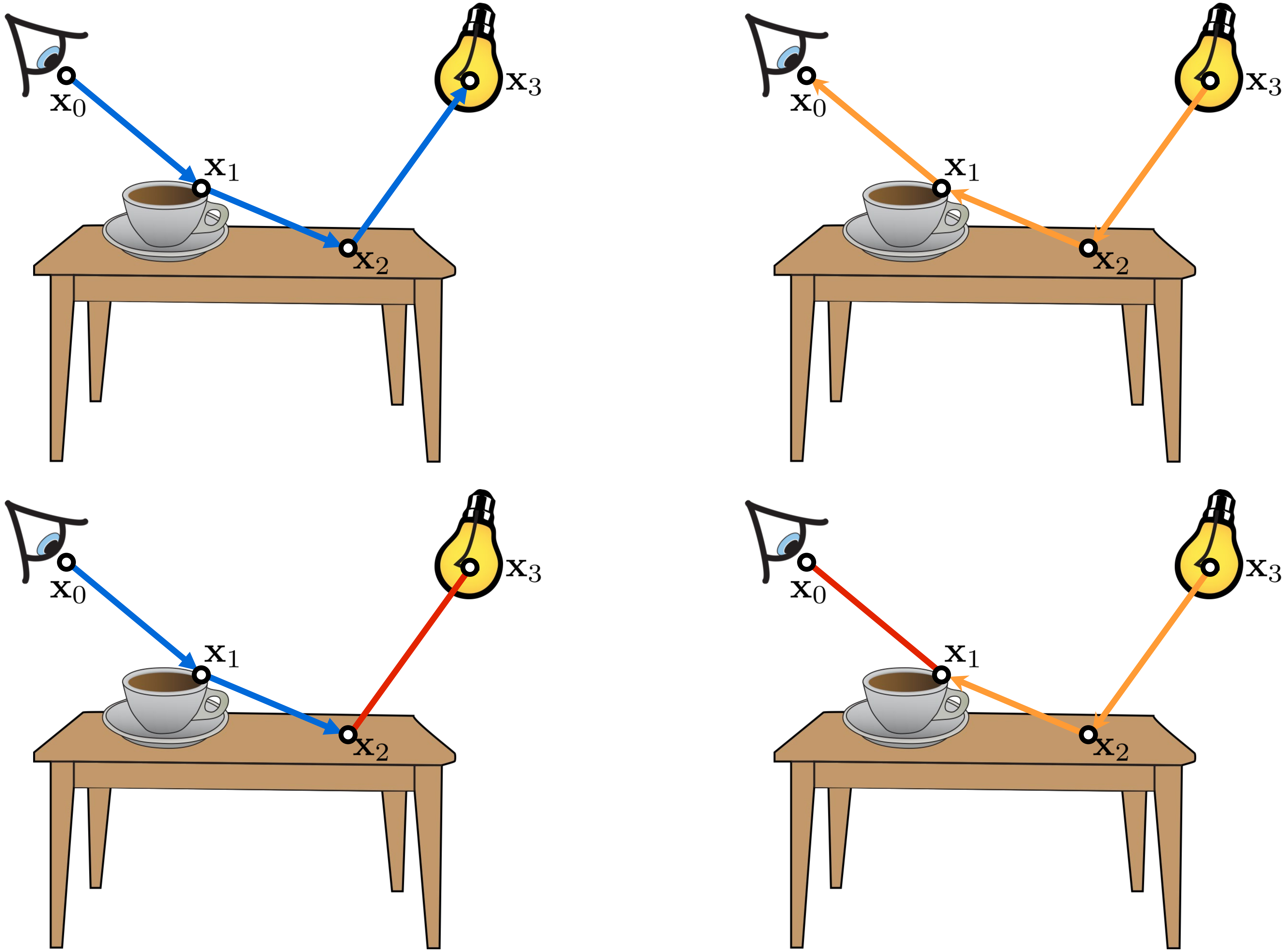
Independent sampling of path vertices

(not very practical though)



$$\begin{aligned} p(\bar{\mathbf{x}}) = & \, p(\mathbf{x}_0) \\ & \times p(\mathbf{x}_1) \\ & \times p(\mathbf{x}_2) \\ & \times p(\mathbf{x}_3) \end{aligned}$$

Can we combine them?



Bidirectional Path Tracing

Bidirectional Path Tracing

[Lafortune and Willems 1993]
[Veach and Guibas 1994]

t - # vertices on camera subpath
s - # vertices on light subpath
ts - # connections

Bidirectional Path Tracing

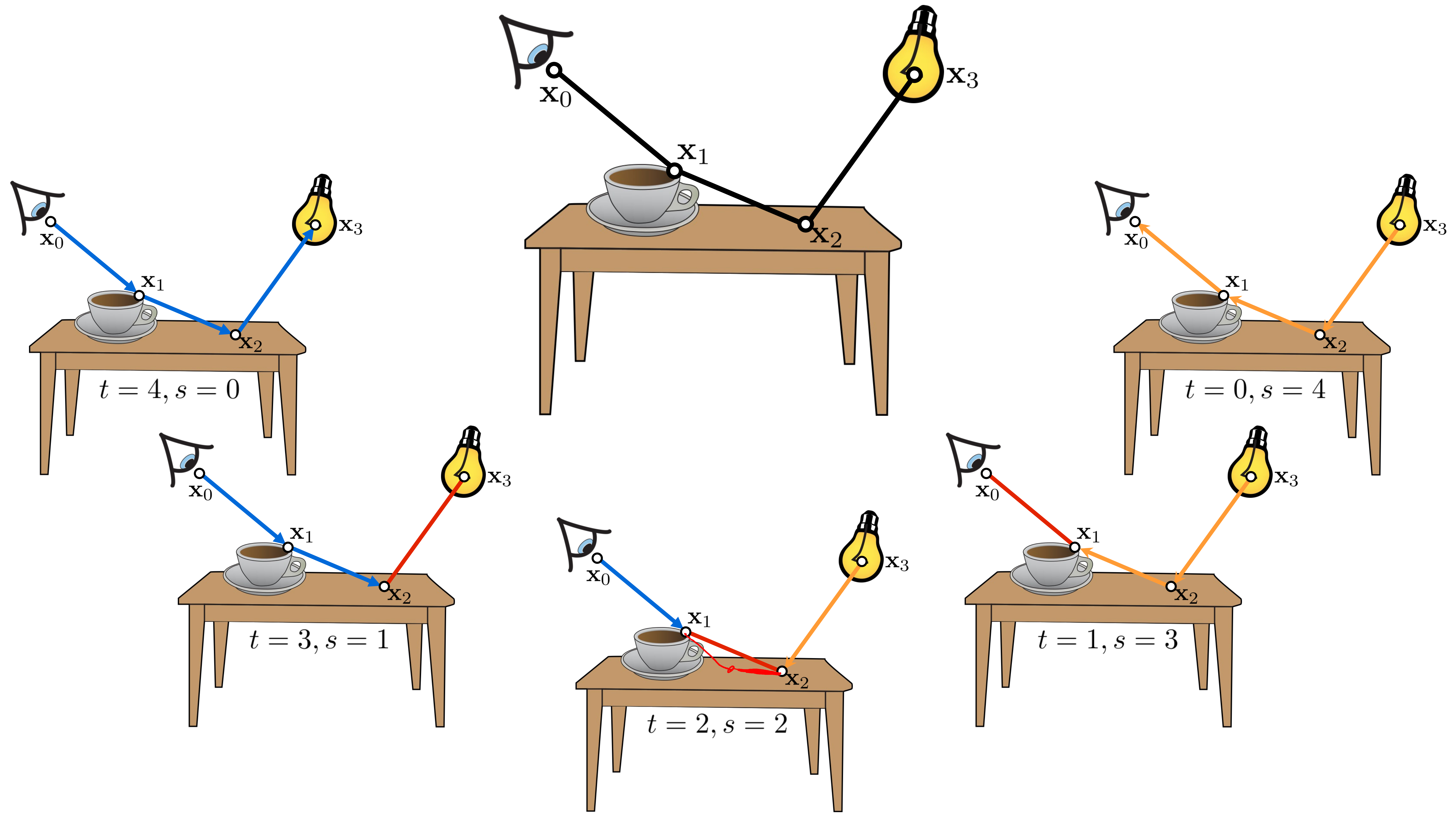
```
color estimate (point x)
{
   lp = sample light subpath
   cp = sample camera subpath for image point x

   for each vertex s in lp
      for each vertex t in cp
         fullPath = join(cp[0..s], lp[0..t])
         splat(fullPath.screenPos,
fullPath.contrib)
}
```

Bidirectional Path Tracing

Key observations:

- Every path (formed by connecting camera sub-path to light sub-path) with k vertices can be constructed using $k+1$ strategies
- For a particular path length, all strategies estimate the same integral
- Each strategy has a different PDF, i.e., each strategy has different strengths and weaknesses
- Let’s combine them using MIS!

Bidirectional Path Tracing



Bidirectional Path Tracing

Images courtesy of W. Jakob

Bidirectional Path Tracing (MIS)

Images courtesy of W. Jakob

Bidirectional Path Tracing

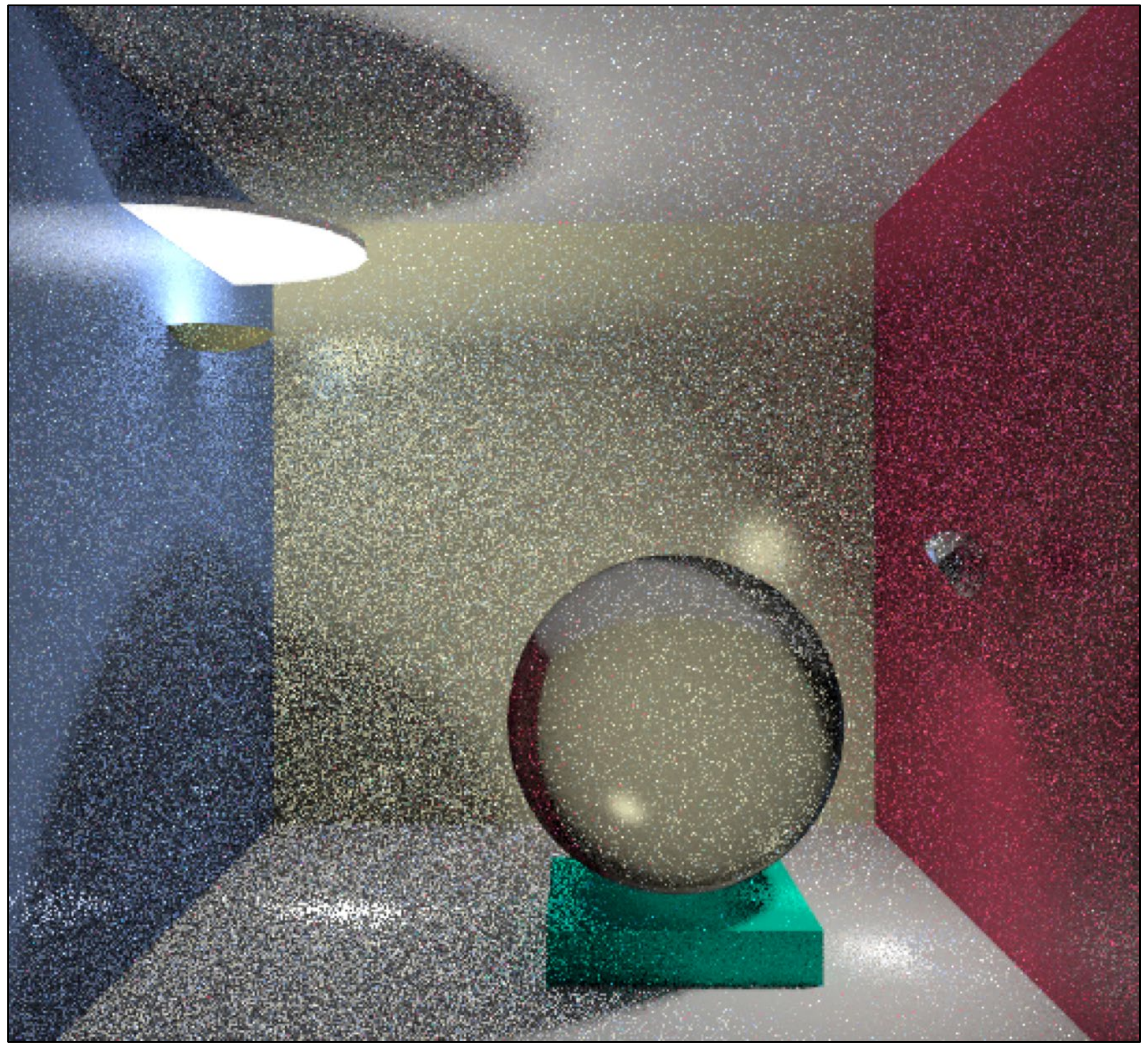
(Unidirectional) path tracing

Bidirectional path tracing

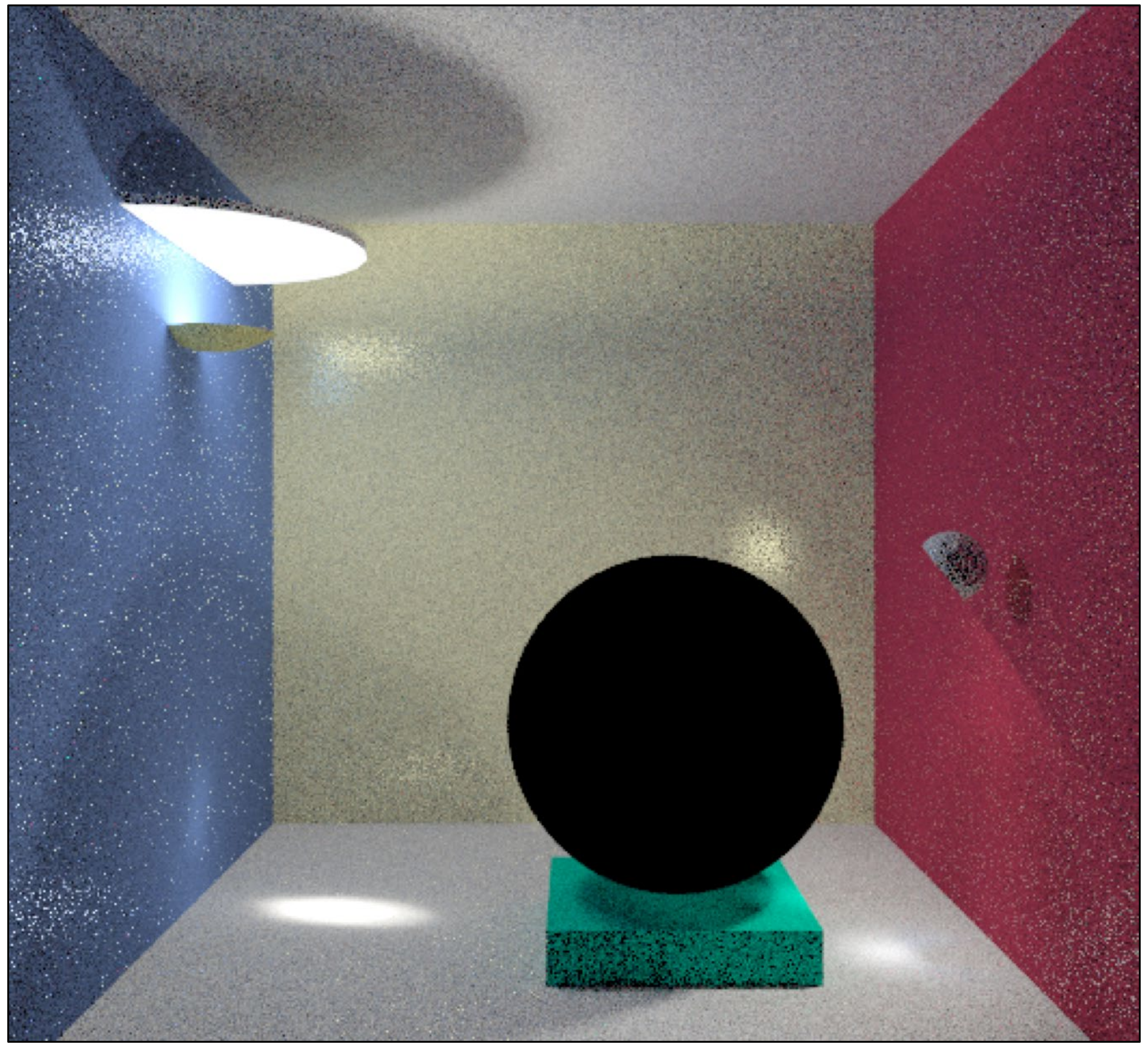
Images courtesy of W. Jakob

Bidirectional Path Tracing

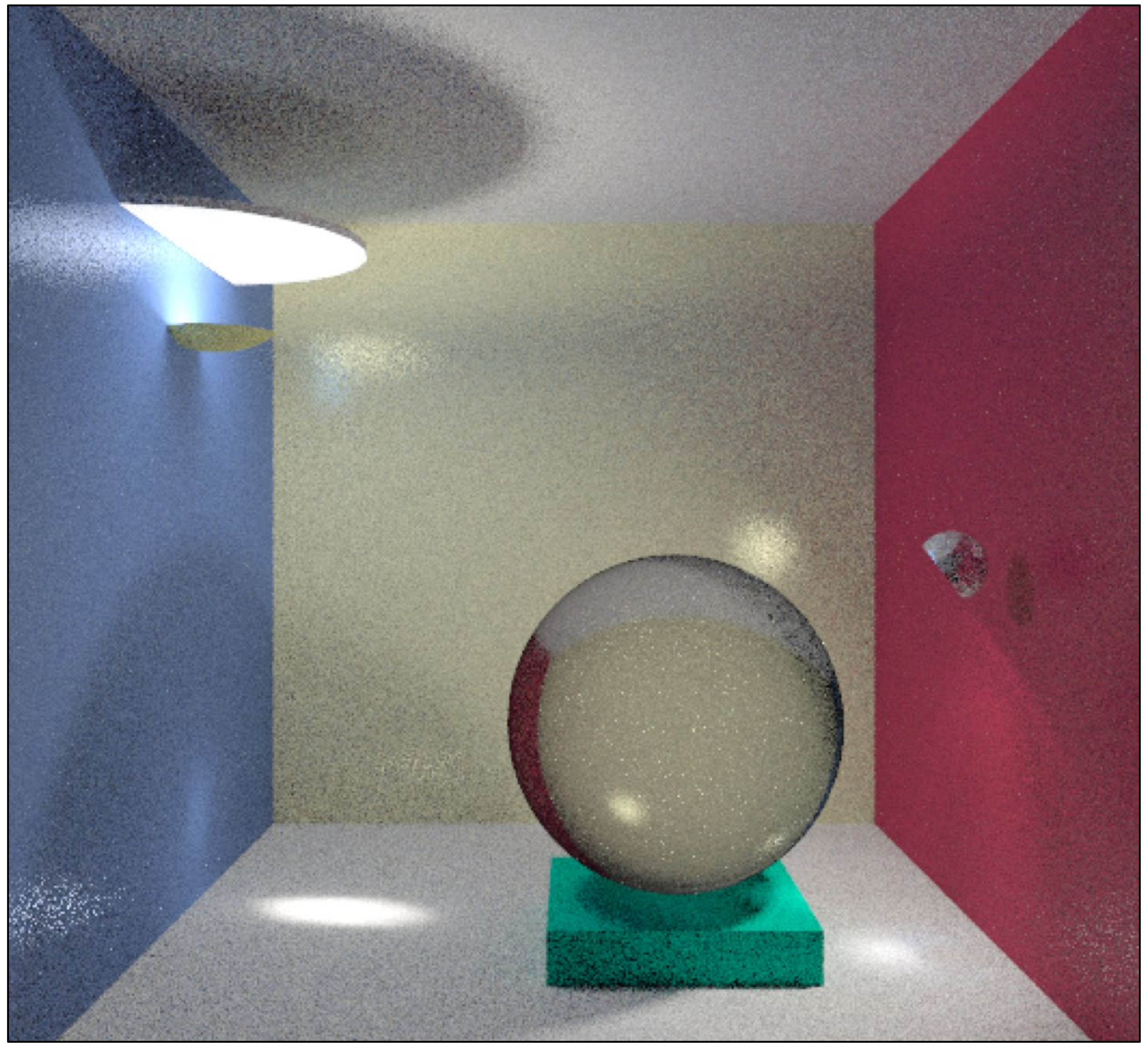
Path tracing



Light tracing



Bidirectional PT



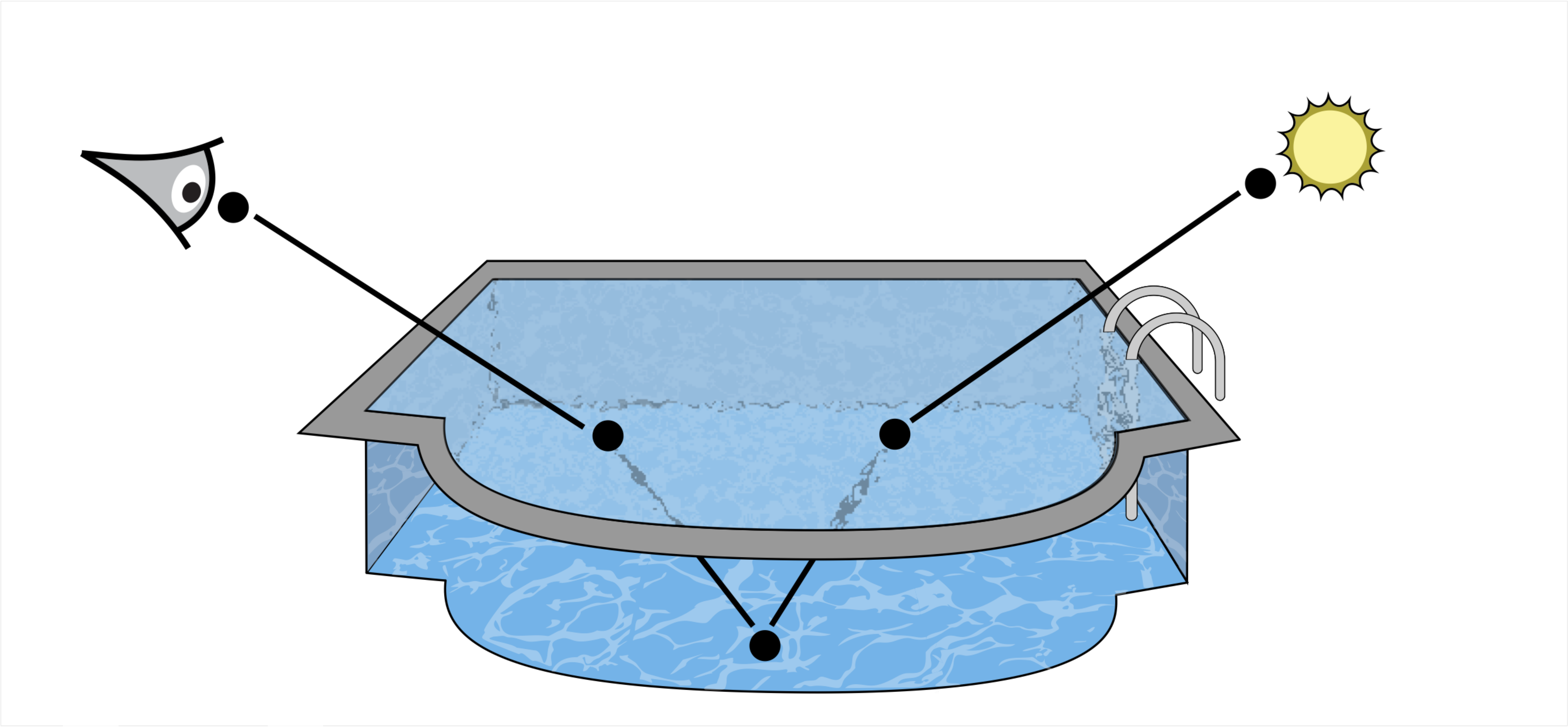
Images courtesy of F. Suykens

Still not robust enough...

Reference | Bidirectional PT

Images courtesy of J. Křivánek

Still not robust enough...



$LSDSE$ paths are difficult for any unbiased method

Still not robust enough...

Extensions

- Combination with photon mapping
  - Unified Path Sampling [Hachisuka et al. 2012]
  - Vertex Connection Merging [Georgiev et al. 2012]
- Metropolis sampling (global PDF)
- Path-space regularization [Kaplanyan et al. 2013]
- Path guiding (learn global PDF)