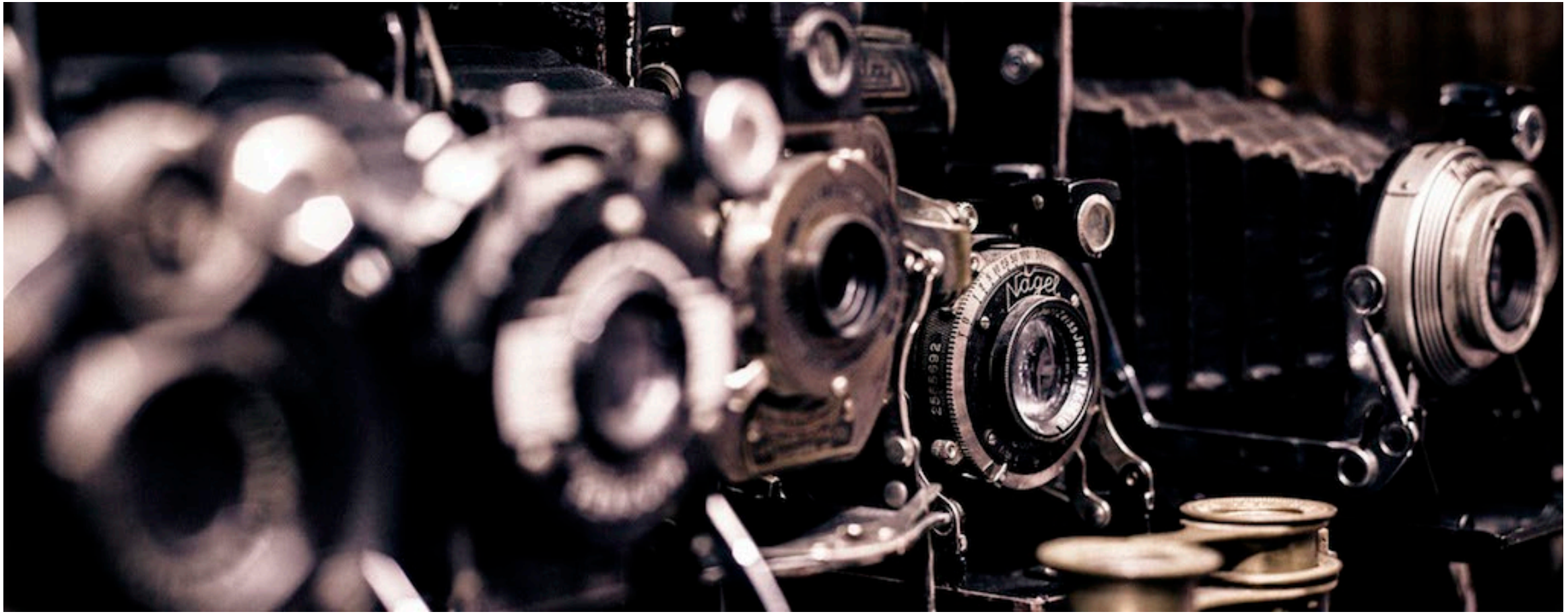
# Geometric camera models and calibration




15-463, 15-663, 15-862
Computational Photography
Fall 2023, Lecture 15

http://graphics.cs.cmu.edu/courses/15-463

# Course announcements

- Homework 5 is due **today**.
  - Any questions?
- Remember to vote on homework 3 competition.
- Homework 6 will be posted tonight.
- Project proposals due on Monday 10/27.
  - See details here: http://graphics.cs.cmu.edu/courses/15-463/final_project.html

# Overview of today's lecture

- Pinholes and lenses.
- Pinhole camera.
- Accidental pinholes.
- Camera matrix.
- Perspective.
- Other camera models.
- Pose estimation.

# Slide credits

Most of these slides were adapted from:

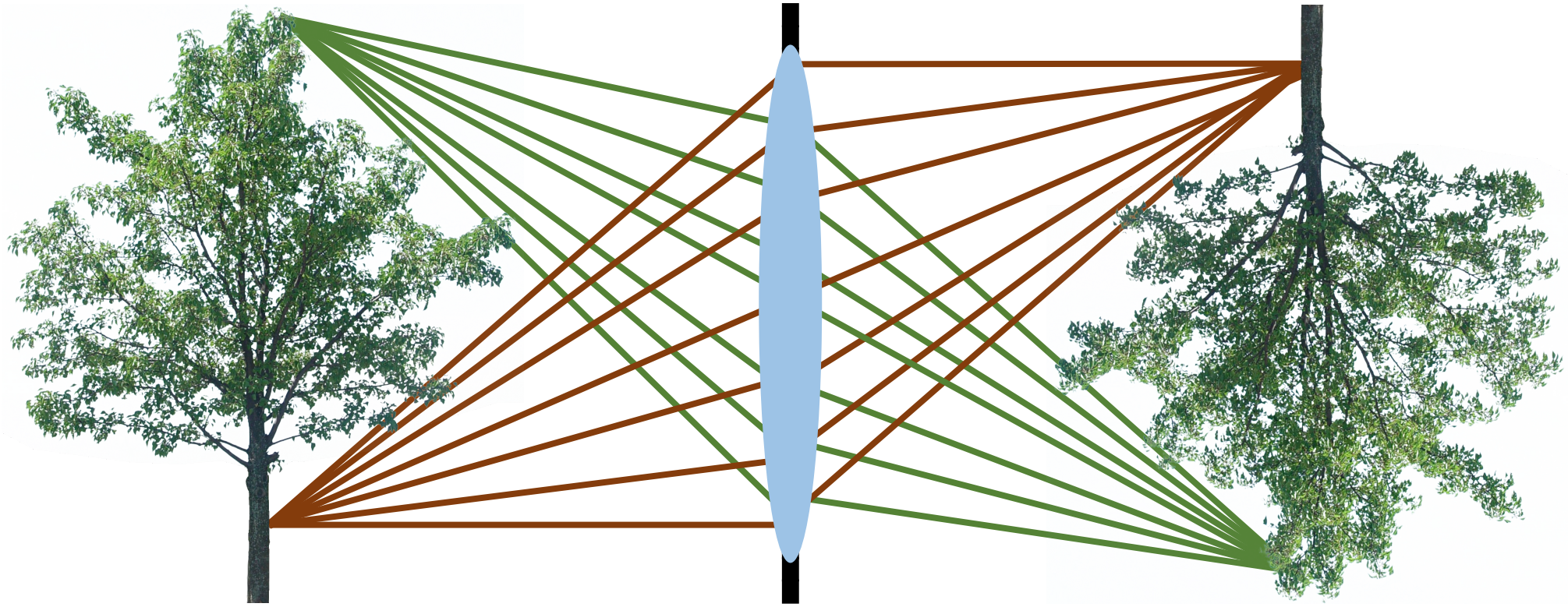
- Kris Kitani (15-463, Fall 2016).

Some slides inspired from:

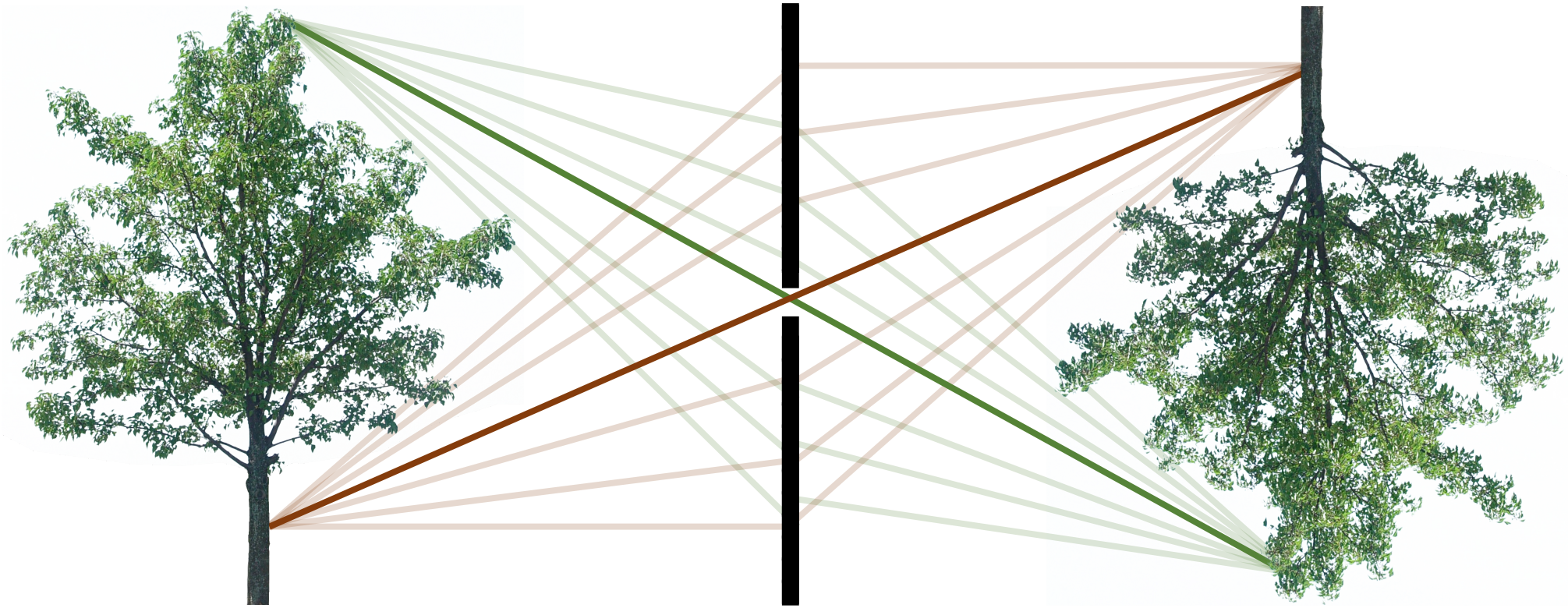
- Fredo Durand (MIT).

# Pinhole and lens cameras

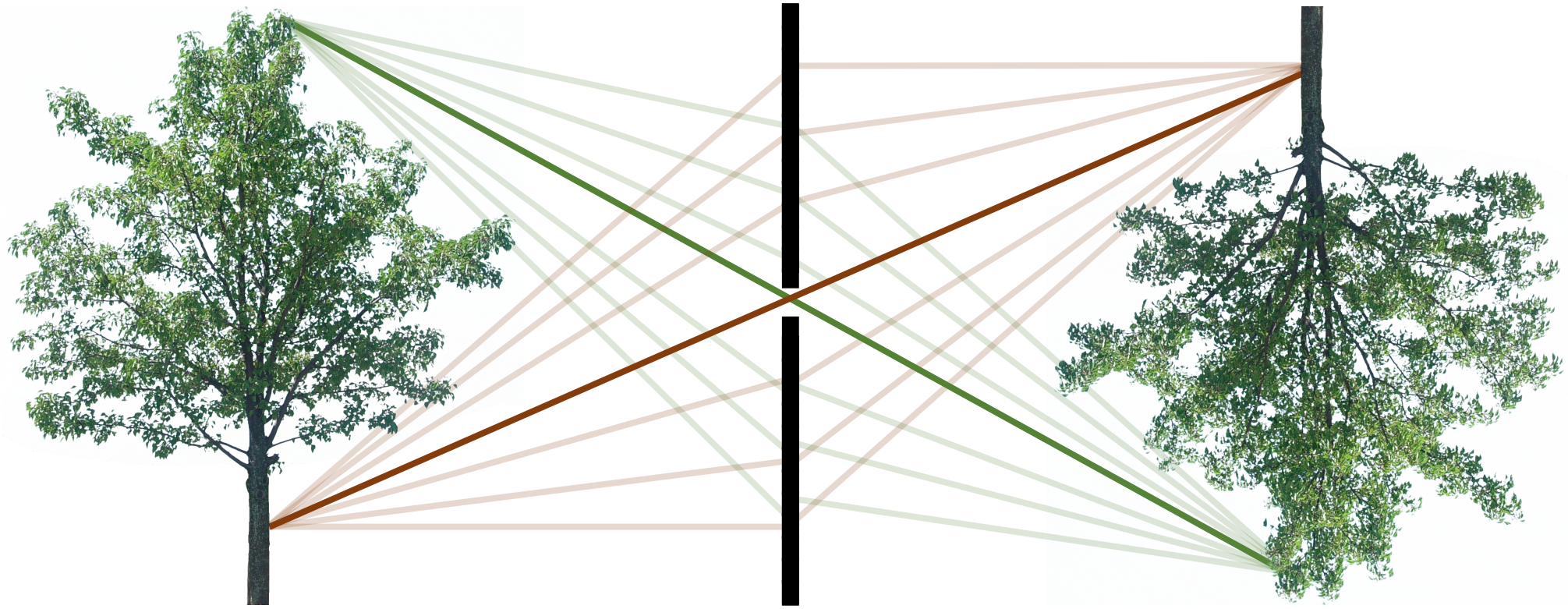
# The lens camera



# The pinhole camera

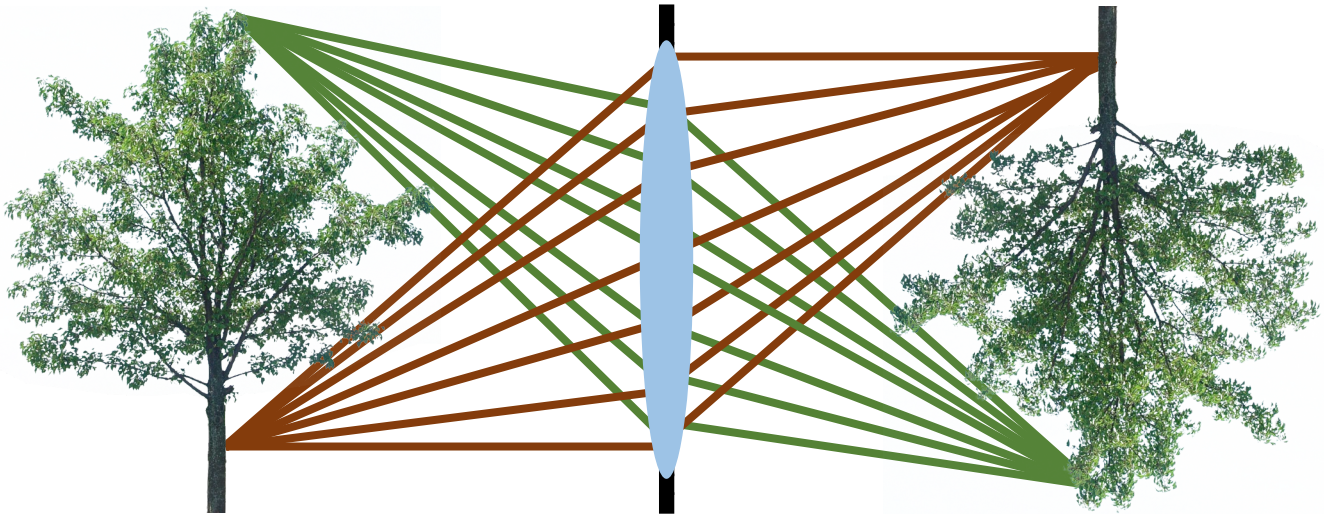


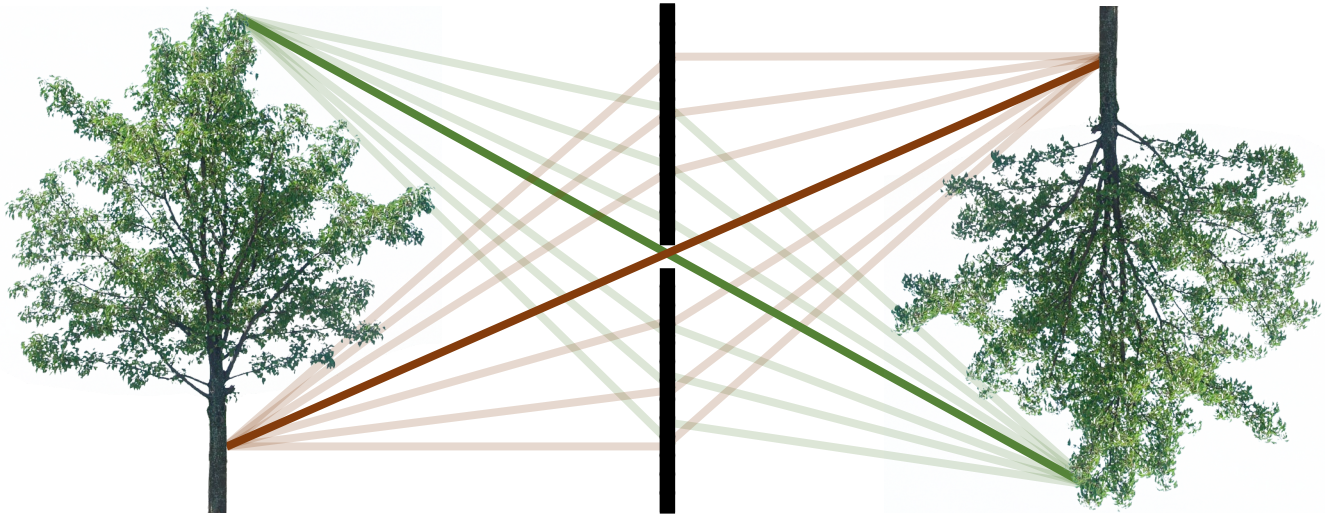
# The pinhole camera



Central rays propagate in the same way for both models!

# Describing both lens and pinhole cameras



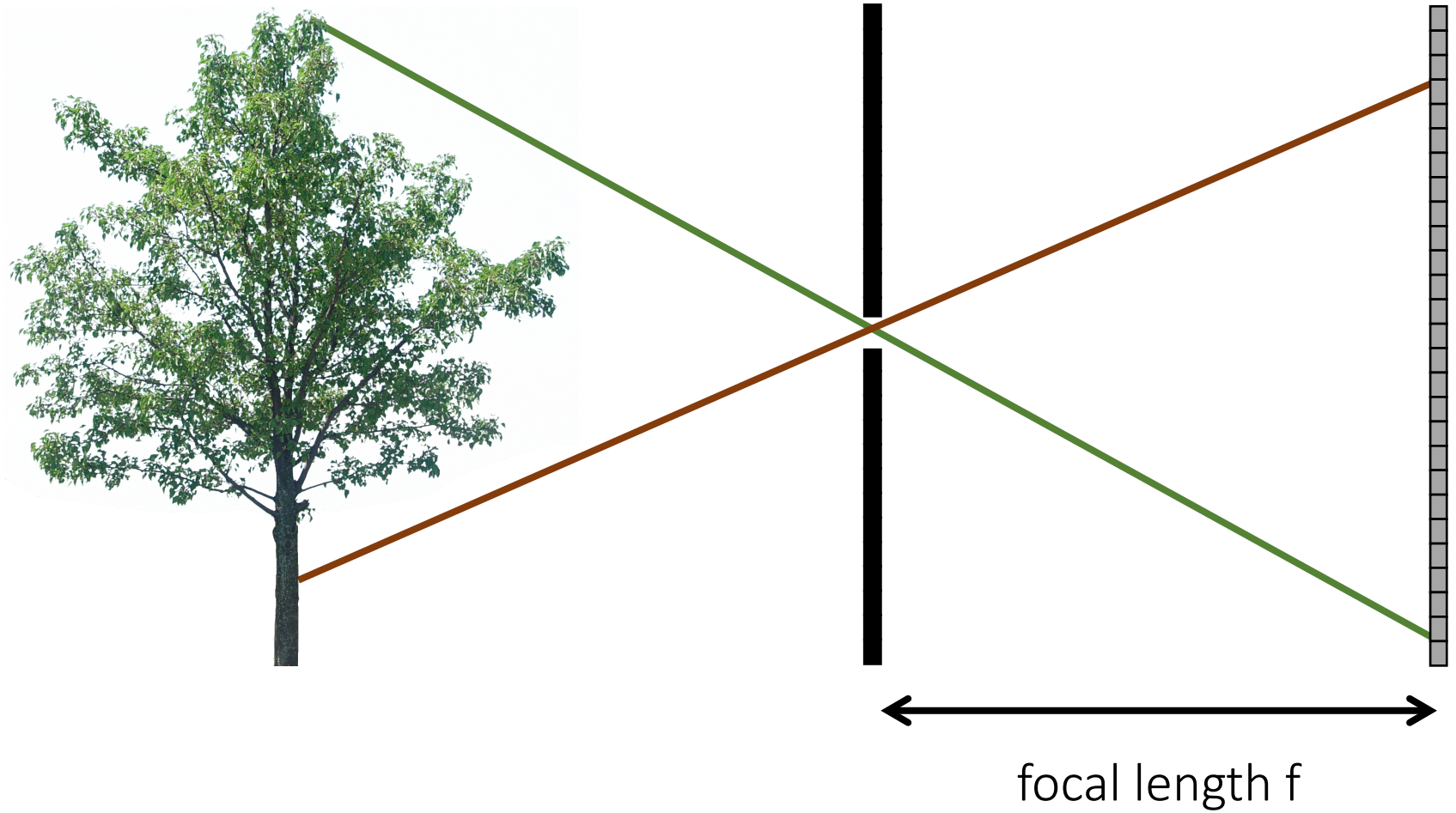


We can derive properties and descriptions that hold for both camera models if:

- We use only central rays.
- We assume the lens camera is in focus.

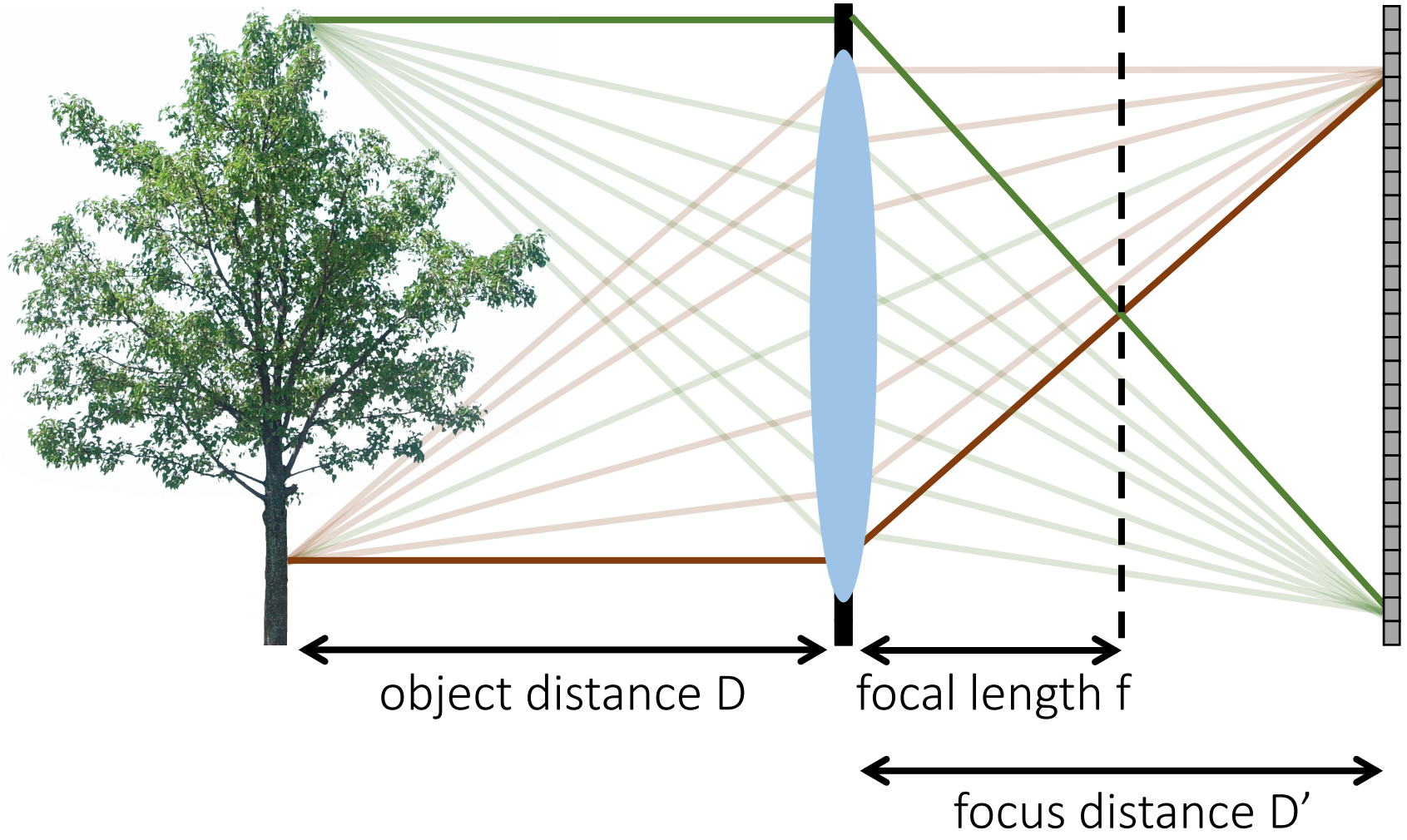
# Important difference: focal length

In a pinhole camera, focal length is distance between aperture and sensor

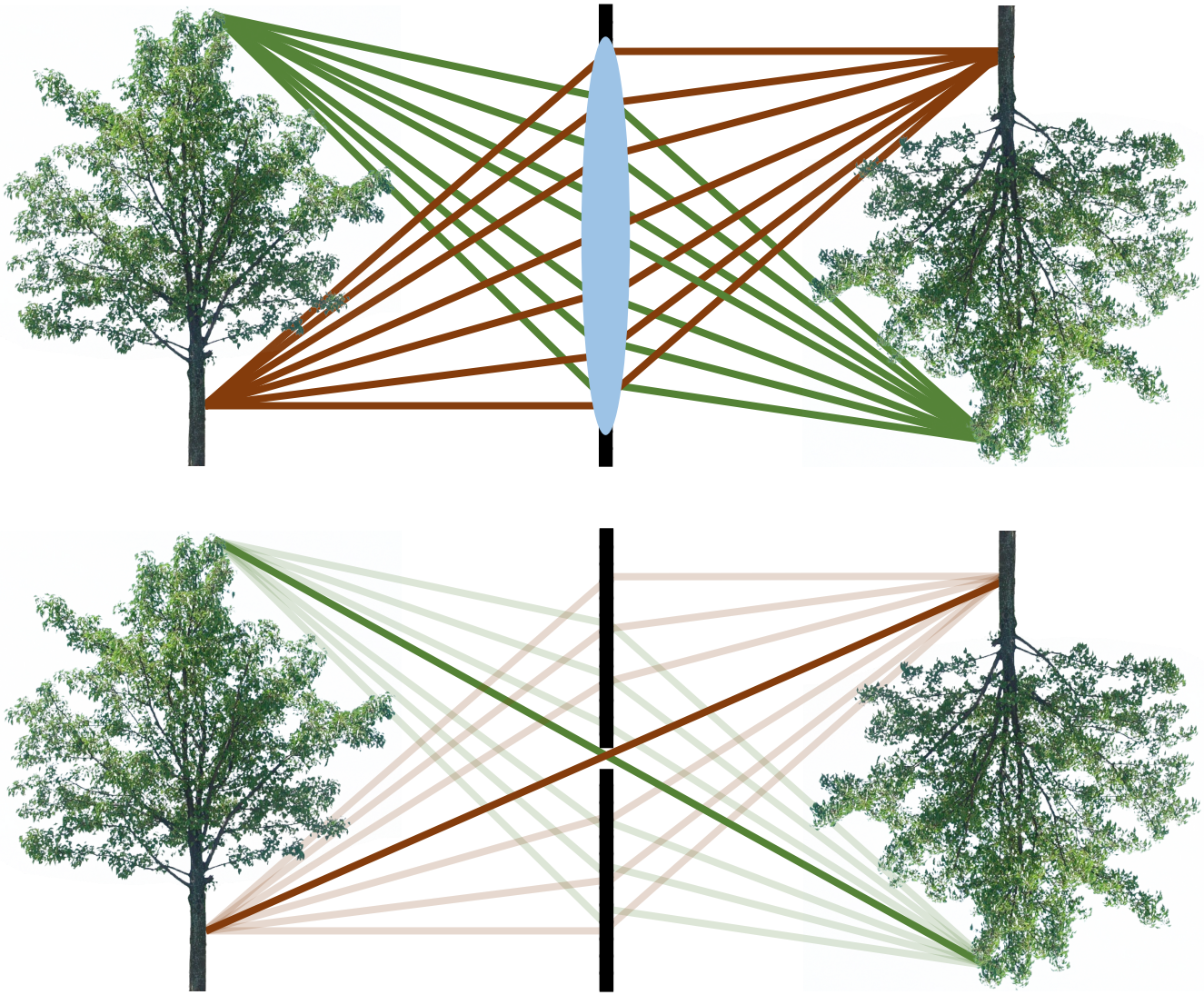


# Important difference: focal length

In a lens camera, focal length is distance where parallel rays intersect



# Describing both lens and pinhole cameras



We can derive properties and descriptions that hold for both camera models if:

- We use only central rays.
- We assume the lens camera is in focus.
- We assume that the focus distance of the lens camera is equal to the focal length of the pinhole camera.

Remember: *focal length* f refers to different things for lens and pinhole cameras.

- In this lecture, we use it to refer to the aperture-sensor distance, as in the pinhole camera case.

# Camera matrix

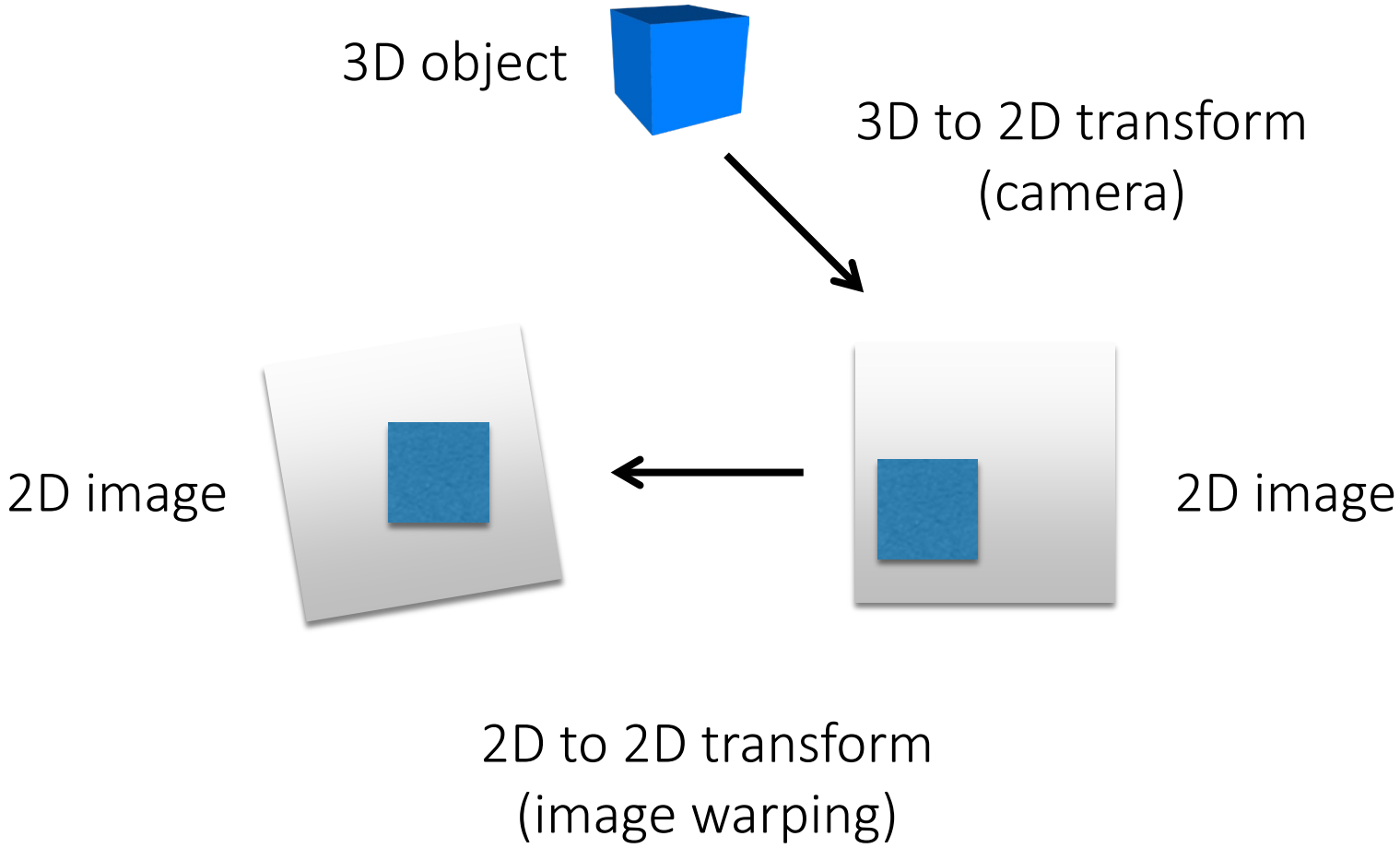
# The camera as a coordinate transformation

A camera is a mapping from:

the 3D world

to:

a 2D image



# The camera as a coordinate transformation

A camera is a mapping from:

the 3D world

to:

a 2D image

homogeneous coordinates

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{X}$$

2D image point

camera matrix

3D world point

What are the dimensions of each variable?

# Reminder: 2D homogeneous coordinates

heterogeneous coordinates

homogeneous coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

add a 1 here

- Represent 2D point with a 3D vector

# Reminder: 2D homogeneous coordinates

heterogeneous coordinates

homogeneous coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \overset{\text{def}}{=\!=} \begin{bmatrix} ax \\ ay \\ a \end{bmatrix}$$

- Represent 2D point with a 3D vector
- 3D vectors are only defined up to scale

# Reminder: 2D homogeneous coordinates

Conversion:

- heterogeneous → homogeneous

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- homogeneous → heterogeneous

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} x/z \\ y/z \end{bmatrix}$$

Scale invariance:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Special points:

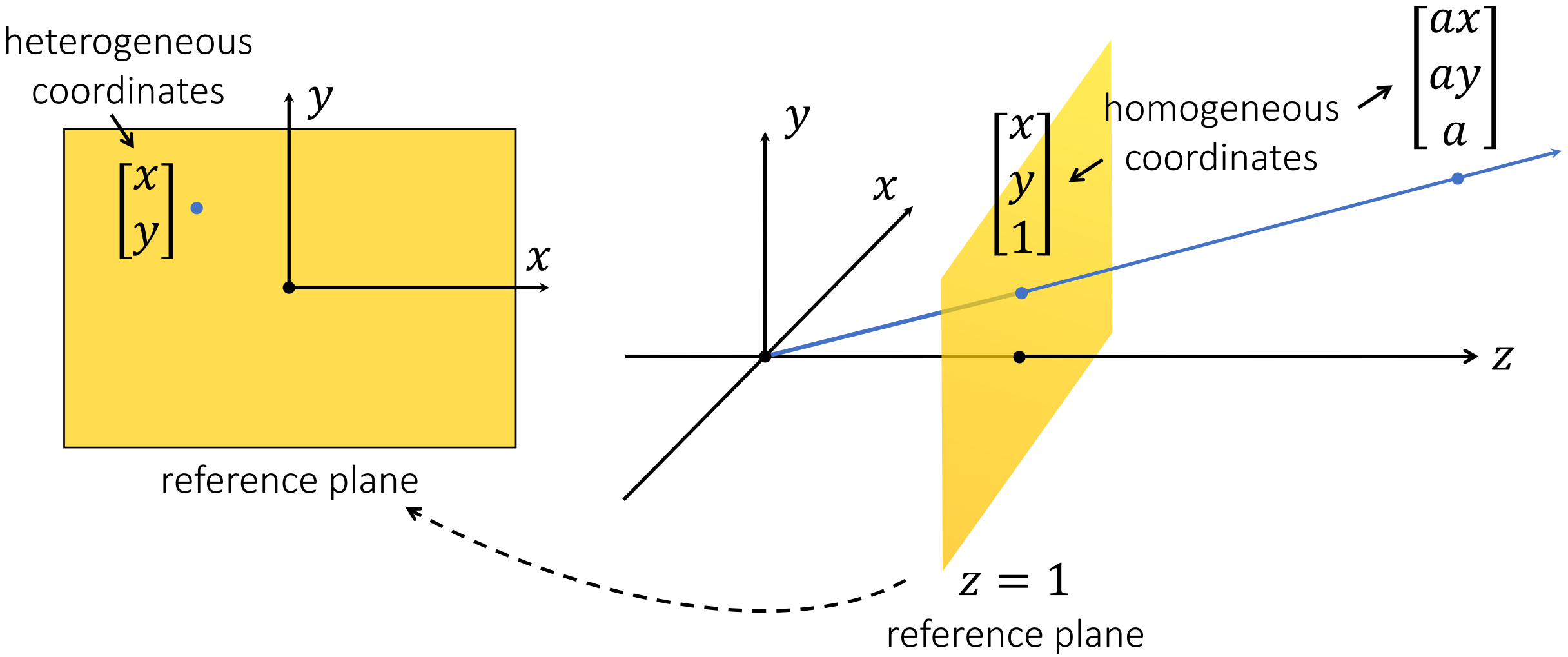
- point at infinity

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

- undefined

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

# Reminder: 2D projective geometry




Through the scale invariance property, homogeneous coordinates map all points on a line passing through the origin to the point where this line intersects the reference plane.

# Reminder: 3D homogeneous coordinates

heterogeneous coordinates

homogeneous coordinates

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \Rightarrow \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} aX \\ aY \\ aZ \\ a \end{bmatrix}$$

- Represent 3D point with a 4D vector
- 4D vectors are only defined up to scale

# Reminder: notation

| | heterogeneous coordinates | homogeneous coordinates |
|---|---|---|
| 2D coordinates | 2D vector $\tilde{\boldsymbol{x}} = \begin{bmatrix} x \\ y \end{bmatrix}$ | 3D vector $\boldsymbol{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ |
| 3D coordinates | 3D vector $\tilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ | 4D vector $\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$ |

# The camera as a coordinate transformation

A camera is a mapping from:

the 3D world

to:

a 2D image

homogeneous coordinates

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{X}$$

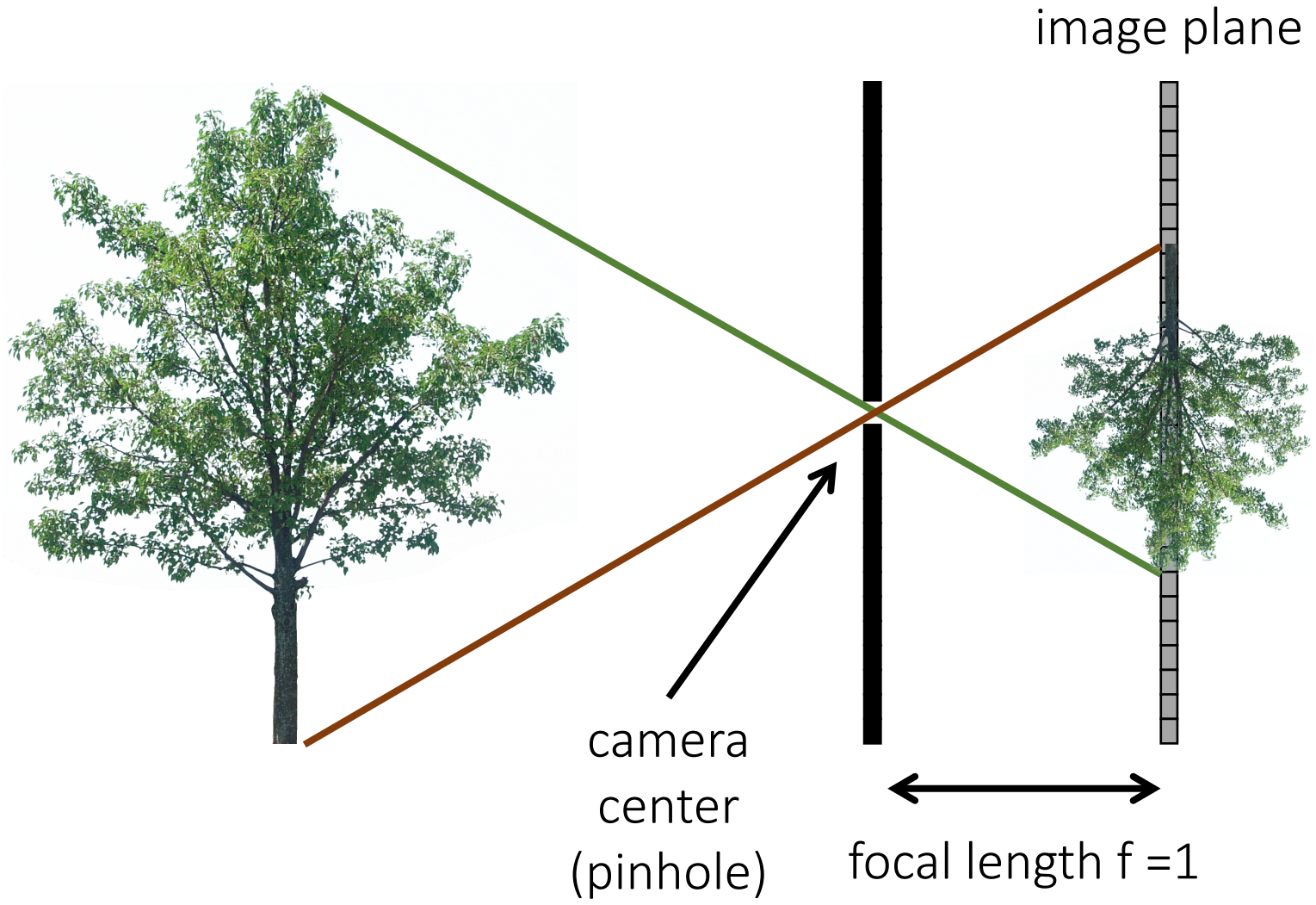
2D image point

camera matrix

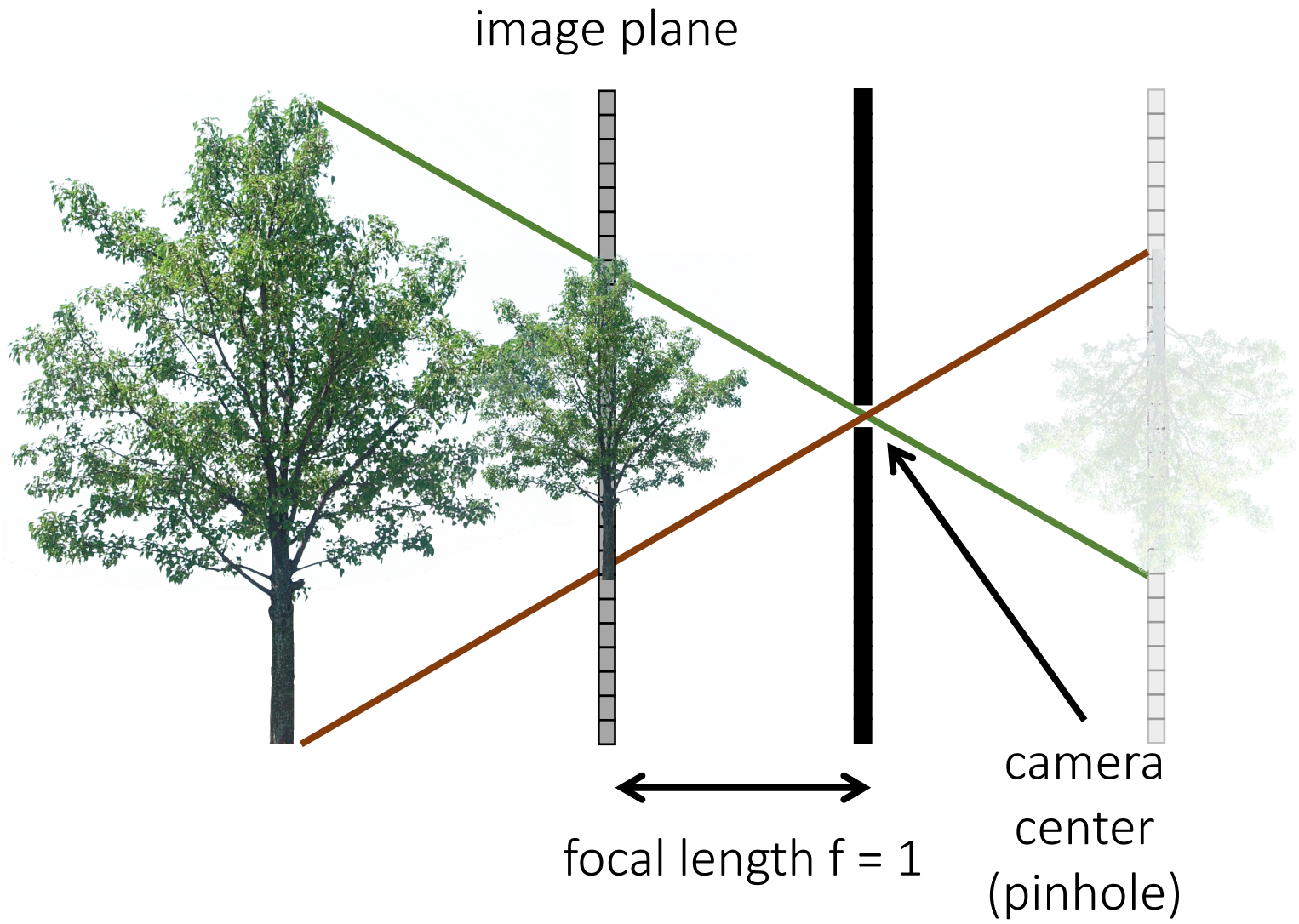
3D world point

What does this transformation look like?

# The pinhole camera



# The (rearranged) pinhole camera



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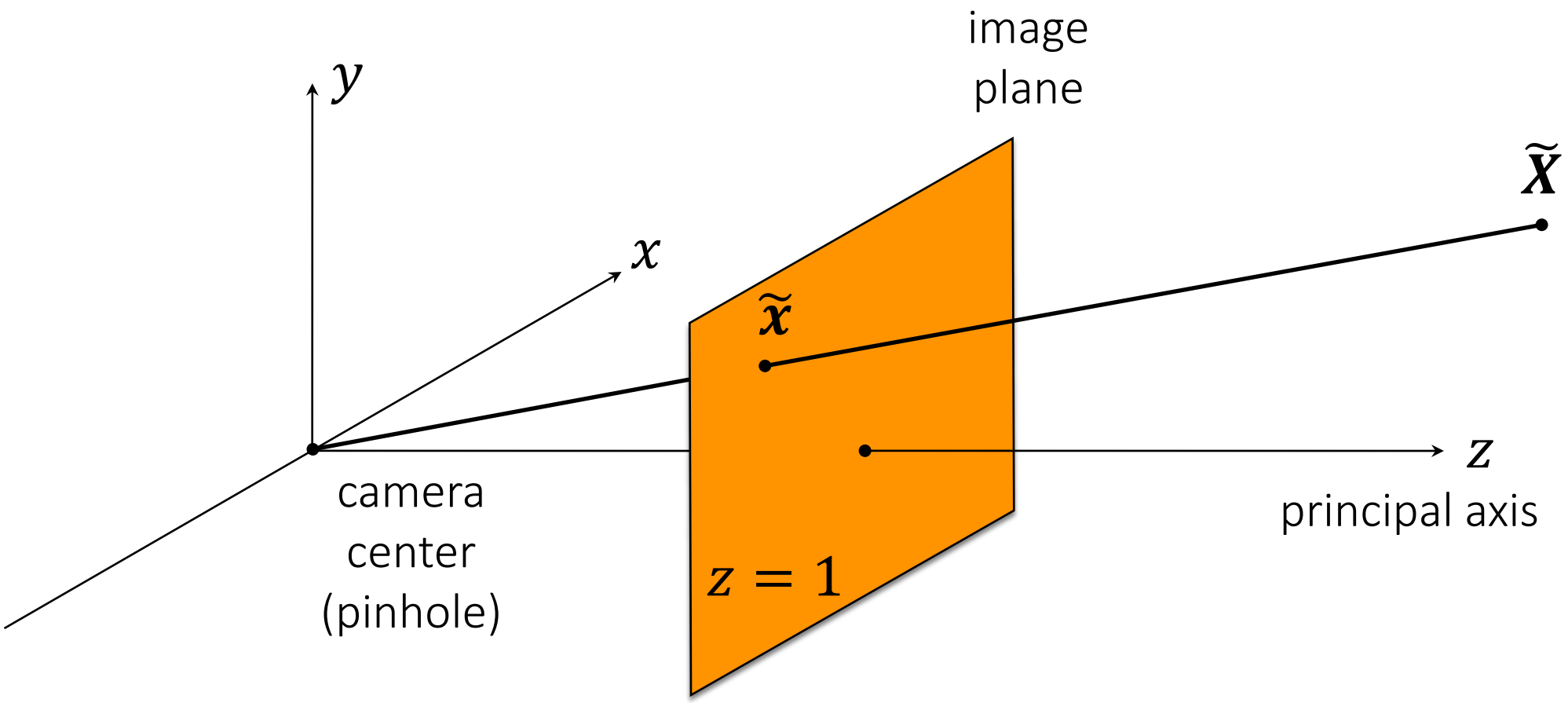
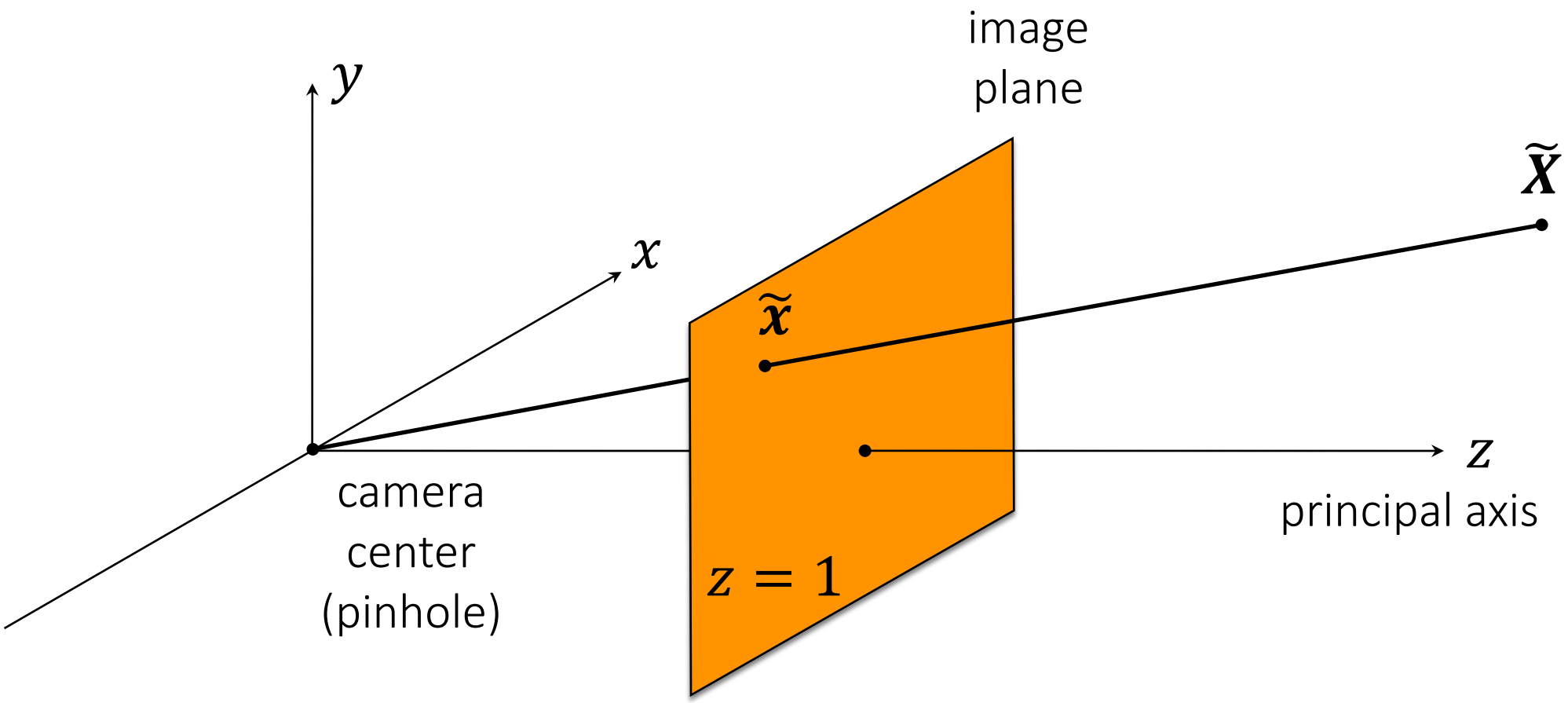


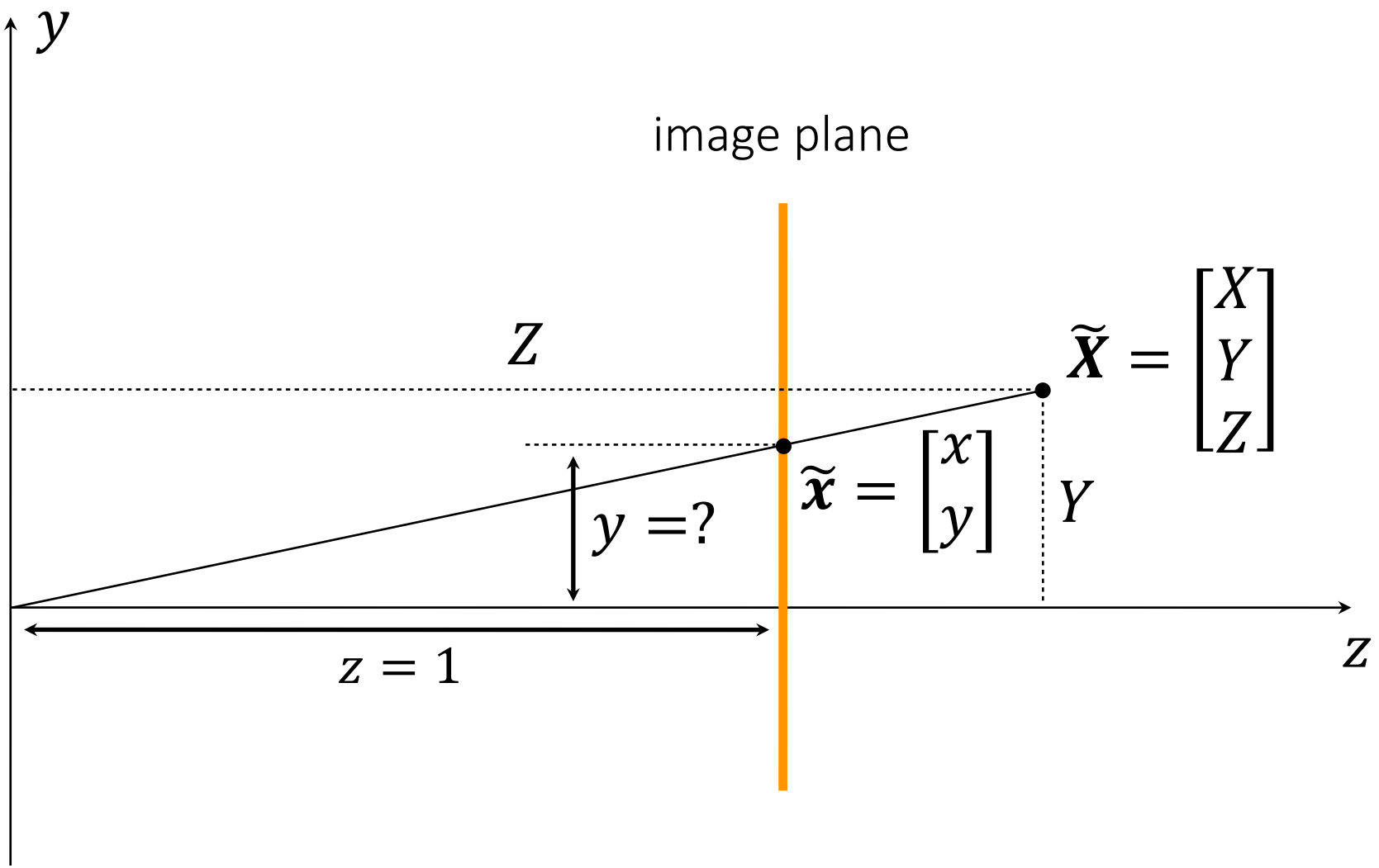

Where did we see a similar picture?

# The (rearranged) pinhole camera




What is the equation for image coordinate $\tilde{\boldsymbol{x}}$ in terms of $\tilde{\boldsymbol{X}}$?

# The 2D view of the (rearranged) pinhole camera

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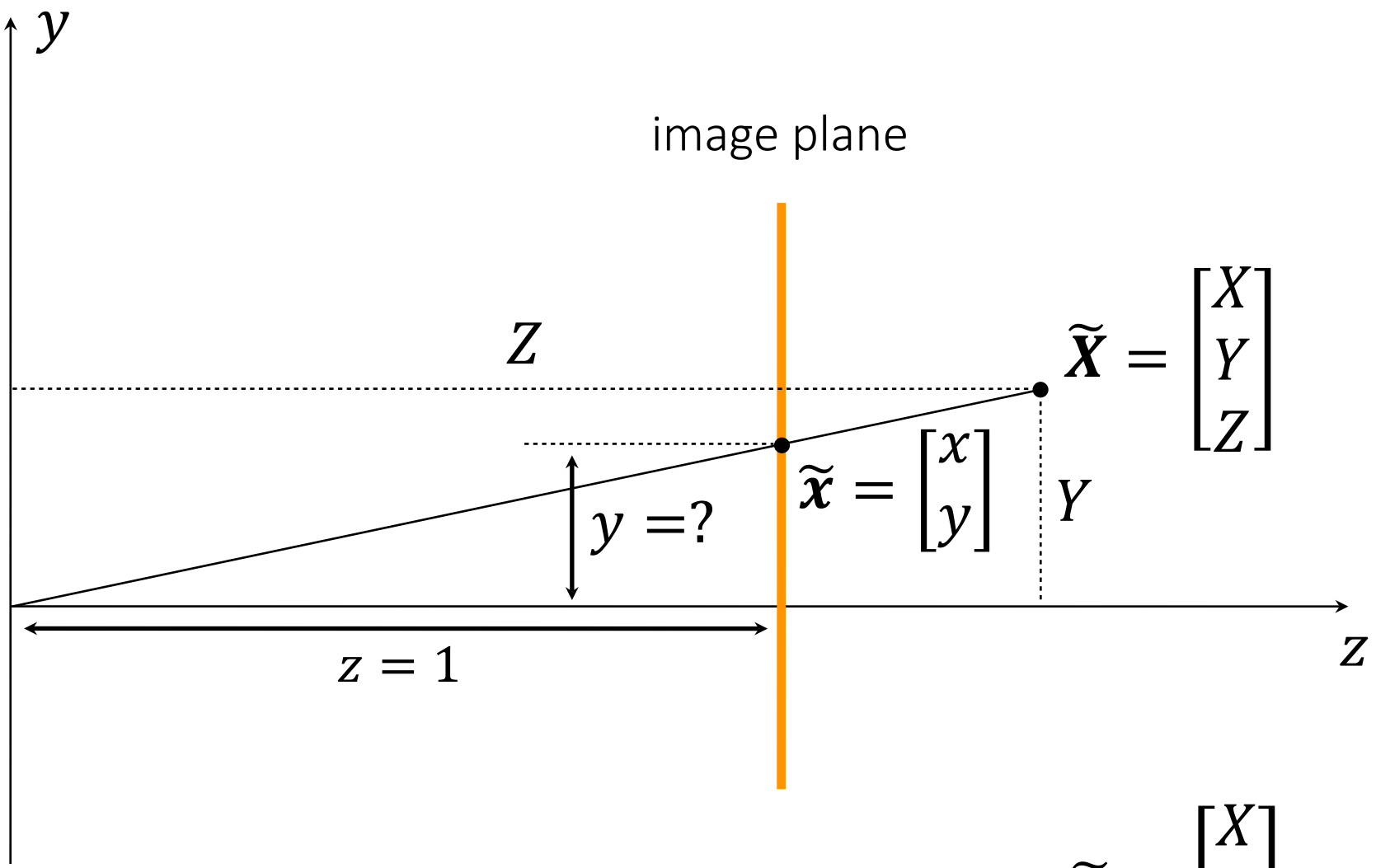
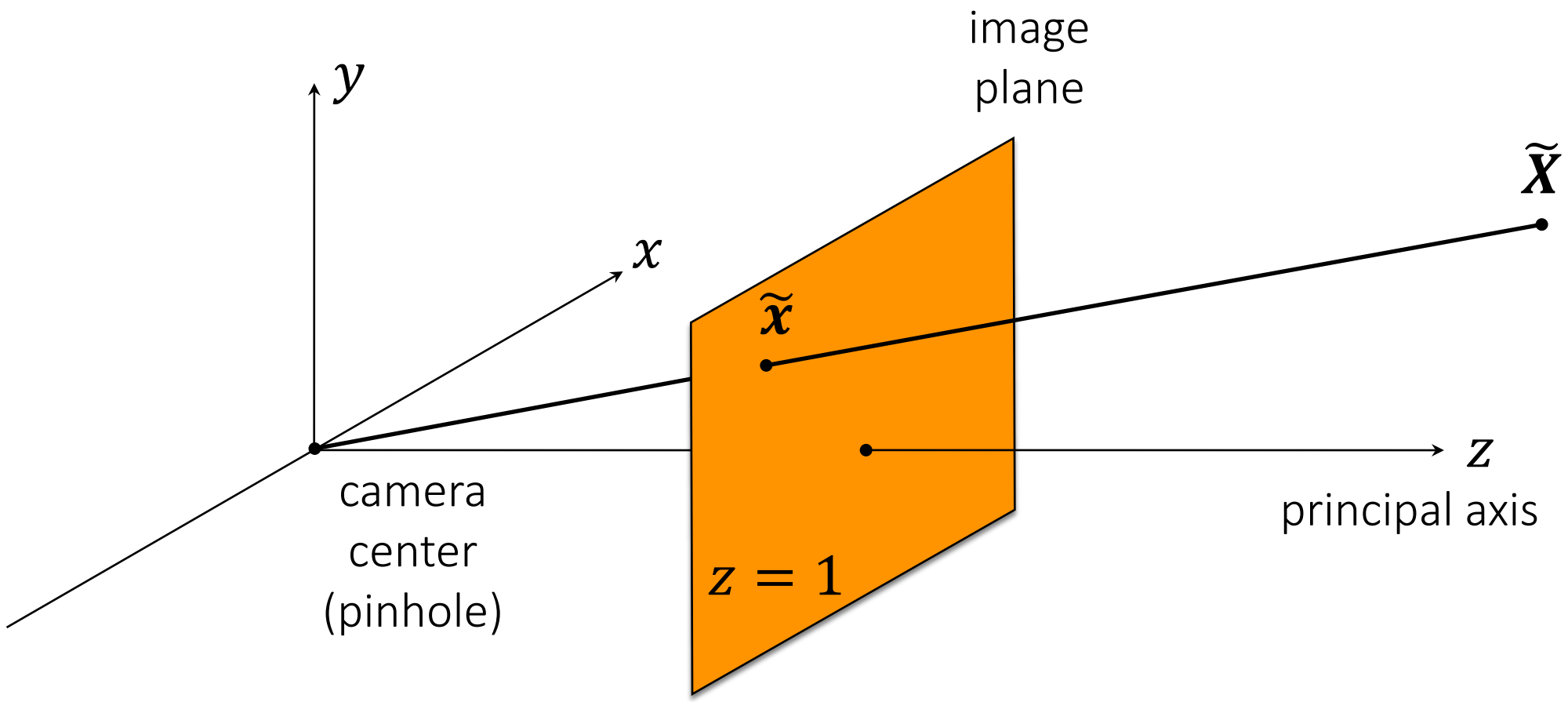



Using similar triangles: $\tilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\boldsymbol{x}} = \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix}$

# The (rearranged) pinhole camera




What is the camera matrix $\boldsymbol{P}$ for a pinhole camera?

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{X}$$

# The pinhole camera matrix

Camera projection relationship expressed:

- in *heterogeneous coordinates*

$$\tilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\boldsymbol{x}} = \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix}$$

- in *homogeneous coordinates*

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \rightarrow \boldsymbol{x} = ?$$

# The pinhole camera matrix

Camera projection relationship expressed:

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- in *homogeneous coordinates*

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \rightarrow \boldsymbol{x} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

General camera model in *homogeneous coordinates*:

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{X}$$

What does the pinhole camera projection look like?

$$\boldsymbol{P} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

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$$\widetilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \widetilde{\boldsymbol{x}} = \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix}$$

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The perspective projection matrix

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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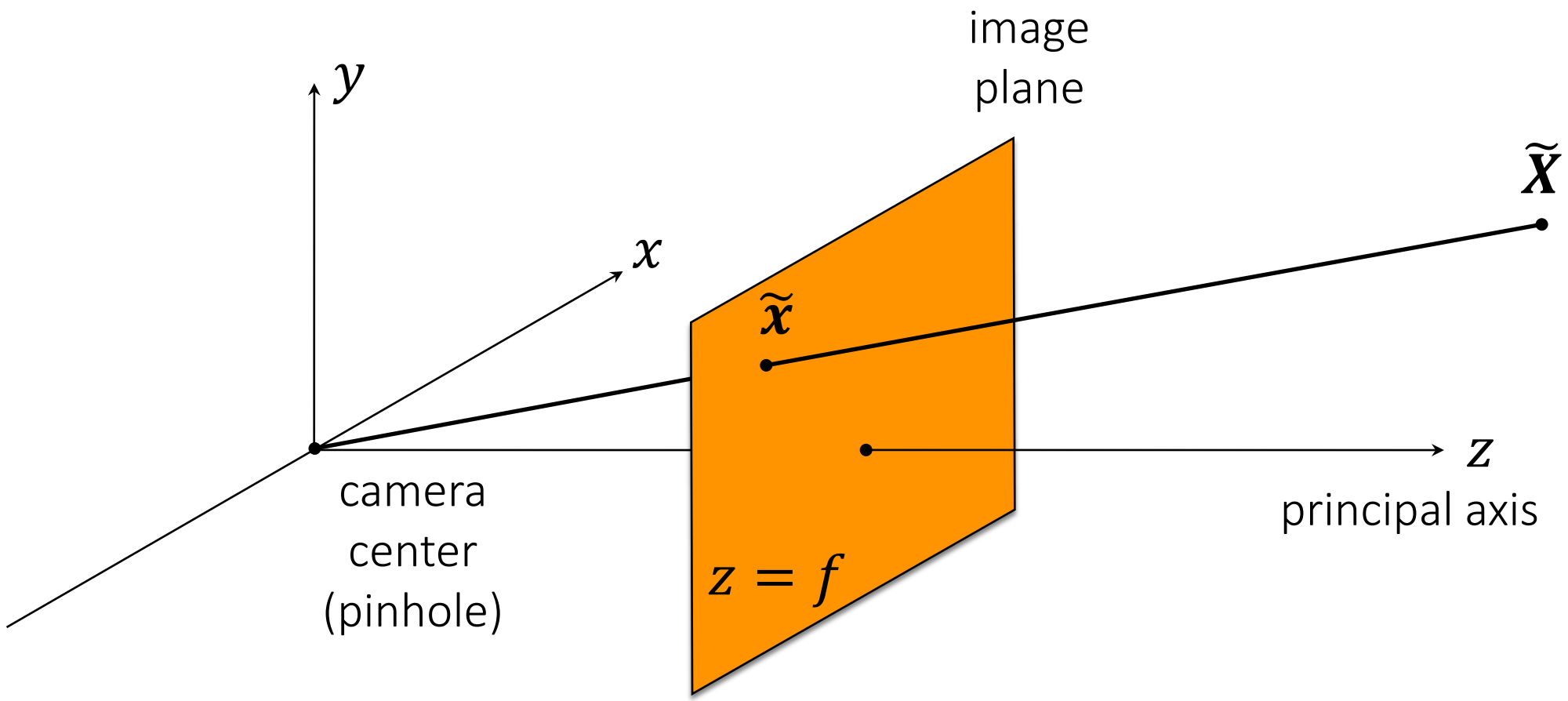
What does the pinhole camera projection look like?

The *perspective projection matrix*

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [\boldsymbol{I} \mid \boldsymbol{0}]$$

alternative way to write the same thing

# More general case: arbitrary focal length

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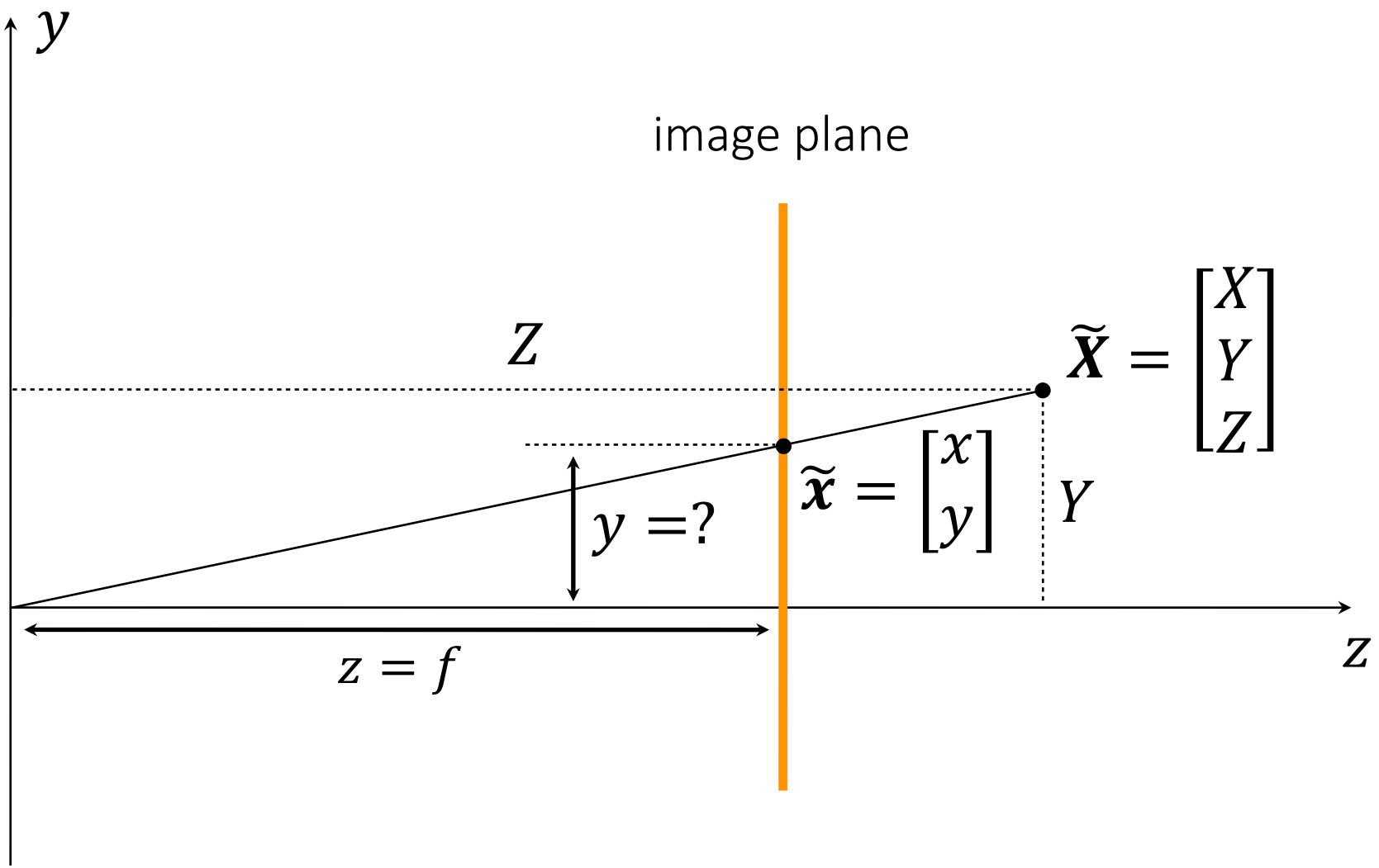
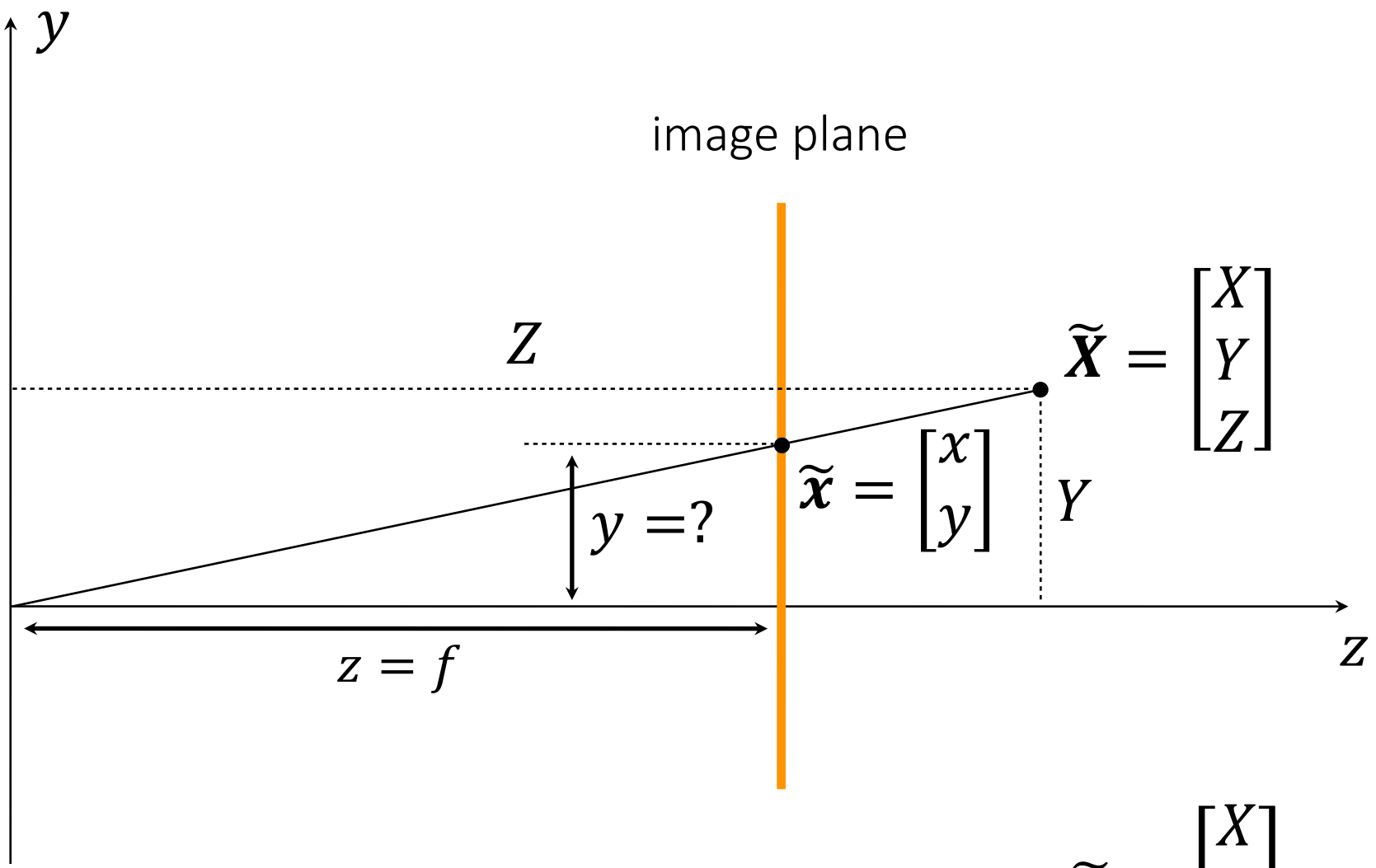
# More general (2D) case: arbitrary focal length




What is the equation for image coordinate $\tilde{x}$ in terms of $\tilde{X}$?

# More general (2D) case: arbitrary focal length




Using similar triangles: $\tilde{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{x} = \begin{bmatrix} fX/Z \\ fY/Z \end{bmatrix}$

# The pinhole camera matrix for arbitrary focal length

Camera projection relationship expressed:

- in *heterogeneous coordinates*

$$\widetilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \widetilde{\boldsymbol{x}} = \begin{bmatrix} fX/Z \\ fY/Z \end{bmatrix}$$

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What does the pinhole camera projection look like?

$$\boldsymbol{P} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

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$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \rightarrow \boldsymbol{x} = \begin{bmatrix} fX \\ fY \\ Z \end{bmatrix}$$

General camera model in *homogeneous coordinates*:

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{X}$$

What does the pinhole camera projection look like?

$$\boldsymbol{P} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

# The pinhole camera matrix for arbitrary focal length

Camera projection relationship expressed:

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General camera model in *homogeneous coordinates*:

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{X}$$

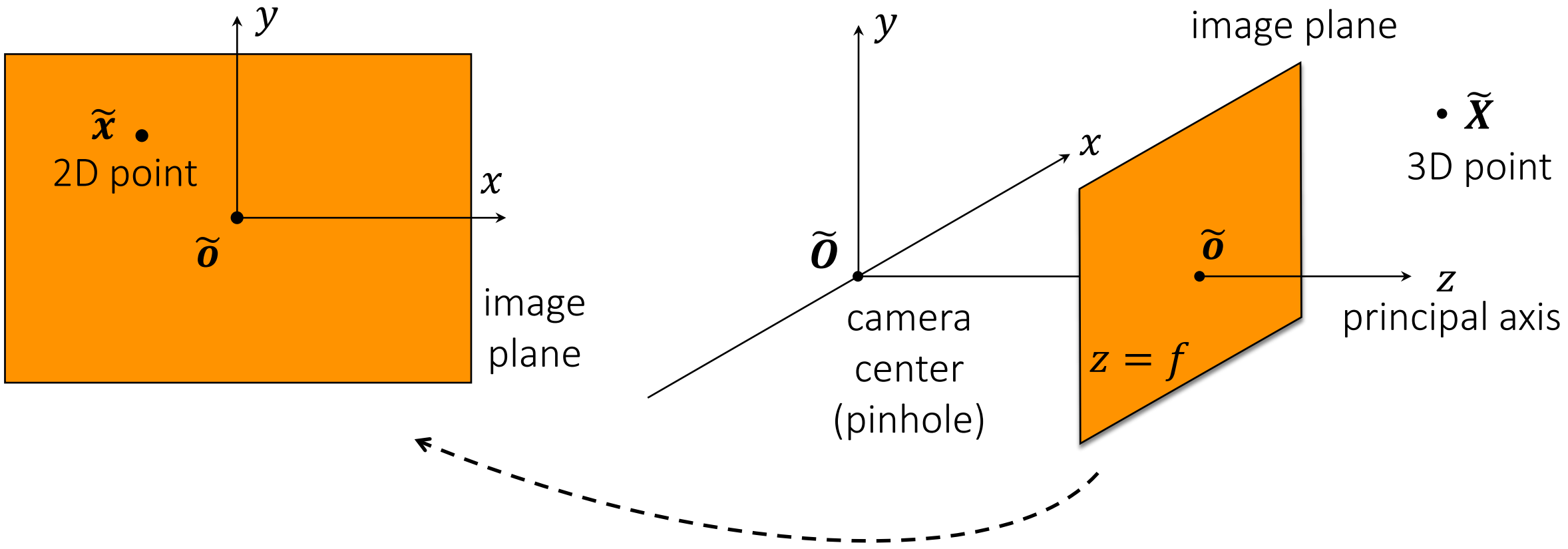
What does the pinhole camera projection look like?

Equivalently we can write:

$$\boldsymbol{P} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

combination of perspective projection and a 2D scaling transformation

# Generalizations: coordinate systems

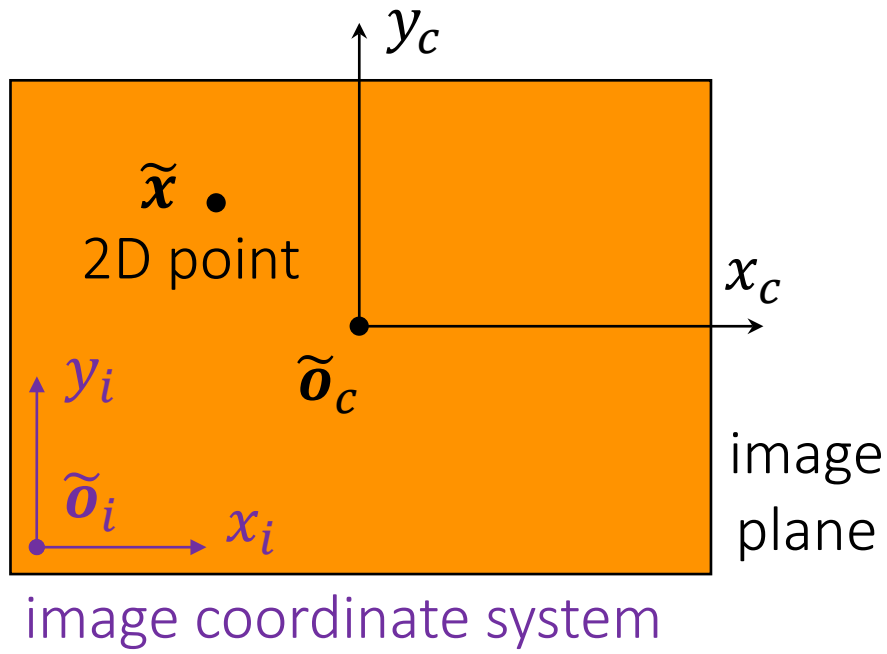


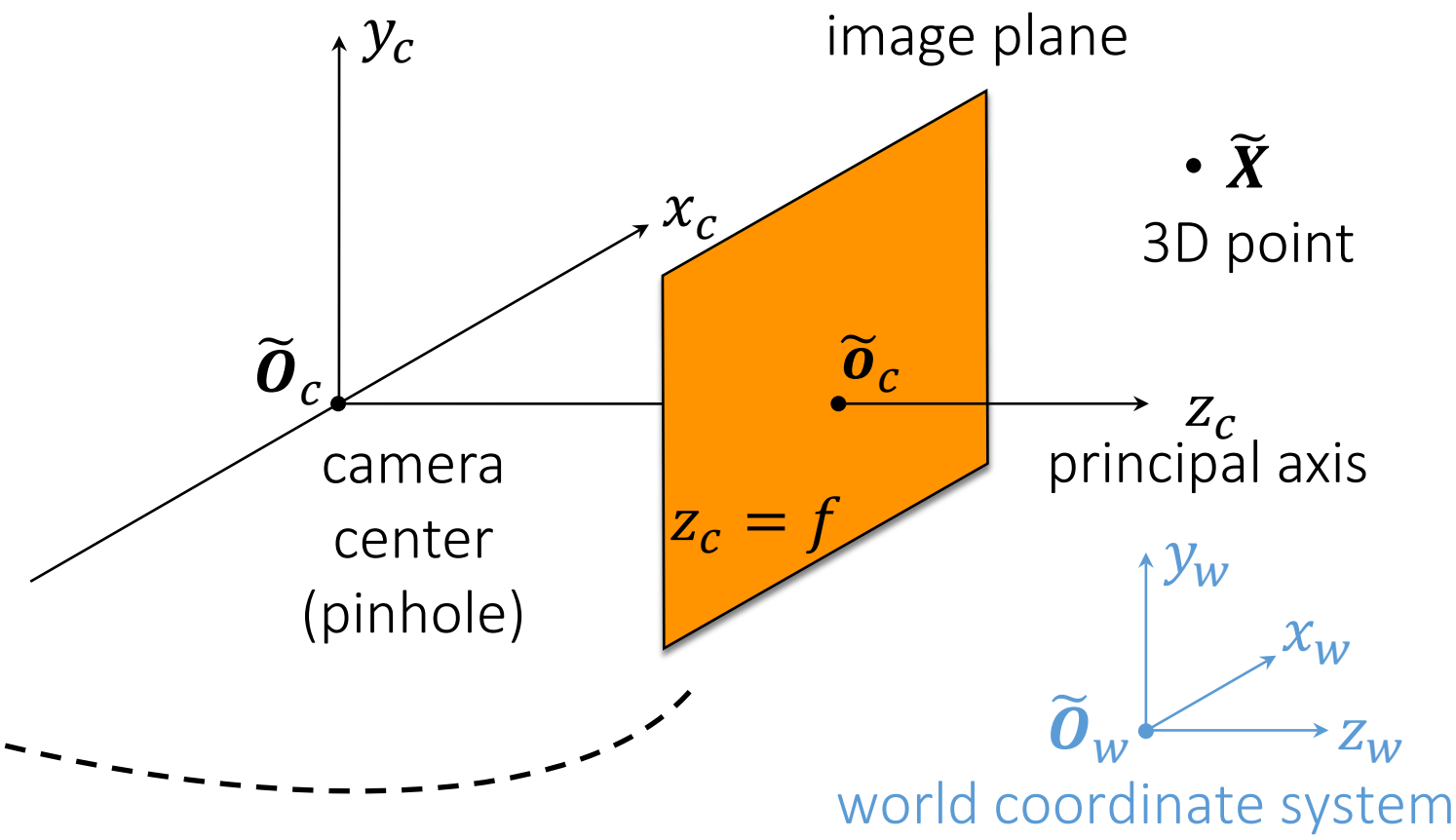

2D camera coordinate system

3D camera coordinate system

- A camera introduces two related coordinate systems, in 3D (world), and in 2D (image plane).

# Generalizations: coordinate systems




2D camera coordinate system

3D camera coordinate system

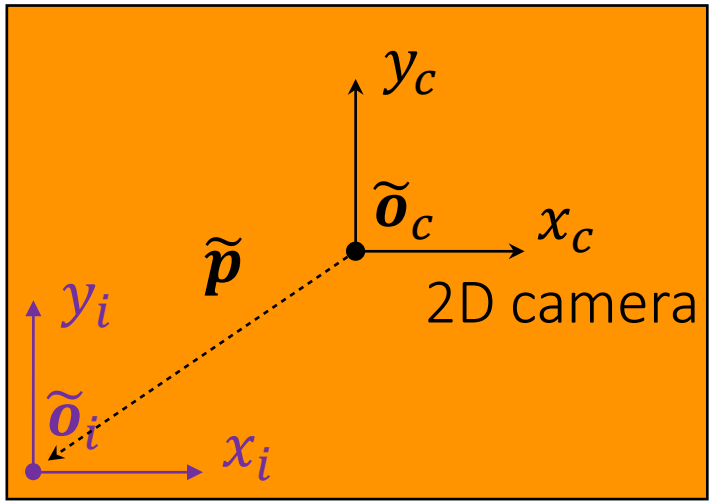
- A camera introduces two related coordinate systems, in 3D (world), and in 2D (image plane).
- These coordinate systems may be different from the coordinate systems of our application.

# Generalization: image coordinate system

In particular, the camera origin and image origin may be different.

- Can you think of a case when this happens?

image plane




2D camera coordinate system

image coordinate system

How does the camera matrix change?

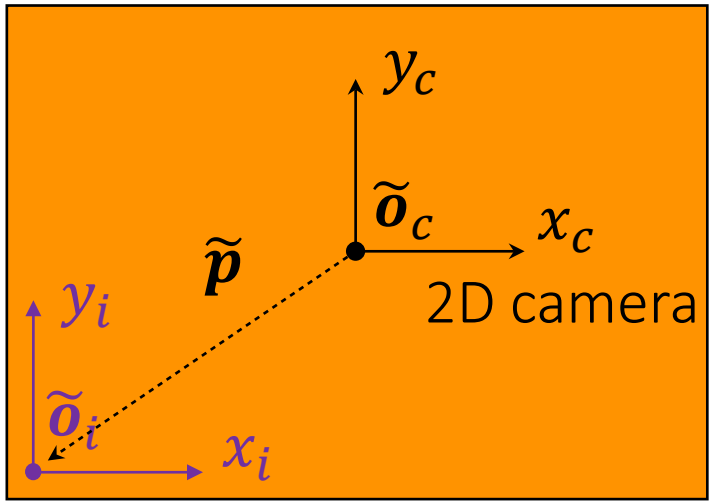
$$\boldsymbol{P} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

# Generalization: image coordinate system

In particular, the camera origin and image origin may be different.

- Can you think of a case when this happens?

image plane




2D camera coordinate system

image coordinate system

shift vector transforming camera origin to image origin

How does the camera matrix change?

$$\boldsymbol{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

# Camera matrix decomposition

We can decompose the camera matrix like this:

$$\boldsymbol{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

What does each part of the matrix represent?

# Camera matrix decomposition

We can decompose the camera matrix like this:

$$\boldsymbol{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

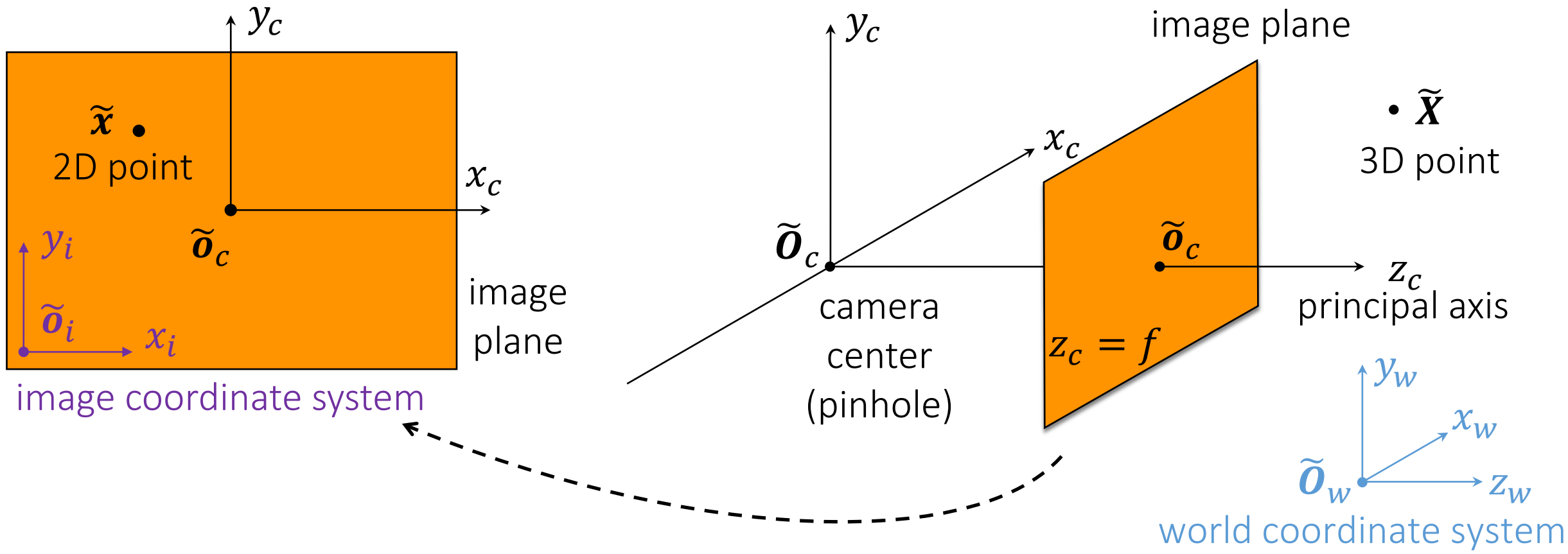
(homogeneous) transformation from 2D to 2D, accounting for non-unit focal length and origin shift

(homogeneous) perspective projection from 3D to 2D, assuming image plane at z = 1 and shared camera/image origin

Also written as:

$$\boldsymbol{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} [\boldsymbol{I} \mid \boldsymbol{0}]$$

# Generalizations: coordinate systems




2D camera coordinate system

3D camera coordinate system

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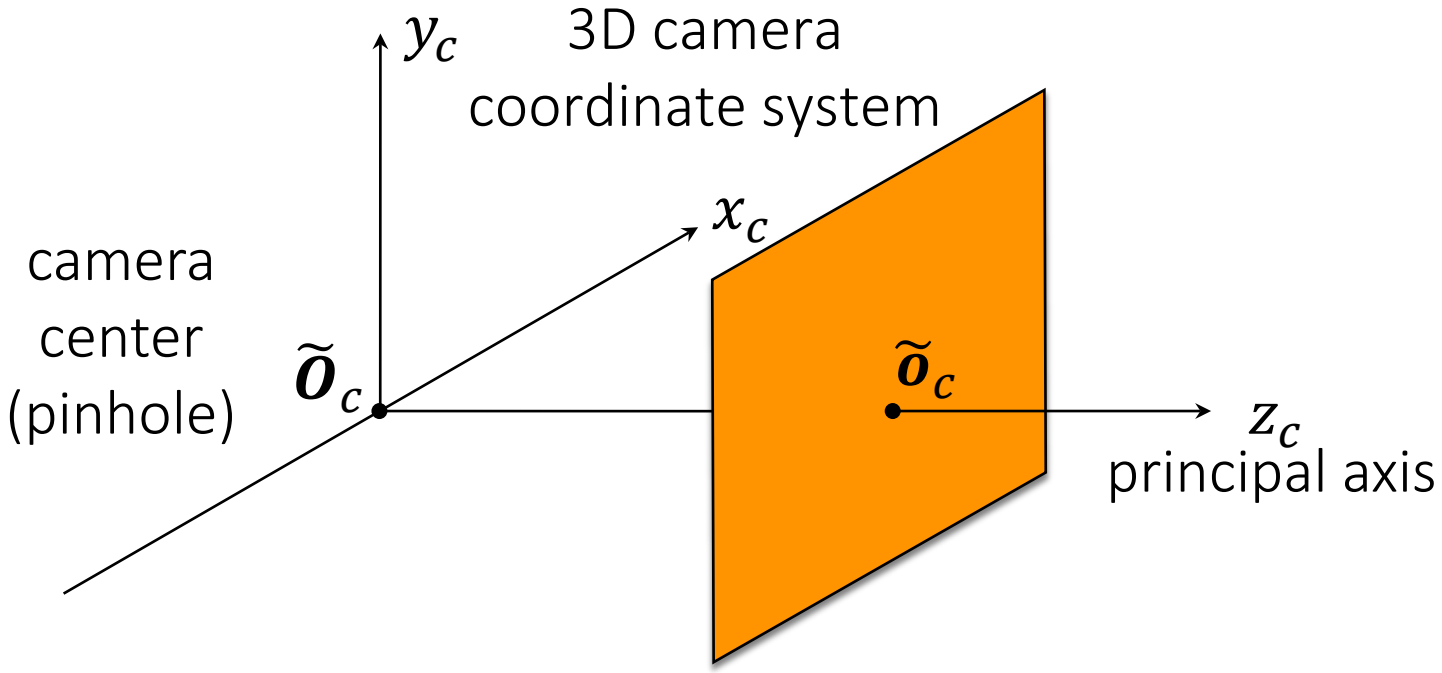
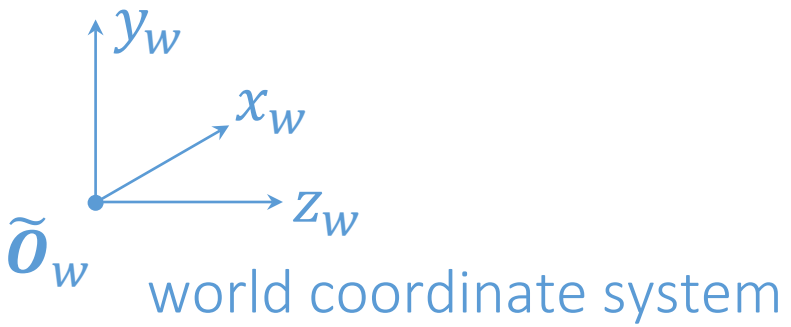
- These coordinate systems may be different from the coordinate systems of our application.

# World-to-camera coordinate system transformation



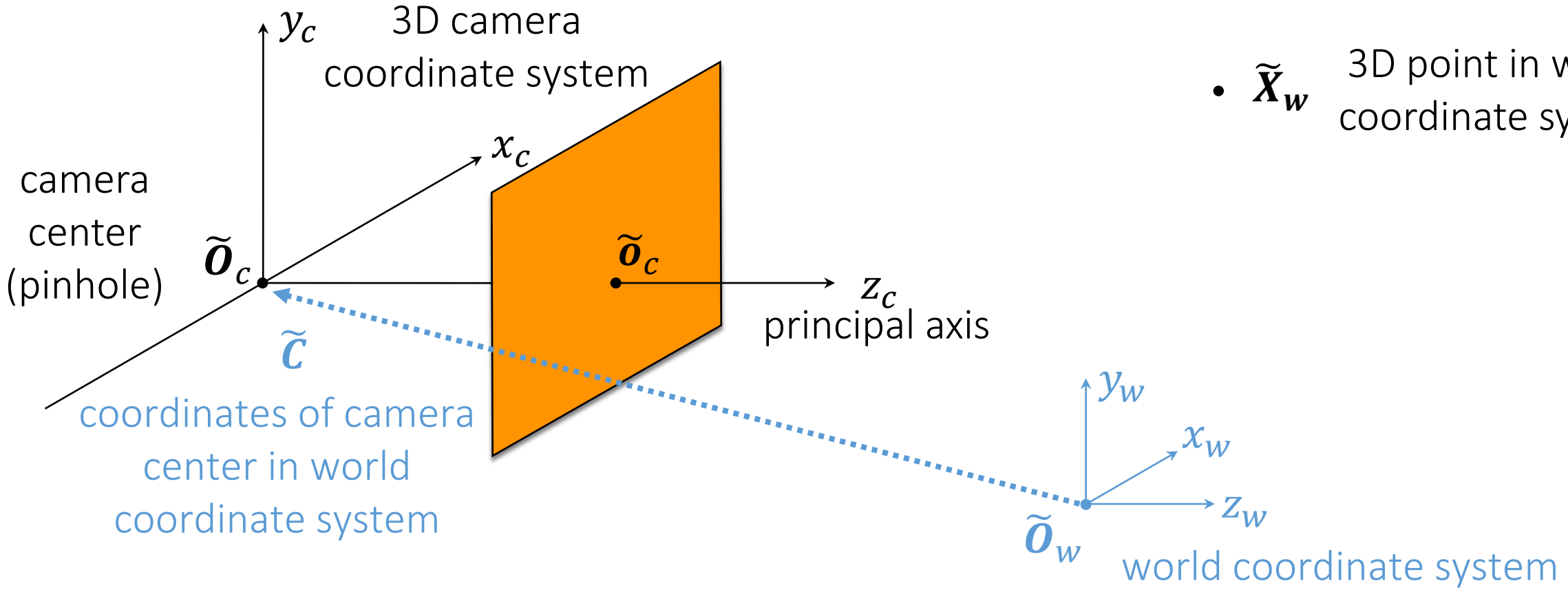

- $\tilde{\boldsymbol{X}}_{\boldsymbol{w}}$ 3D point in world coordinate system




How do we express $\tilde{\boldsymbol{X}}$ in the 3D camera coordinate system?

$$\tilde{\boldsymbol{X}}_{\boldsymbol{w}}$$

# World-to-camera coordinate system transformation



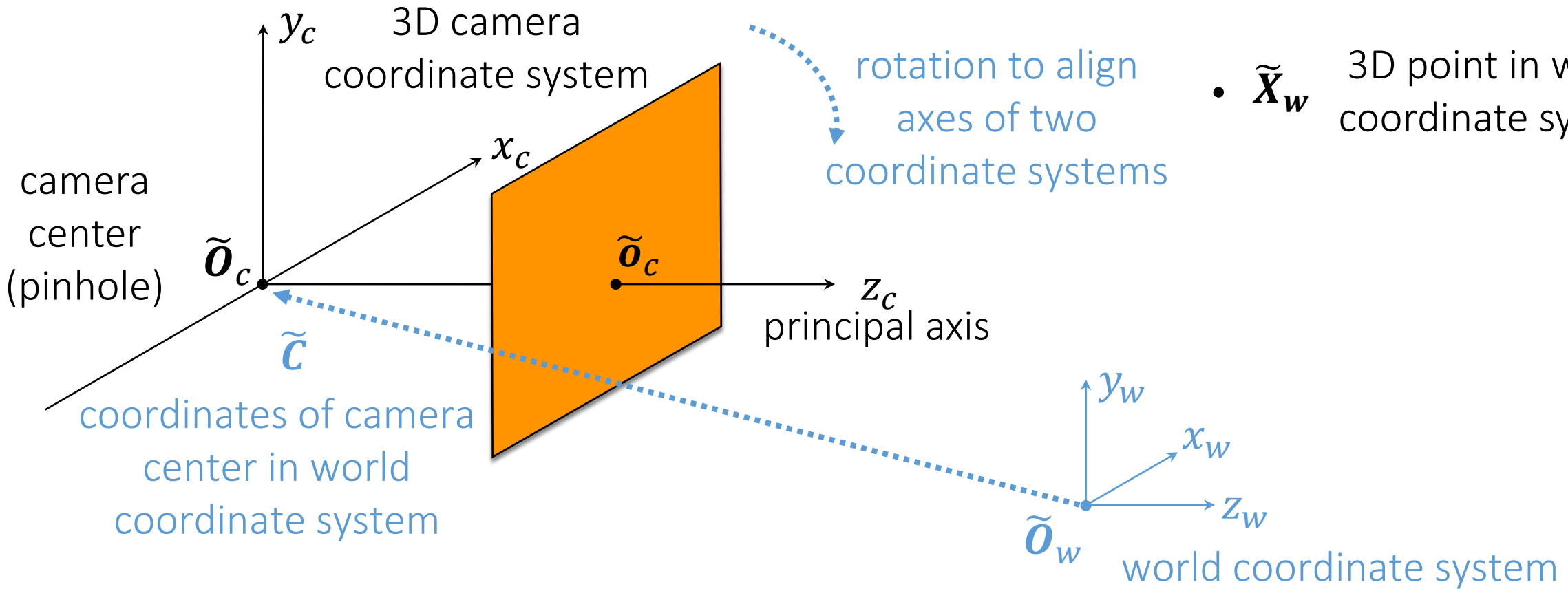

- $\tilde{X}_w$ 3D point in world coordinate system

How do we express $\tilde{X}$ in the 3D camera coordinate system?

$$\tilde{X}_w - \tilde{C}$$

translate

# World-to-camera coordinate system transformation




How do we express $\tilde{X}$ in the 3D camera coordinate system?

$$R \cdot (\tilde{X}_w - \tilde{C})$$

rotate translate

# Modeling the 3D coordinate system transformation

In heterogeneous coordinates, we have:

$$\widetilde{\boldsymbol{X}}_{\boldsymbol{c}} = \boldsymbol{R} \cdot \left(\widetilde{\boldsymbol{X}}_{\boldsymbol{w}} - \widetilde{\boldsymbol{C}}\right)$$

How do we write this transformation in homogeneous coordinates?

# Modeling the 3D coordinate system transformation

In heterogeneous coordinates, we have:

$$\widetilde{\boldsymbol{X}}_{\boldsymbol{c}} = \boldsymbol{R} \cdot \left(\widetilde{\boldsymbol{X}}_{\boldsymbol{w}} - \widetilde{\boldsymbol{C}}\right)$$

In homogeneous coordinates, we have:

$$\boldsymbol{X}_{\boldsymbol{c}} = \begin{bmatrix} \boldsymbol{R} & -\boldsymbol{R}\widetilde{\boldsymbol{C}} \\ \boldsymbol{0} & 1 \end{bmatrix} \boldsymbol{X}_{\boldsymbol{w}}$$

# Incorporating the transform in the camera matrix

The previous camera matrix is for homogeneous 3D coordinates in camera coordinate system:

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{X}_{\boldsymbol{c}} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} [\boldsymbol{I} \mid \boldsymbol{0}]\boldsymbol{X}_{\boldsymbol{c}}$$

We also just derived:

$$\boldsymbol{X}_{\boldsymbol{c}} = \begin{bmatrix} \boldsymbol{R} & -\boldsymbol{R}\widetilde{\boldsymbol{C}} \\ \boldsymbol{0} & 1 \end{bmatrix} \boldsymbol{X}_{\boldsymbol{w}}$$

# Putting it all together

We can write everything into a single projection:

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{X}_{\boldsymbol{w}}$$

The camera matrix now looks like:

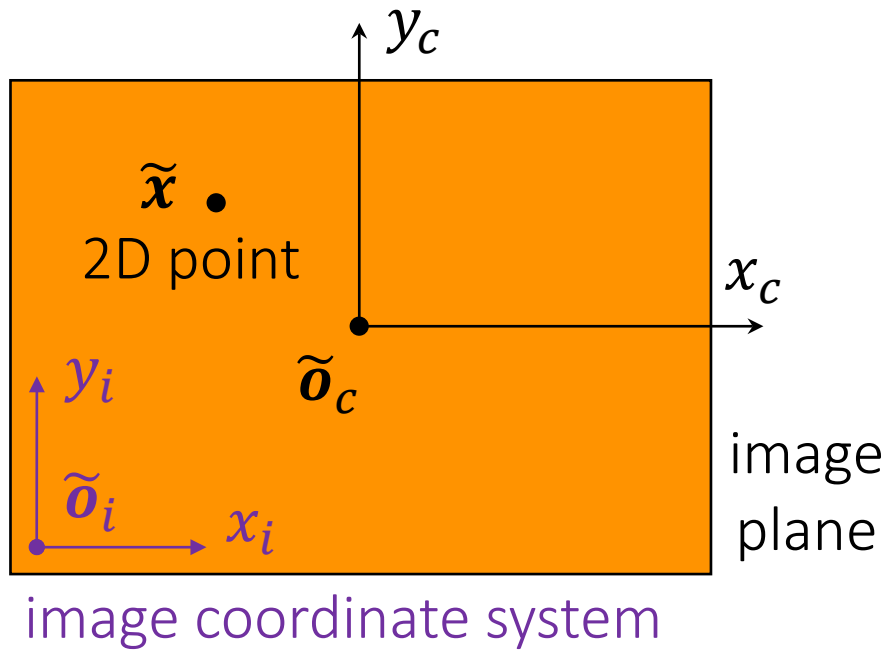
$$\boldsymbol{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} [\boldsymbol{I} \mid \boldsymbol{0}] \begin{bmatrix} \boldsymbol{R} & -\boldsymbol{R}\widetilde{\boldsymbol{C}} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

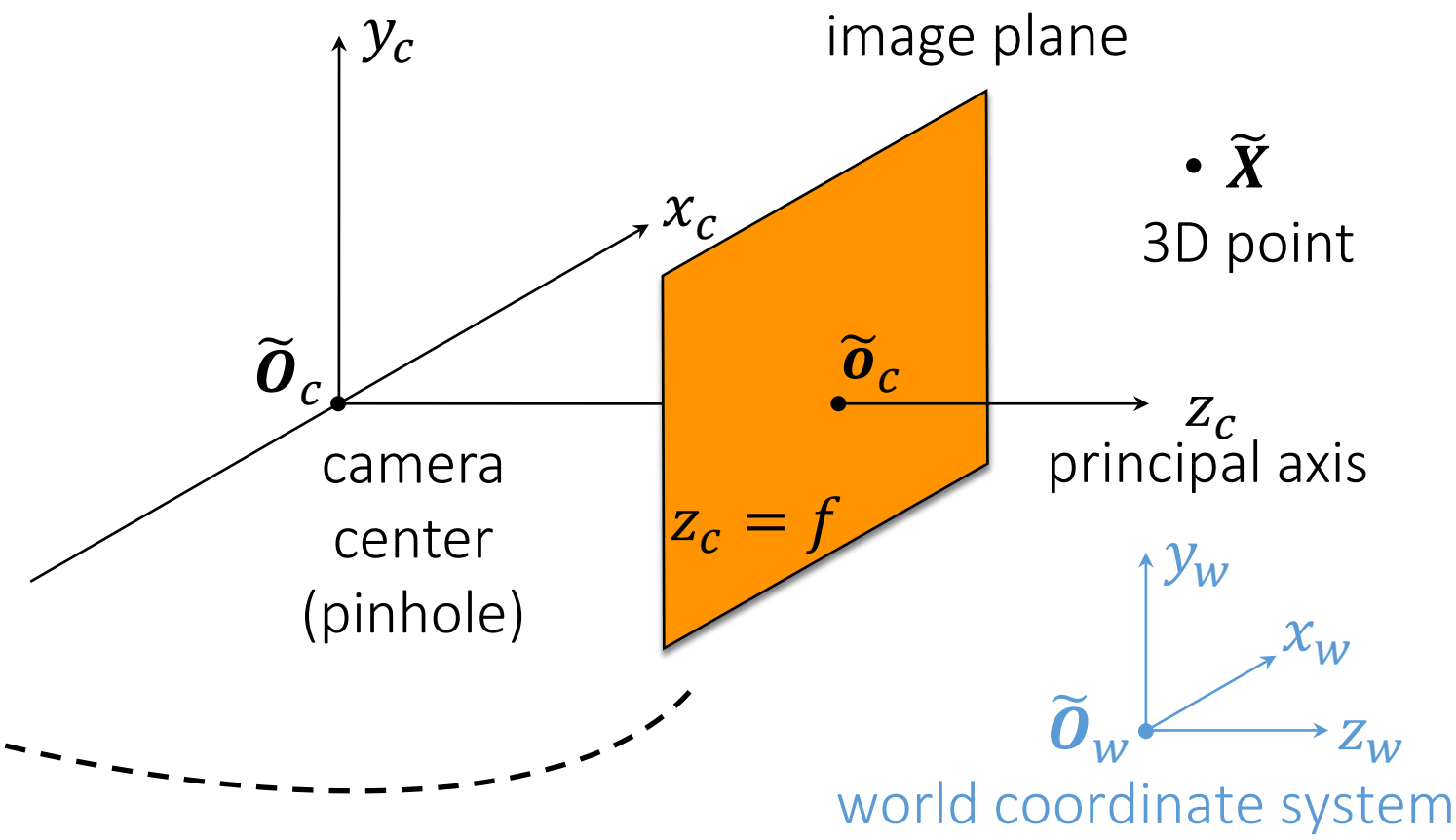
*intrinsic parameters* (3 x 3): correspond to camera internals (2D image-to-image transformation)

*perspective projection* (3 x 4): maps 3D to 2D points (camera-to-image transformation)

*extrinsic parameters* (4 x 4): correspond to camera externals (3D world-to-camera transformation)

# Generalizations: coordinate systems




2D camera coordinate system

3D camera coordinate system

- A camera introduces two related coordinate systems, in 3D (world), and in 2D (image plane).
- These coordinate systems may be different from the coordinate systems of our application.

# Putting it all together

We can write everything into a single projection:

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{X}_{\boldsymbol{w}}$$

The camera matrix now looks like:

$$\boldsymbol{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} [\boldsymbol{R} \mid -\boldsymbol{R}\widetilde{\boldsymbol{C}}]$$

It is common to combine the perspective projection and extrinsics in one matrix.

*intrinsic parameters* (3 x 3): correspond to camera internals (2D image-to-image transformation)

*extrinsic parameters* (3 x 4): correspond to camera externals (3D world-to-camera transformation) *and* perspective projection

# The pinhole camera matrix

More compactly, we can write the pinhole camera matrix as:

$$\boldsymbol{P} = \boldsymbol{K}[\boldsymbol{R} \mid \boldsymbol{t}]$$

where

$$\boldsymbol{K} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{R} = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \qquad \boldsymbol{t} = -\boldsymbol{R}\widetilde{\boldsymbol{C}} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

2D Euclidean transform

3D rotation

3D translation

intrinsic parameters

extrinsic parameters

# More general pinhole camera matrices

The following is the standard pinhole camera matrix we saw.

$$\boldsymbol{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

How many degrees of freedom does this matrix have?

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How many degrees of freedom does this matrix have?

- 9 degrees of freedom (3 for intrinsics, 3 for rotation, 3 for translation).

# More general pinhole camera matrices

The following is the standard pinhole camera matrix we saw.

$$\boldsymbol{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

How many degrees of freedom does this matrix have?

- 9 degrees of freedom (3 for intrinsics, 3 for rotation, 3 for translation).

We can get more general pinhole cameras with more degrees of freedom by generalizing the intrinsics matrix, while leaving everything else the same..

# More general pinhole camera matrices

*CCD camera*: pixels may not be square.

$$\boldsymbol{P} = \begin{bmatrix} a_x & 0 & p_x \\ 0 & a_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

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How many degrees of freedom does this matrix have?

- 10 degrees of freedom (4 for intrinsics, 3 for rotation, 3 for translation).

# More general pinhole camera matrices

*Finite projective camera*: sensor may be skewed.

$$\boldsymbol{P} = \begin{bmatrix} a_x & s & p_x \\ 0 & a_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

How many degrees of freedom does this matrix have?

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How many degrees of freedom does this matrix have?

- 11 degrees of freedom (5 for intrinsics, 3 for rotation, 3 for translation).

Can we get a *perspective projection* camera with more degrees of freedom?

# More general pinhole camera matrices

*Finite projective camera*: sensor may be skewed.

The finite projective camera is the most general camera implementing perspective projection.

$$\boldsymbol{P} = \begin{bmatrix} a_x & s & p_x \\ 0 & a_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

How many degrees of freedom does this matrix have?

- 11 degrees of freedom (5 for intrinsics, 3 for rotation, 3 for translation).

Can we get a *perspective projection* camera with more degrees of freedom?

- No, as the entire camera matrix $\boldsymbol{P}$ has 12 elements (3x4) and is defined up to scale.

# More general pinhole camera matrices

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# Perspective distortion

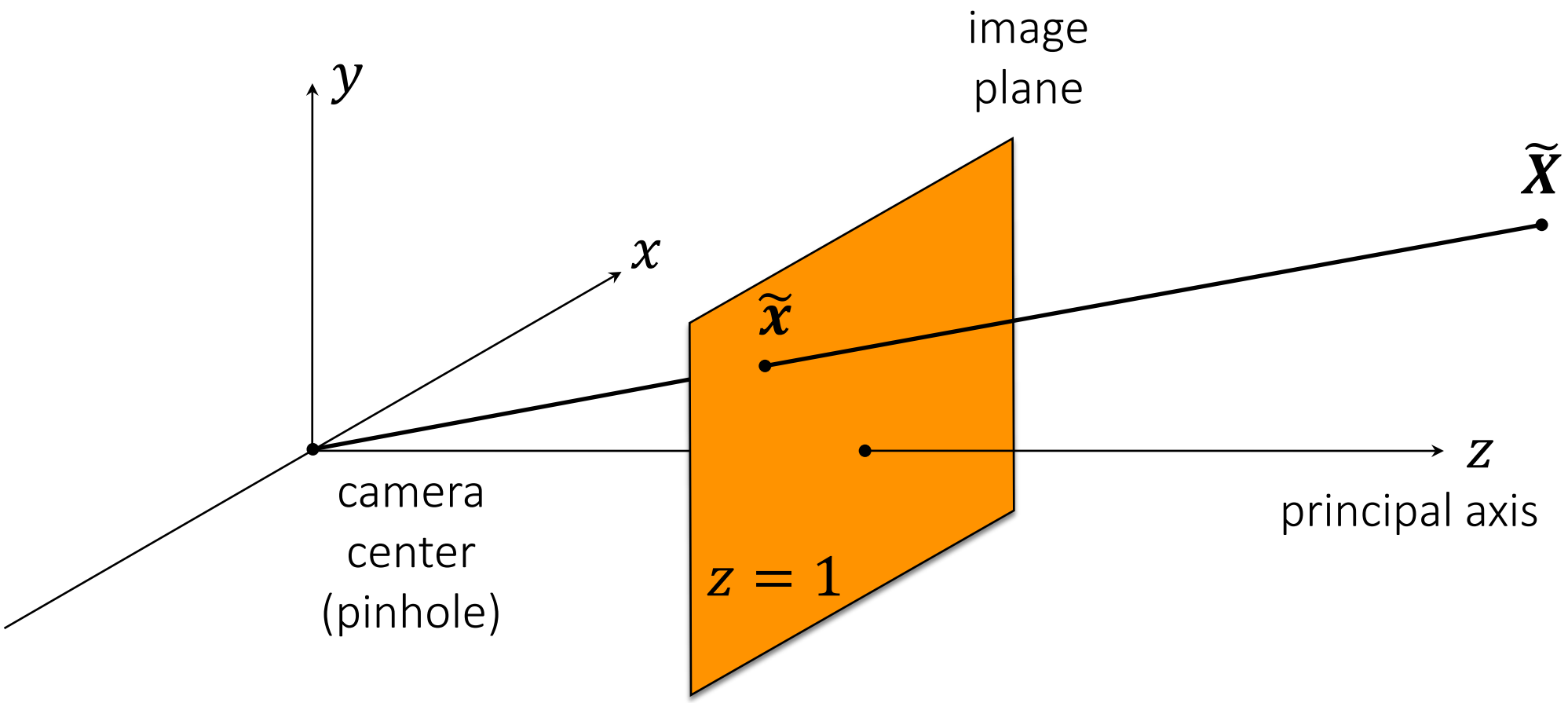
# Finite projective camera

Let’s ignore intrinsics and extrinsics for now.

$$\boldsymbol{P} = \begin{bmatrix} a_x & s & p_x \\ 0 & a_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

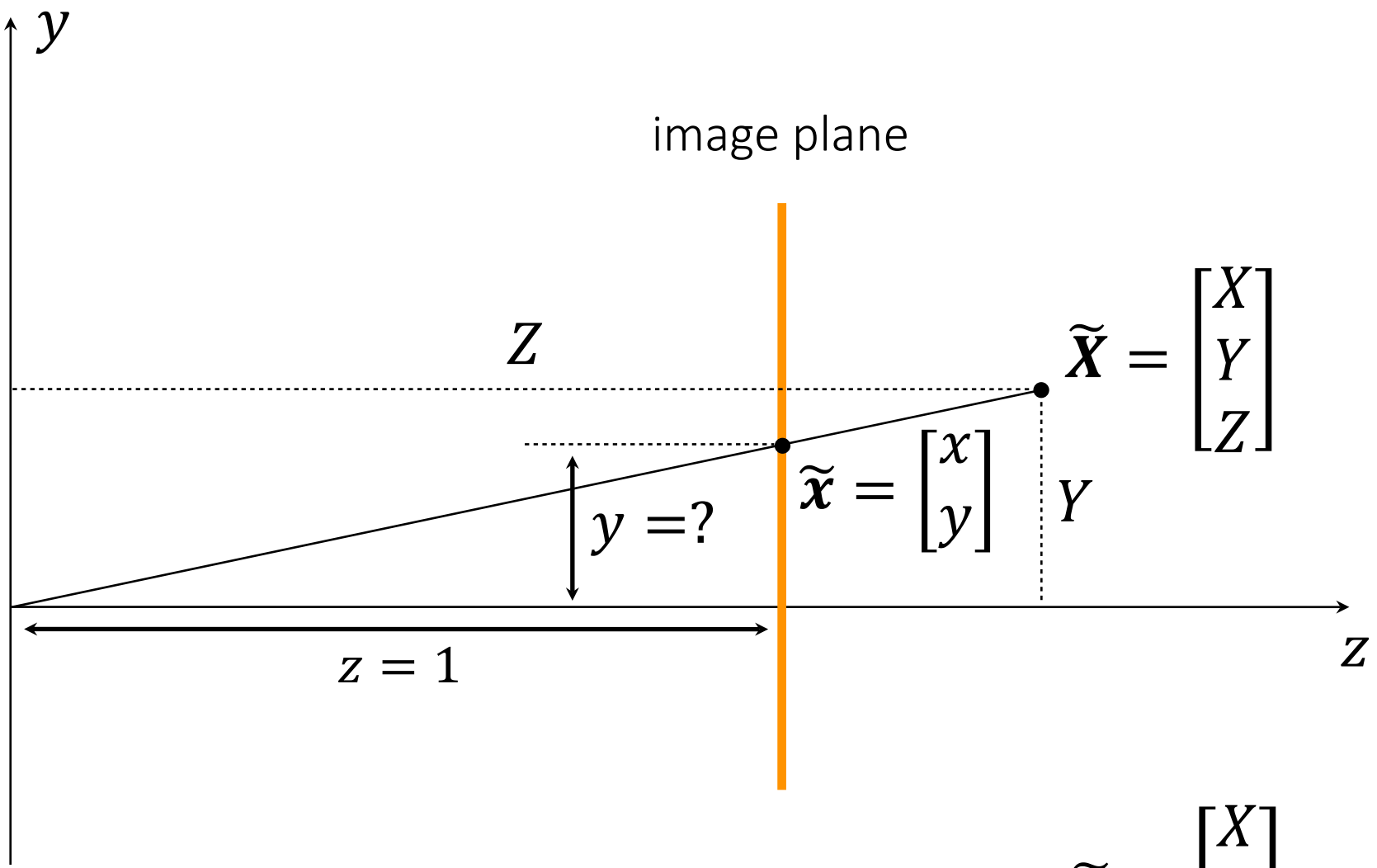
What is the effect of the perspective projection matrix?

# The (rearranged) pinhole camera



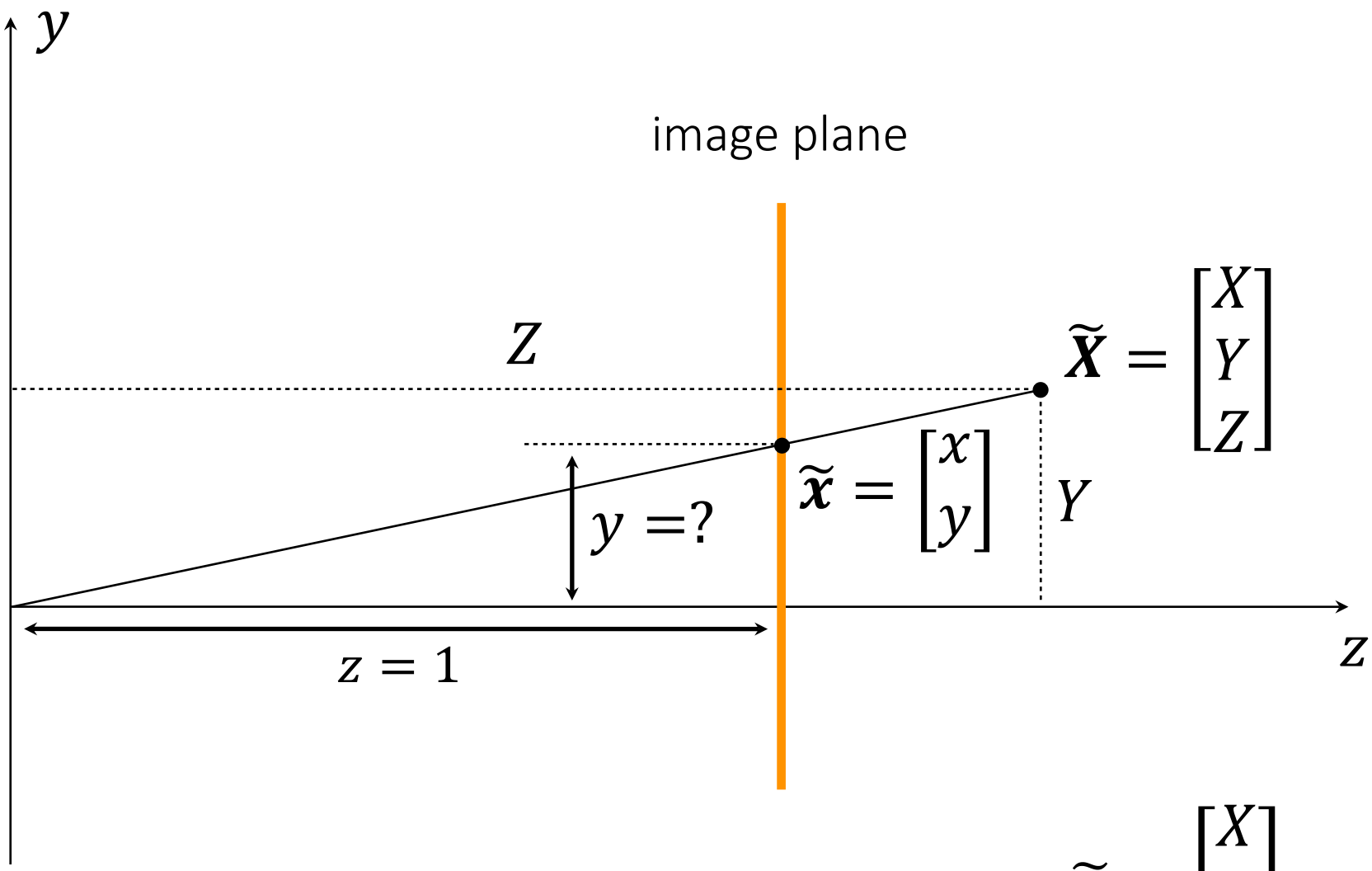

What is the equation for image coordinate $\tilde{\boldsymbol{x}}$ in terms of $\tilde{\boldsymbol{X}}$?

# The 2D view of the (rearranged) pinhole camera




Using similar triangles: $\tilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\boldsymbol{x}} = \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix}$

# The 2D view of the (rearranged) pinhole camera



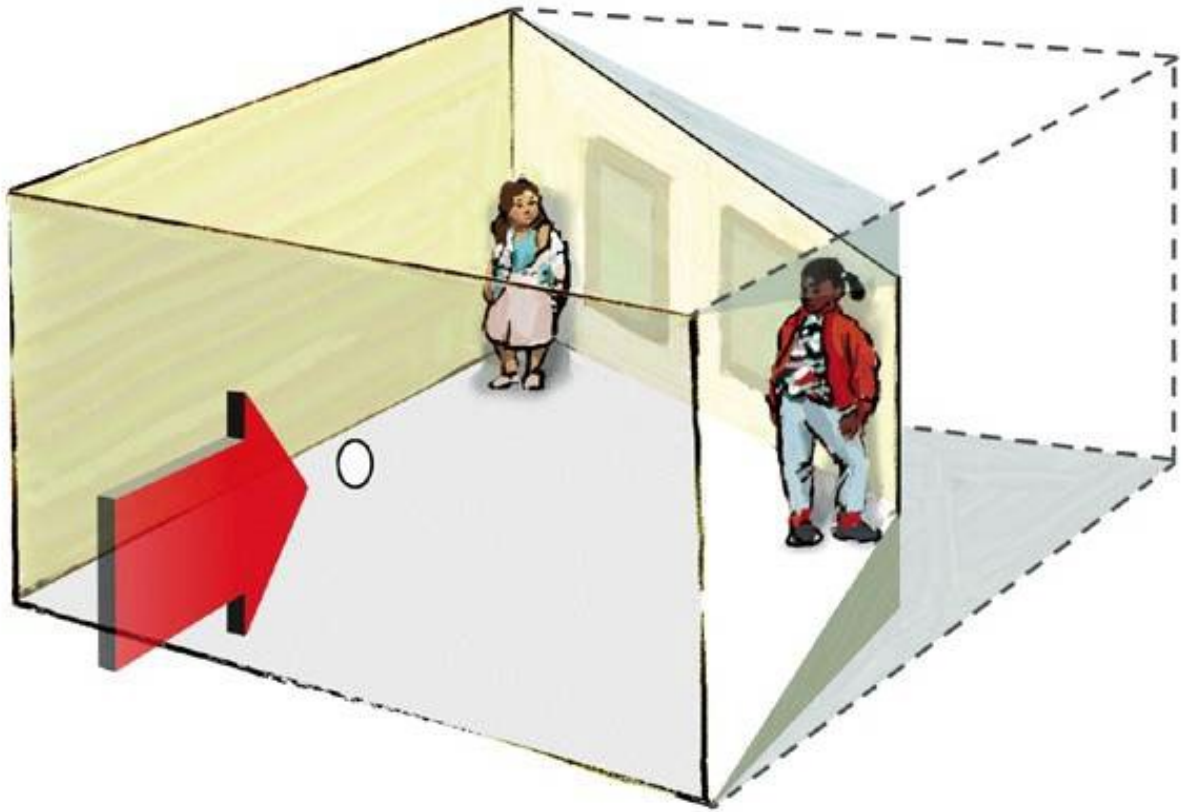

*Perspective distortion*: magnification changes with depth

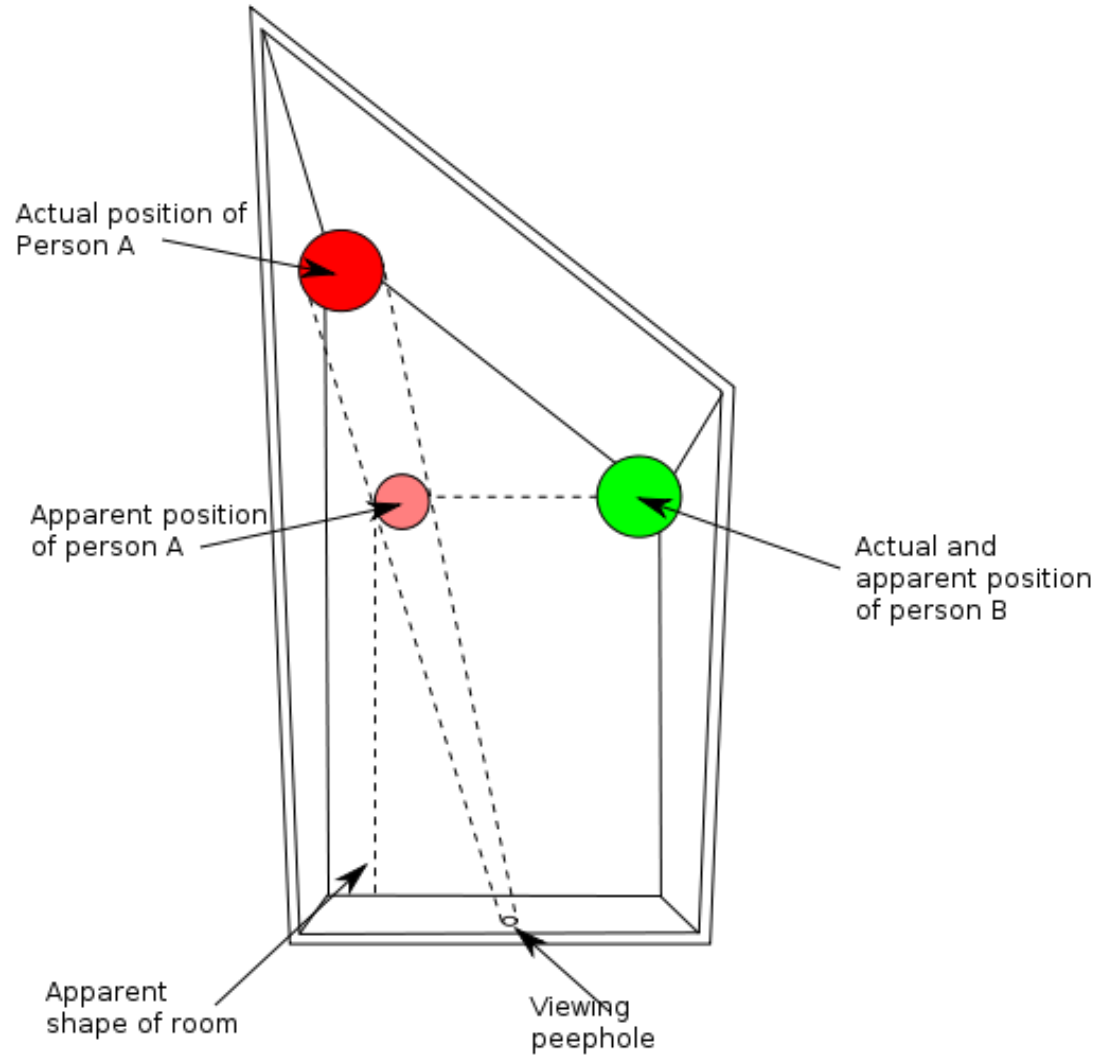
Using similar triangles: $\tilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\boldsymbol{x}} = \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix}$

# Forced perspective

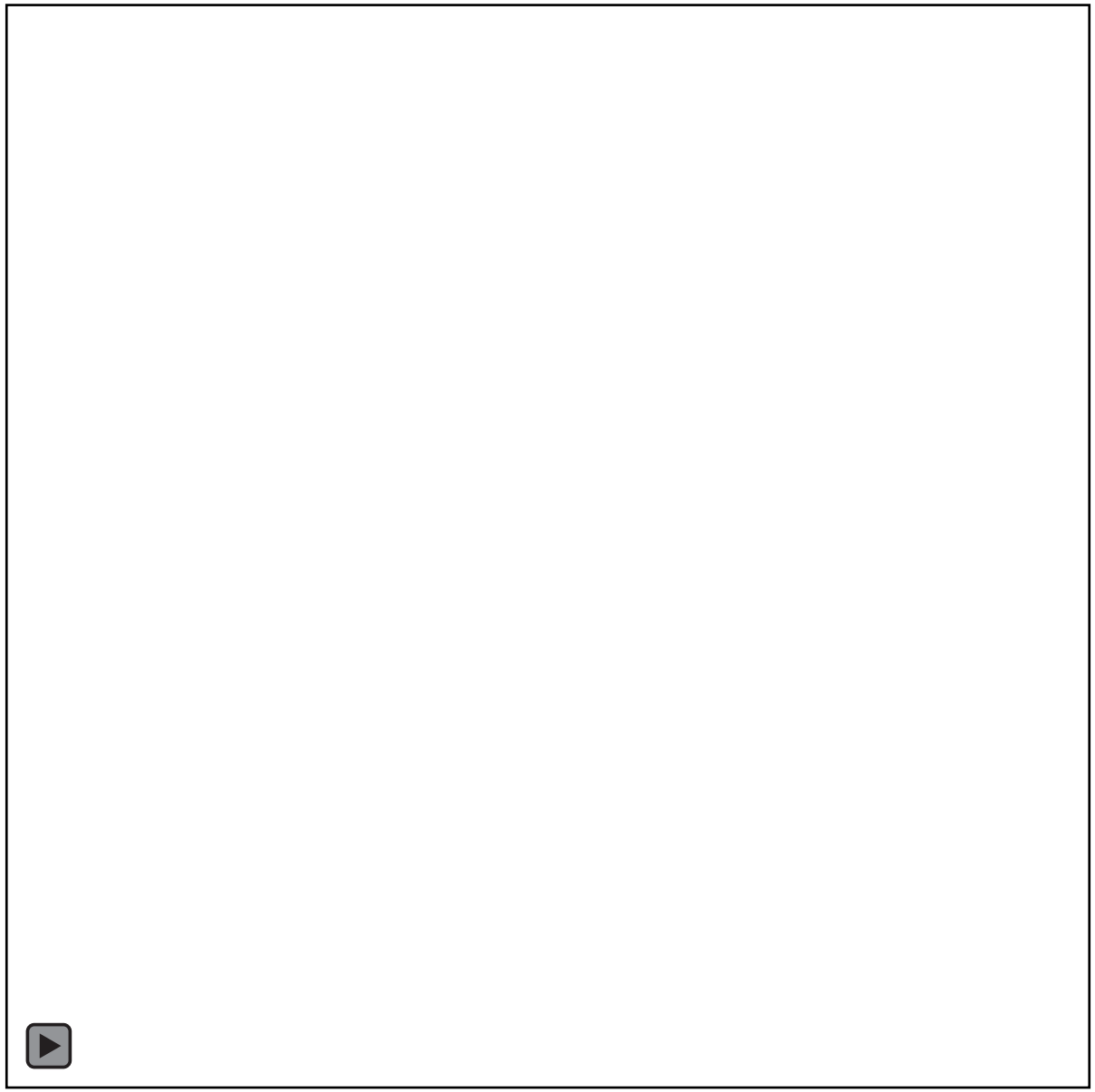
# The Ames room illusion

# The Ames room illusion





# The arrow illusion

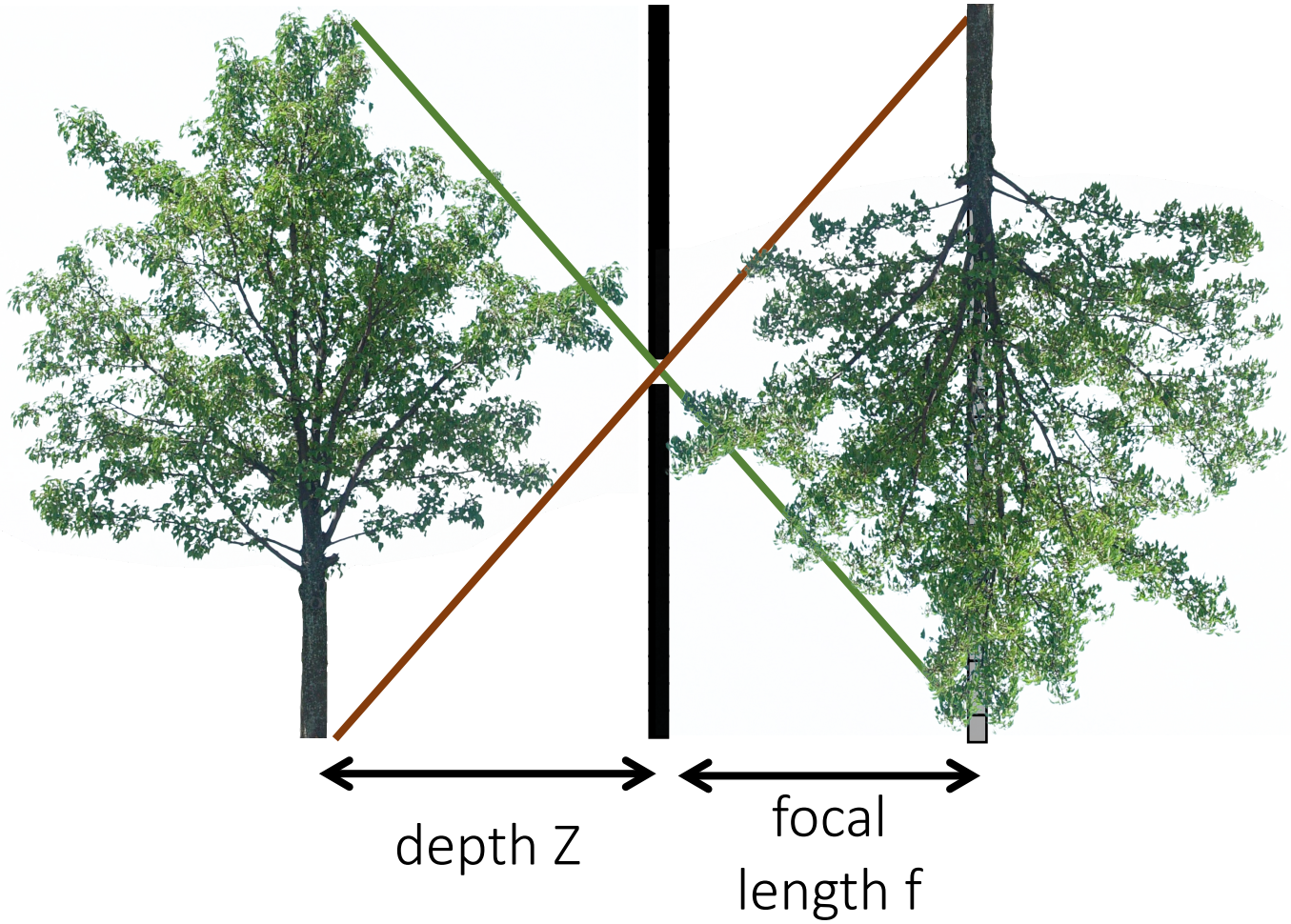


# Is there a camera without perspective distortion?

# Other camera models

# What if…

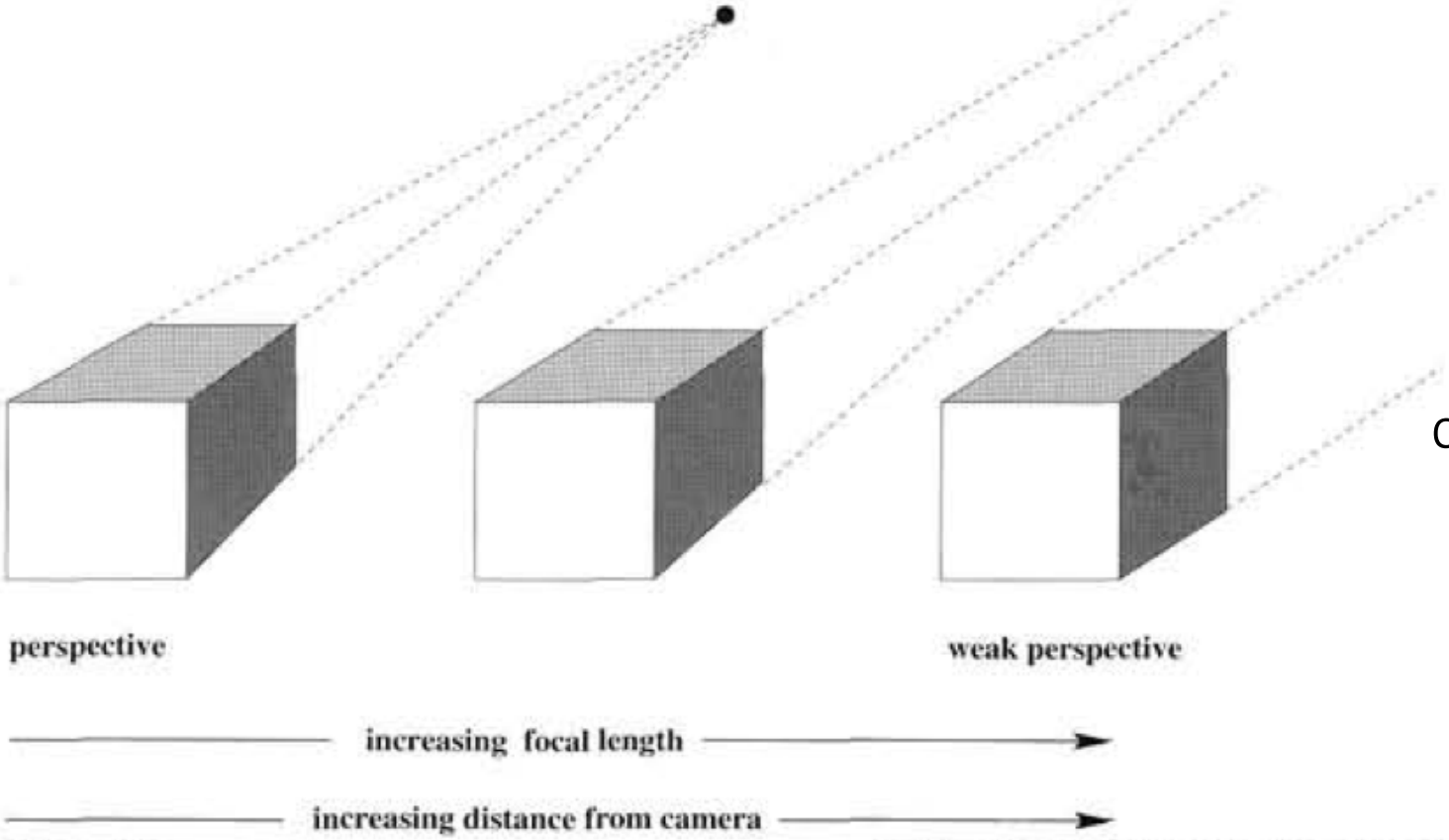
real-world object



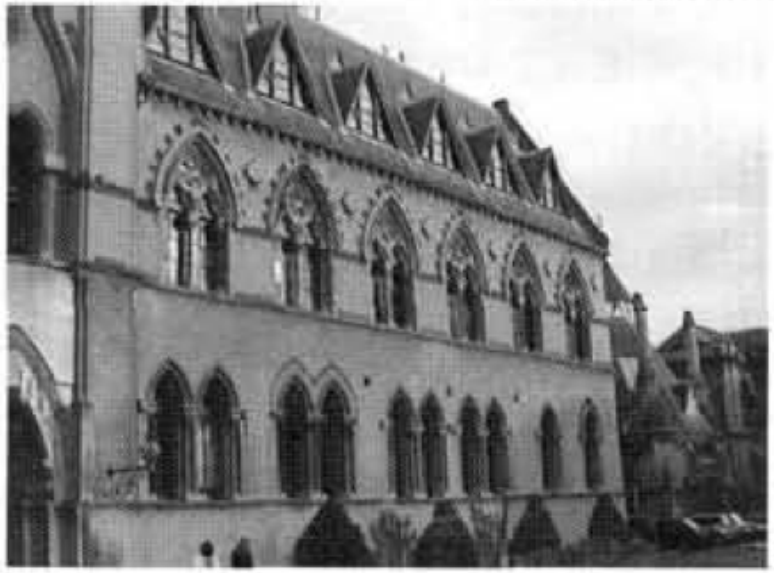

… we continue increasing Z and f while maintaining same magnification?

$$f \to \infty \text{ and } \frac{f}{Z} = \text{constant}$$

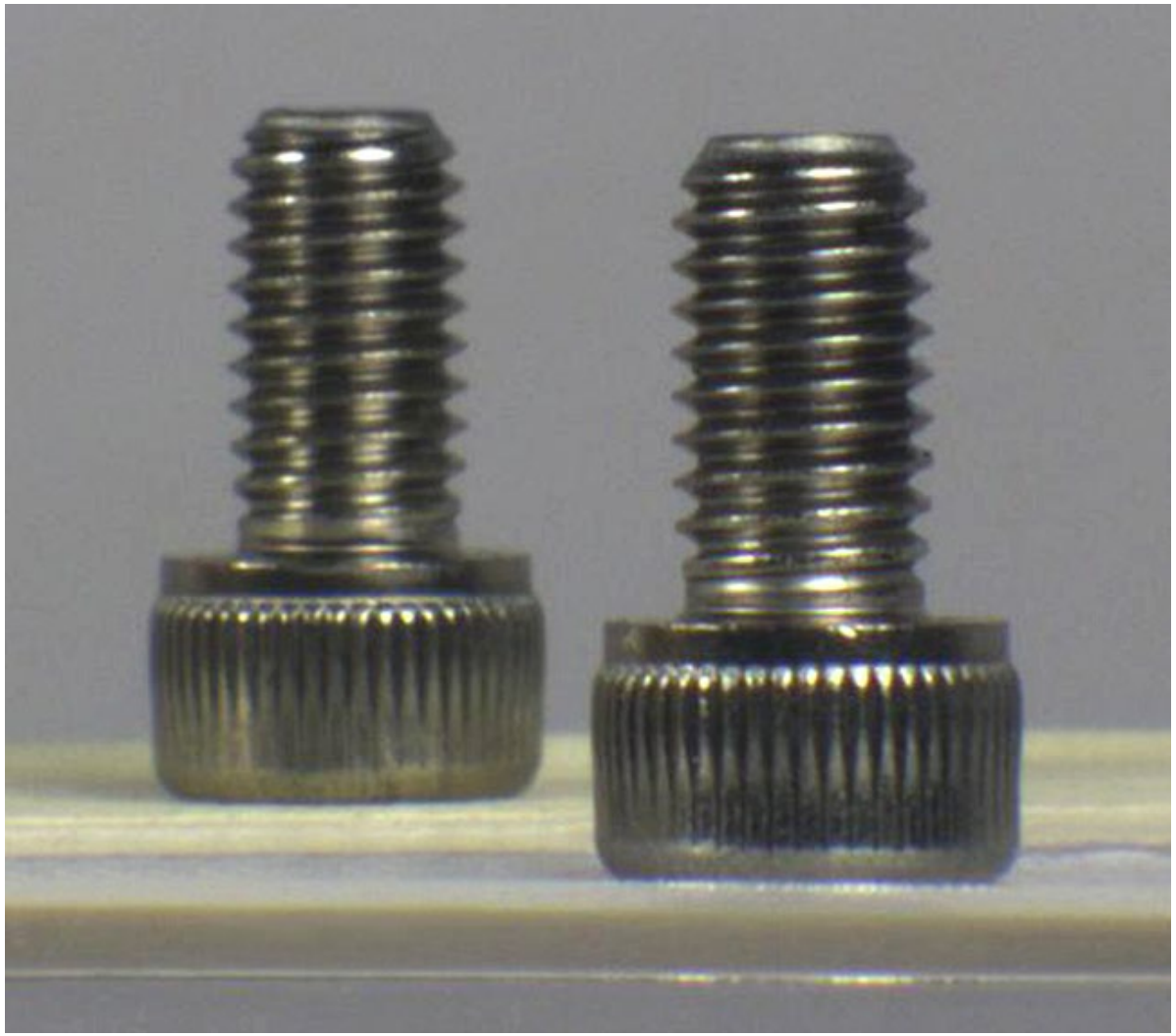
Perspective camera: camera is *close* to object and has *small* focal length



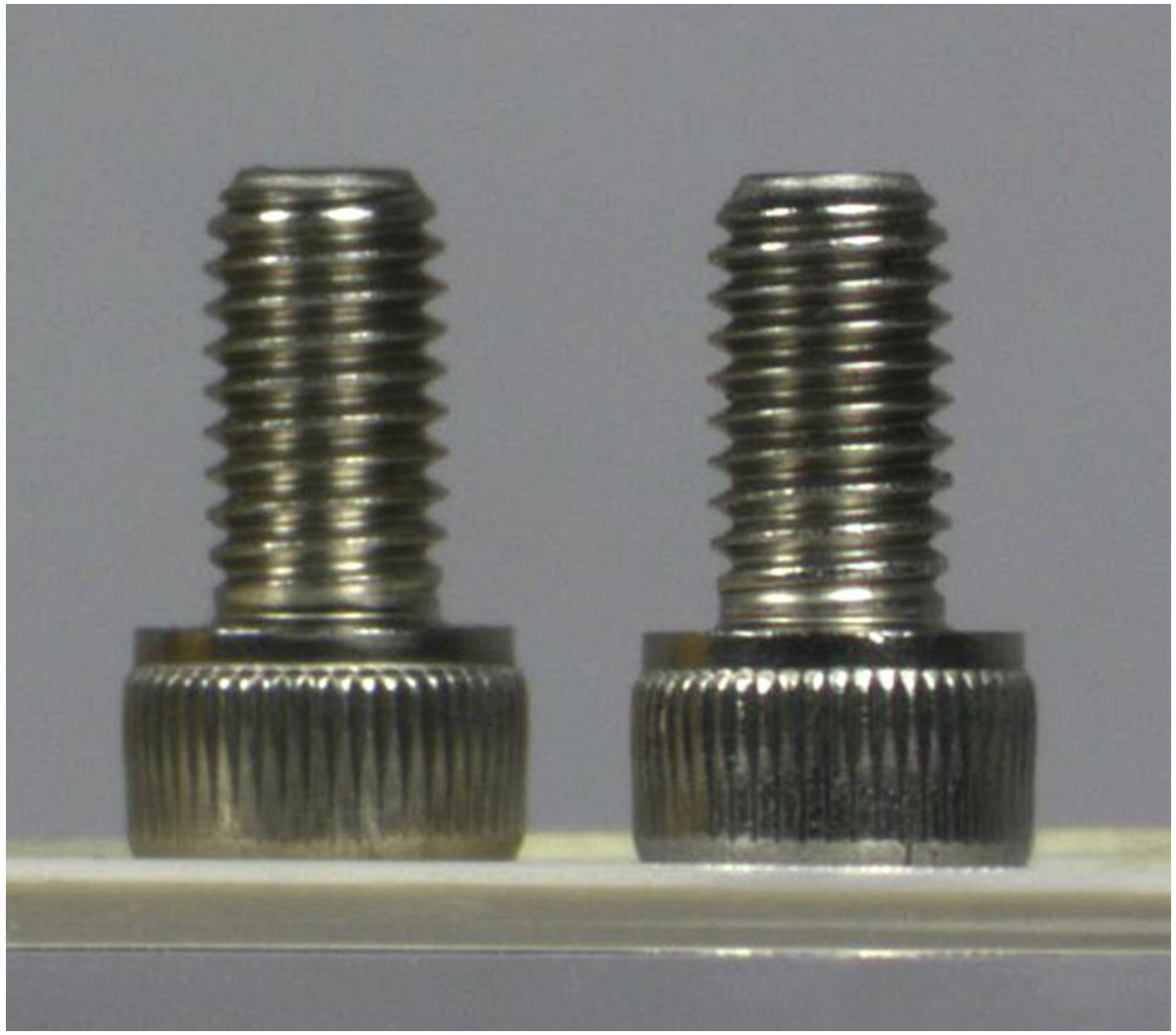

Weak-perspective camera: camera is *far* from object and has *large* focal length



# Different cameras

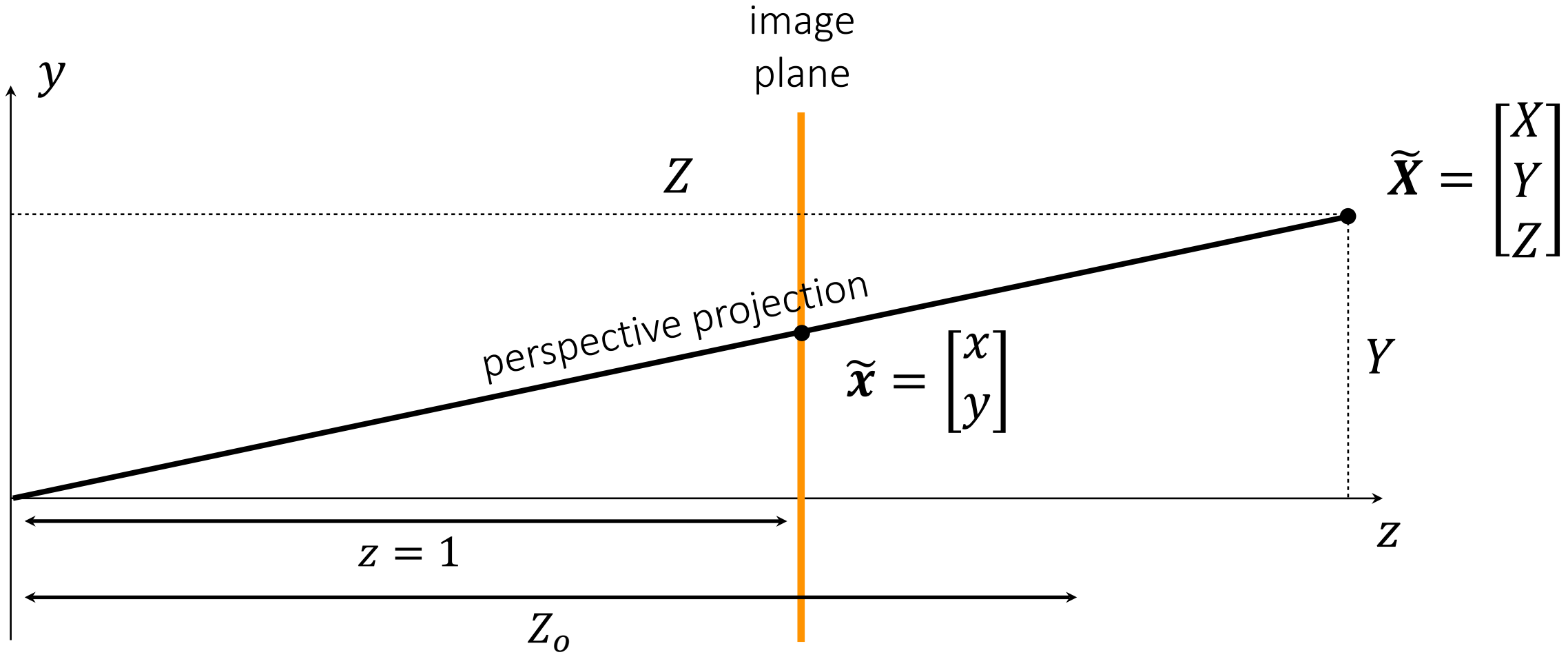


perspective camera



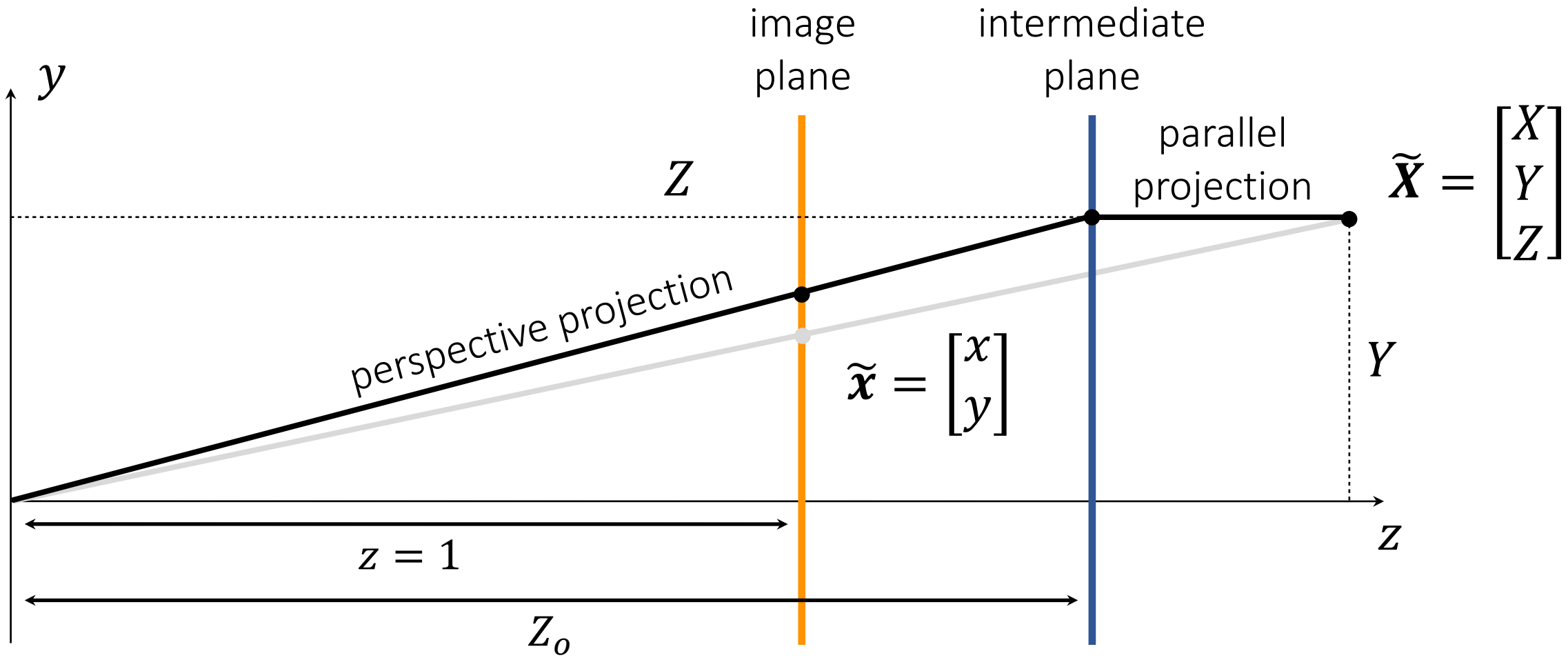
weak perspective camera

# Perspective versus weak-perspective camera




perspective projection $\tilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\boldsymbol{x}} = \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix}$

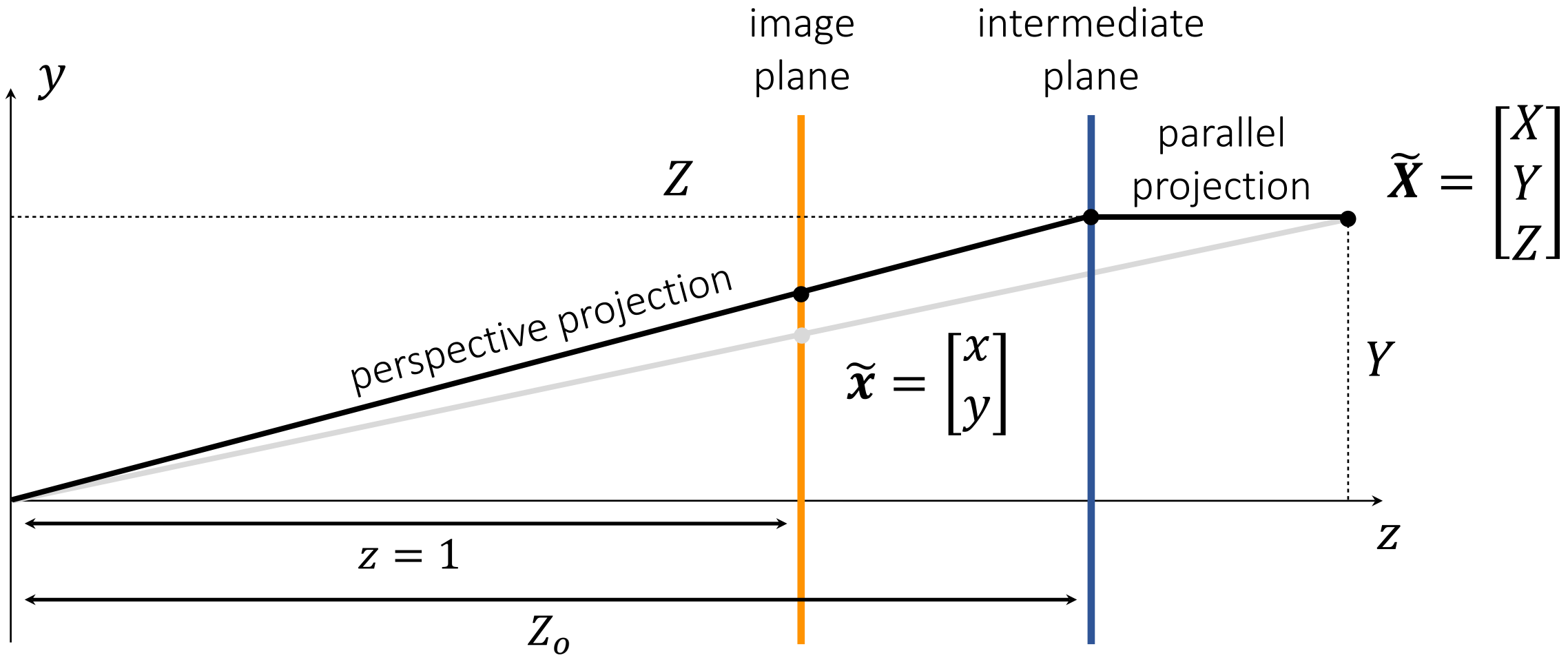
# Perspective versus weak-perspective camera




perspective projection $\tilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\boldsymbol{x}} = \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix}$

weak-perspective projection $\tilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\boldsymbol{x}} = ?$

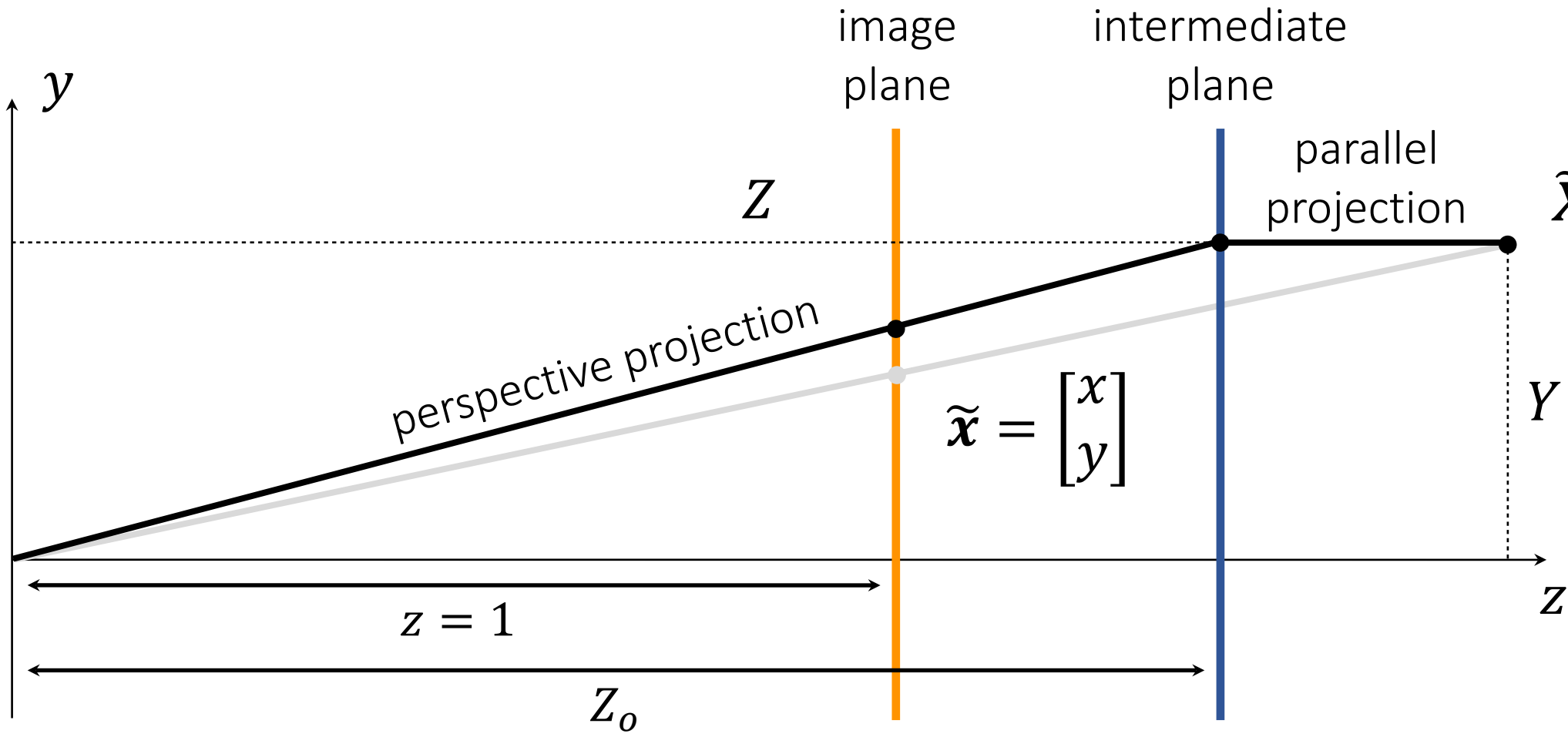
# Perspective versus weak-perspective camera




perspective projection $\tilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\boldsymbol{x}} = \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix}$

weak-perspective projection $\tilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\boldsymbol{x}} = \begin{bmatrix} X/Z_o \\ Y/Z_o \end{bmatrix}$

# Perspective versus weak-perspective camera




magnification *independent* of depth, depends only on $Z_o$

perspective projection $\tilde{\mathbf{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\mathbf{x}} = \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix}$

weak-perspective projection $\tilde{\mathbf{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\mathbf{x}} = \begin{bmatrix} X/Z_o \\ Y/Z_o \end{bmatrix}$

# Comparing camera projection matrices

Let's ignore intrinsics and extrinscis for now.

- The *perspective projection matrix* can be written as:

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- What would the *weak-perspective projection matrix* look like?

# Comparing camera projection matrices

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$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- The *weak-perspective projection matrix* can be written as:

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & Z_o \end{bmatrix}$$

# Comparing camera matrices

Let's now incorporate intrinsics and extrinsics.

- The *finite projective camera matrix* can be written as:

$$\boldsymbol{P} = \boldsymbol{K}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{R} & \boldsymbol{t}\\ \boldsymbol{0} & 1\end{bmatrix}$$

- What would the matrix of the so-called *affine camera* look like?

# Comparing camera matrices

Let's now incorporate intrinsics and extrinsics.

- The *finite projective camera matrix* can be written as:

$$\boldsymbol{P} = \boldsymbol{K}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{R} & \boldsymbol{t}\\ \boldsymbol{0} & 1\end{bmatrix}$$

Change only the projection matrix, and use the exact same matrices for intrinsics and extrinsics.

- The *affine camera matrix* can be written as:

$$\boldsymbol{P} = \boldsymbol{K}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & Z_o\end{bmatrix}\begin{bmatrix}\boldsymbol{R} & \boldsymbol{t}\\ \boldsymbol{0} & 1\end{bmatrix}$$

# Special case: orthographic projection

Let's now incorporate intrinsics and extrinsics.

- The *finite projective camera matrix* can be written as:

$$\boldsymbol{P} = \boldsymbol{K}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{R} & \boldsymbol{t}\\ \boldsymbol{0} & 1\end{bmatrix}$$

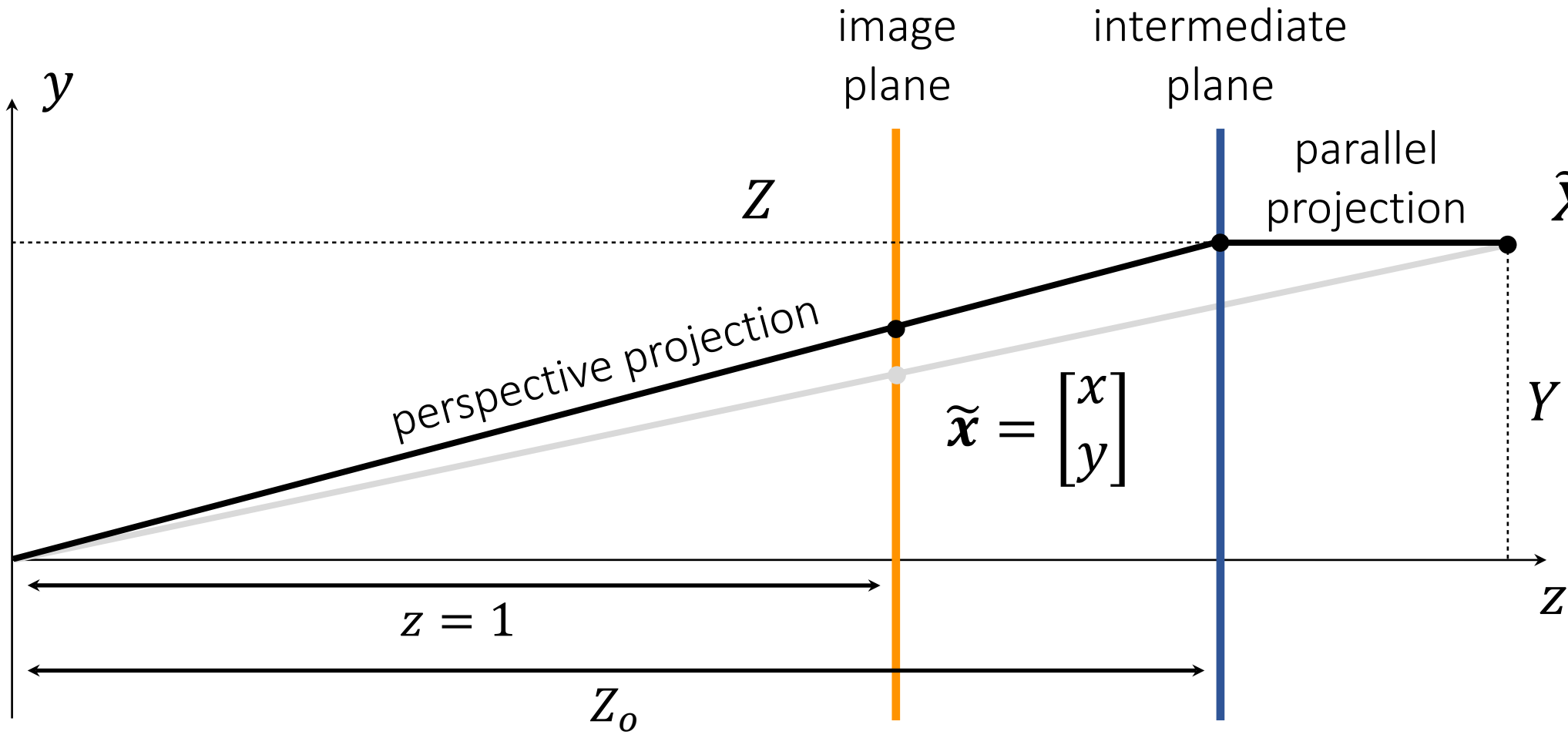
Change only the projection matrix, and use the exact same matrices for intrinsics and extrinsics.

- The *affine camera matrix* can be written as:

What's the effect of setting $Z_o = 1$?

$$\boldsymbol{P} = \boldsymbol{K}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{R} & \boldsymbol{t}\\ \boldsymbol{0} & 1\end{bmatrix}$$

# Perspective versus weak-perspective camera

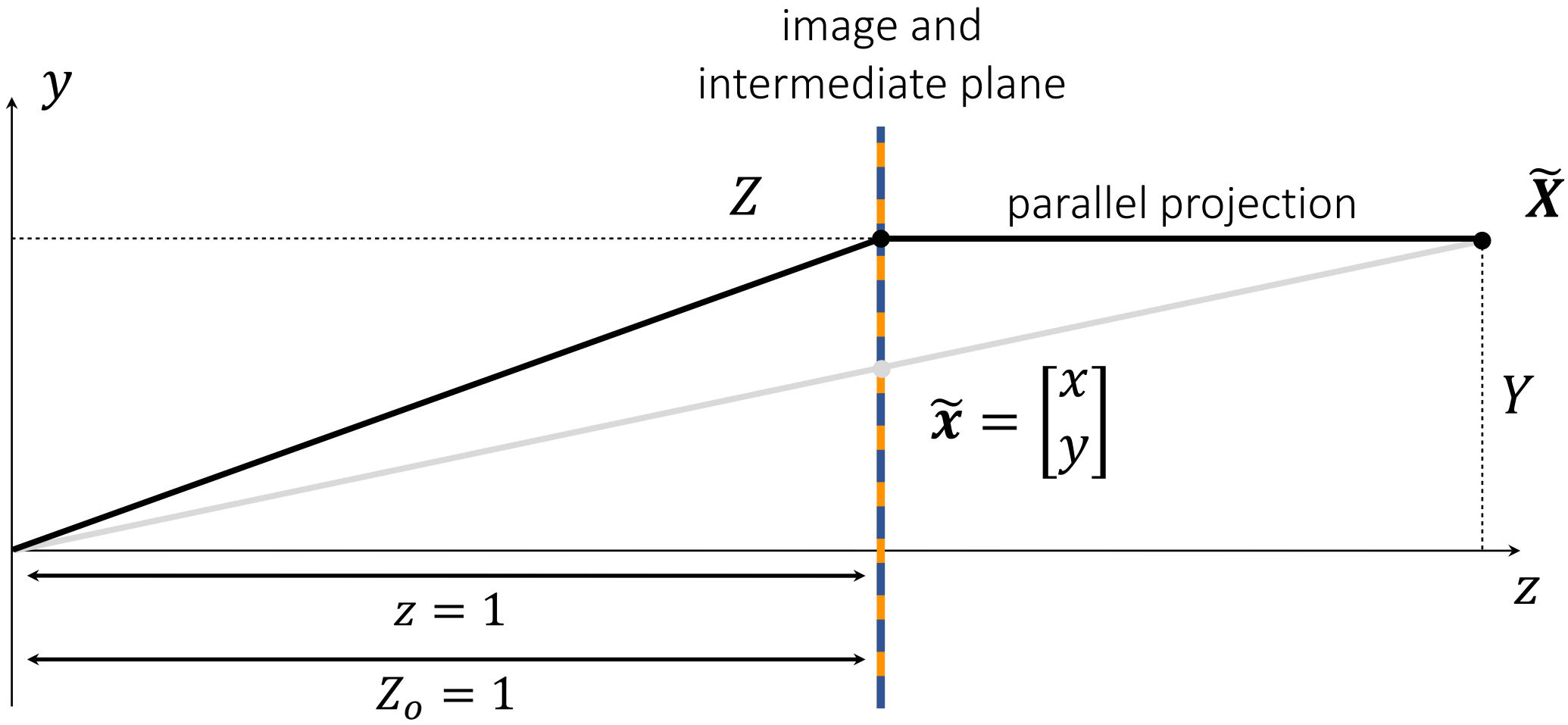



magnification *independent* of depth, depends only on $Z_o$

perspective projection $\tilde{\mathbf{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\mathbf{x}} = \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix}$

weak-perspective projection $\tilde{\mathbf{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\mathbf{x}} = \begin{bmatrix} X/Z_o \\ Y/Z_o \end{bmatrix}$

# Perspective versus orthographic camera




magnification equal to 1 (real-life size)

perspective projection $\tilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\boldsymbol{x}} = \begin{bmatrix} X/Z \\ Y/Z \end{bmatrix}$

weak-perspective projection $\tilde{\boldsymbol{X}} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow \tilde{\boldsymbol{x}} = \begin{bmatrix} X \\ Y \end{bmatrix}$

# When can we assume a weak-perspective camera?

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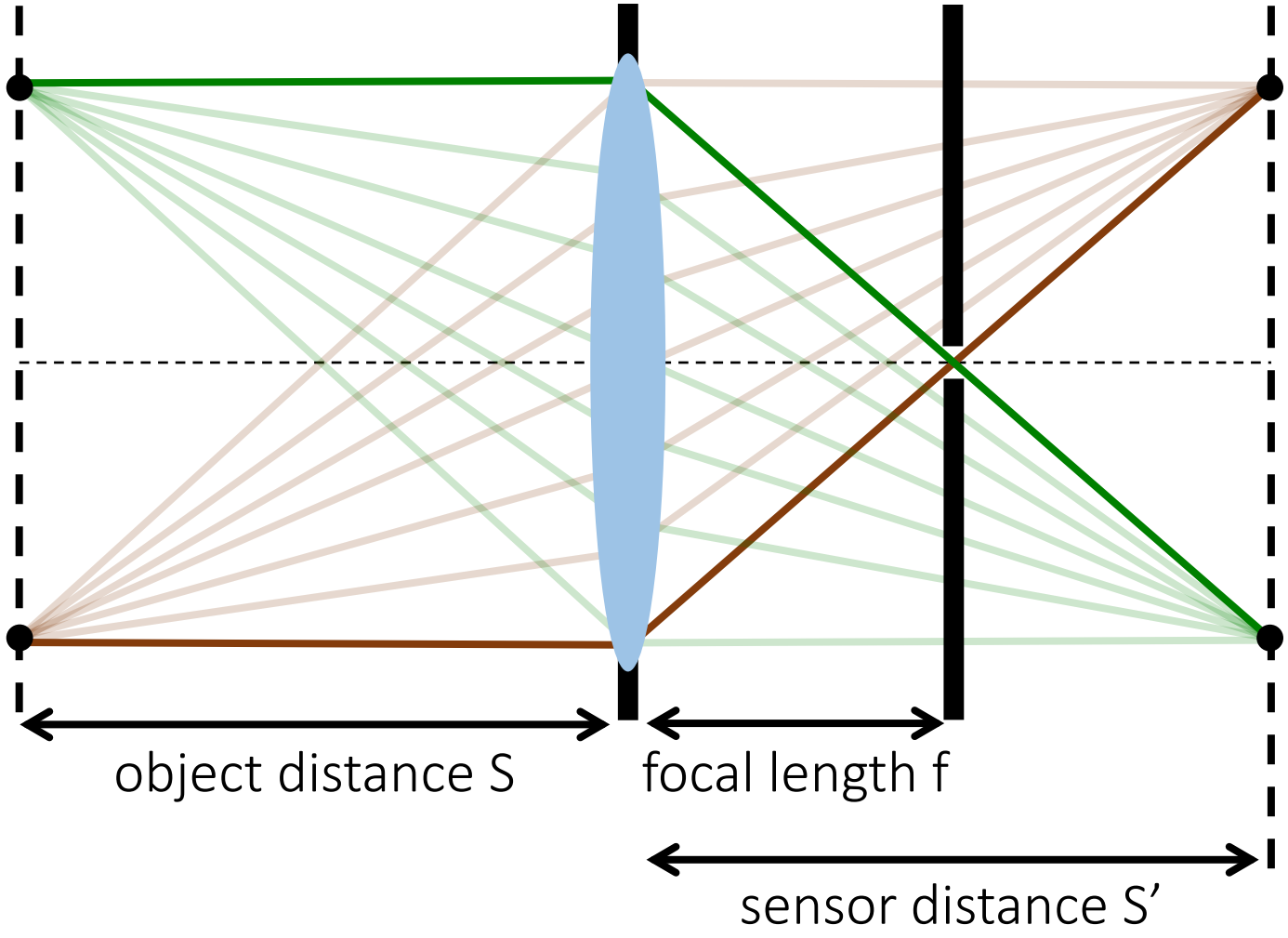
1. When the scene (or parts of it) is very far away.

Weak-perspective projection applies to the mountains.

# When can we assume a weak-perspective camera?

2. When we use a telecentric lens.

Place a pinhole at focal length, so that only rays parallel to primary ray pass through.




What does $Z_o$ equal in this case?

# When can we assume a weak-perspective camera?

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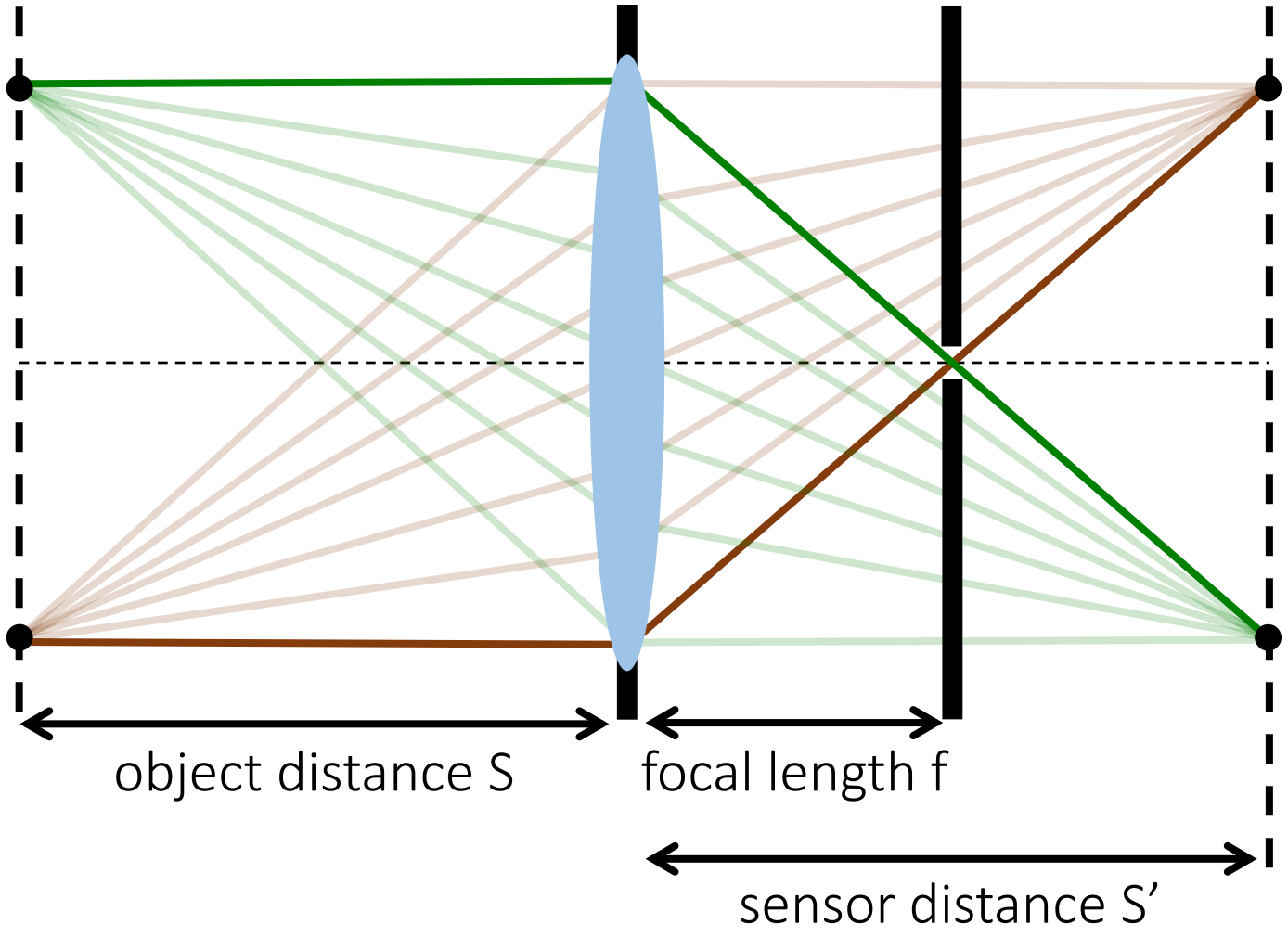
Place a pinhole at focal length, so that only rays parallel to primary ray pass through.

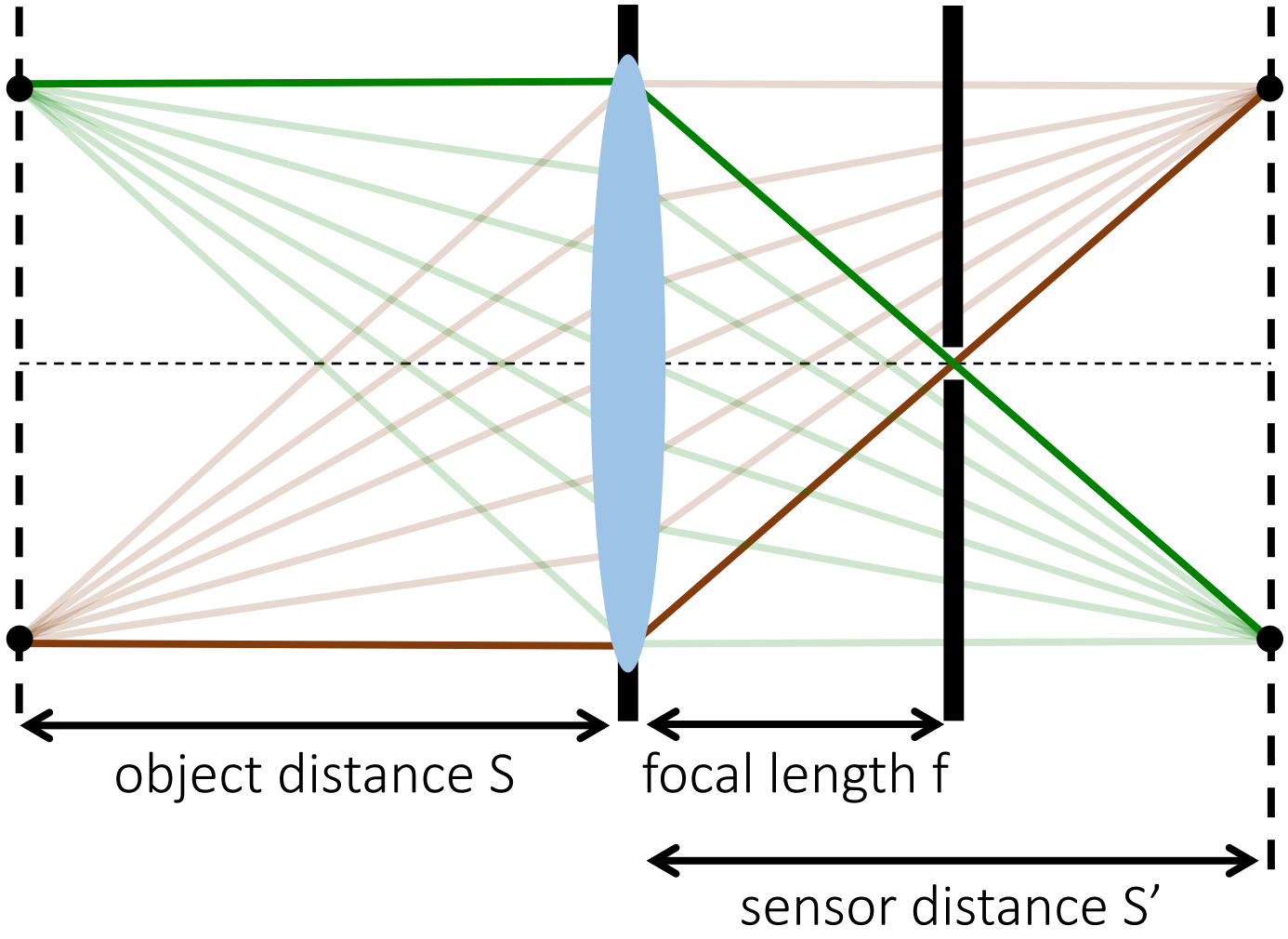



$$Z_o = f$$

remember that *focal length* f refers to different things in pinhole and lens cameras

# Orthographic projection using a telecentric lens

How do we make the telecentric lens act as an orthographic camera?



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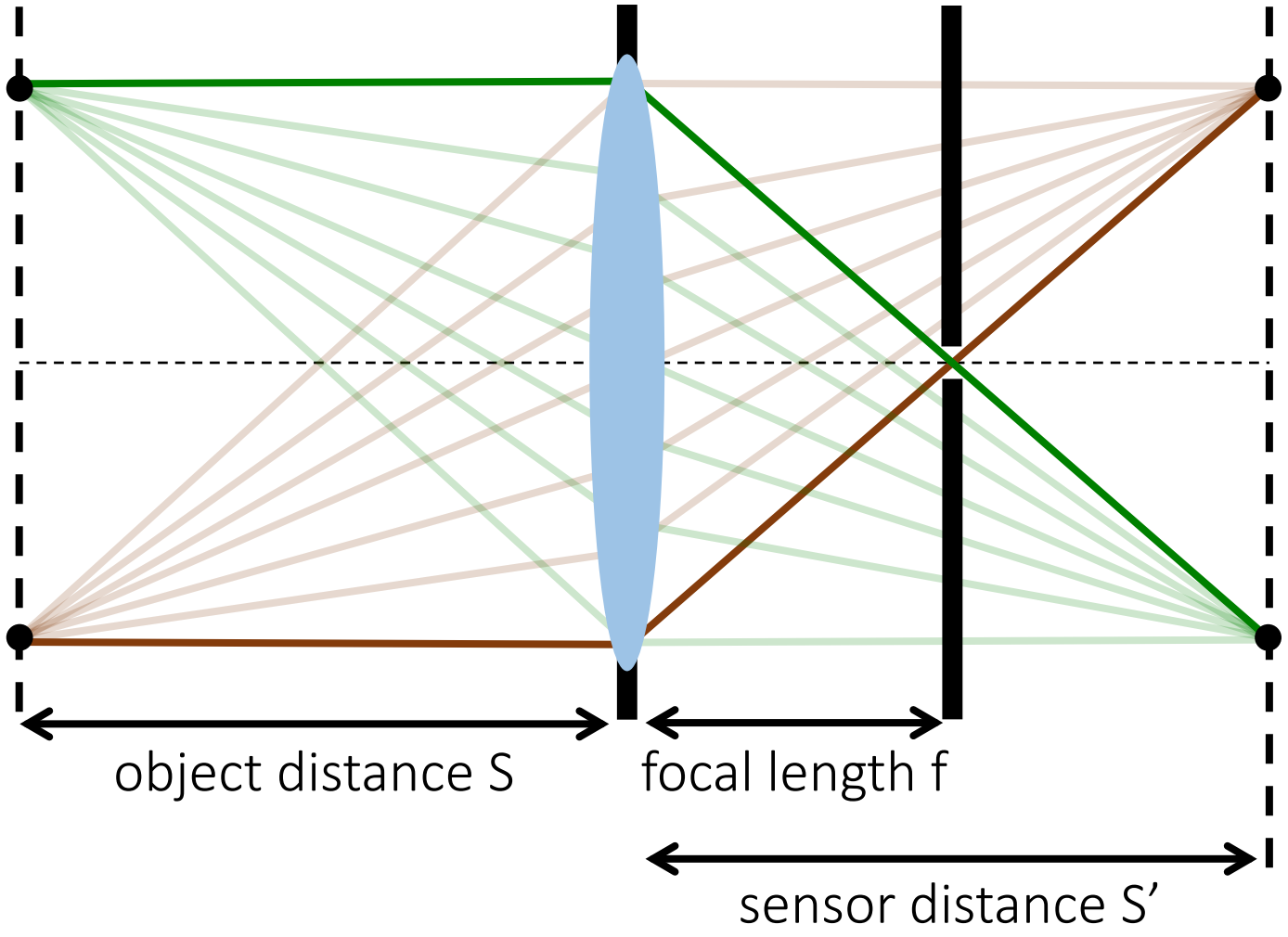
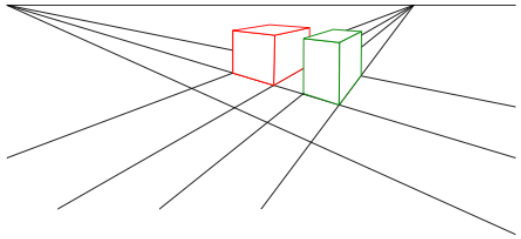



We set the sensor distance as:

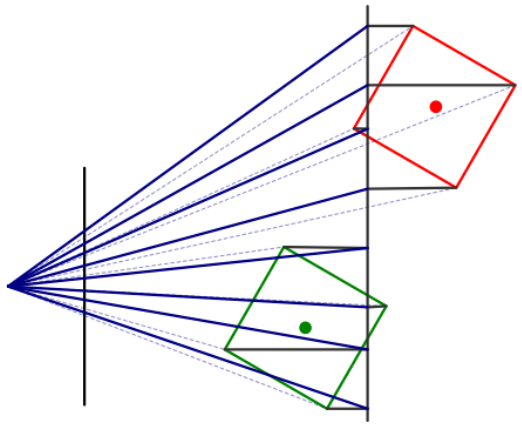
$$S' = 2f$$

in order to achieve unit magnification.

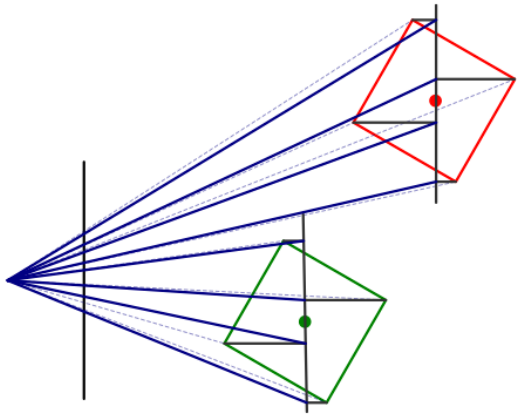
# Many other types of cameras



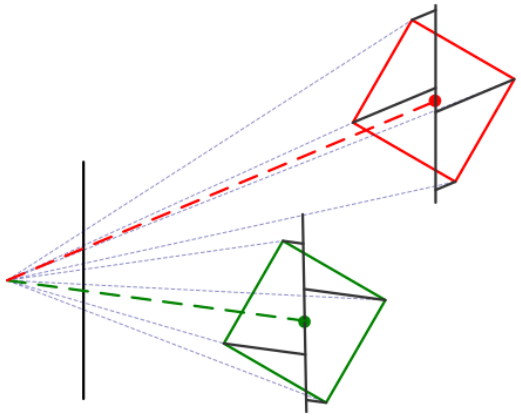
(a) 3D view



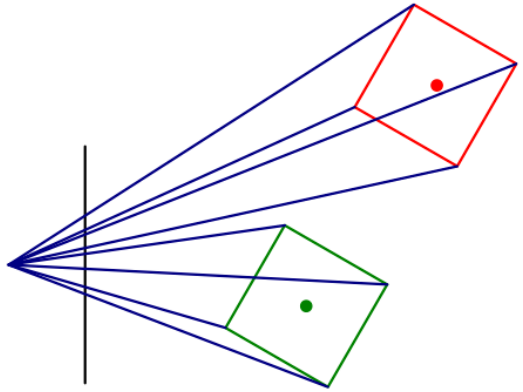
(b) orthography



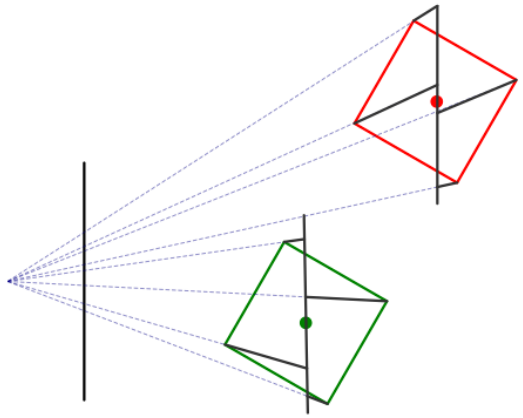
(c) scaled orthography



(d) para-perspective



(e) perspective



(f) object-centered

# Geometric camera calibration

# Geometric camera calibration

Given a set of matched points

$$\{\mathbf{X}_i, \boldsymbol{x}_i\}$$

point in 3D space — point in the image

and camera model

$$\boldsymbol{x} = \boldsymbol{f}(\mathbf{X}; \boldsymbol{p}) = \mathbf{P}\mathbf{X}$$

projection model — parameters — Camera matrix

Find the (pose) estimate of

$$\mathbf{P}$$

We'll use a **perspective** camera model for pose estimation

# Mapping between 3D point and image points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What are the unknowns?

# Mapping between 3D point and image points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} — & \boldsymbol{p}_1^\top & — \\ — & \boldsymbol{p}_2^\top & — \\ — & \boldsymbol{p}_3^\top & — \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{X} \\ | \end{bmatrix}$$

## Heterogeneous coordinates

$$x' = \frac{\boldsymbol{p}_1^\top \boldsymbol{X}}{\boldsymbol{p}_3^\top \boldsymbol{X}} \qquad y' = \frac{\boldsymbol{p}_2^\top \boldsymbol{X}}{\boldsymbol{p}_3^\top \boldsymbol{X}}$$

(non-linear relation between coordinates)

*How can we make these relations linear?*

*How can we make these relations linear?*

$$x' = \frac{\boldsymbol{p}_1^\top \boldsymbol{X}}{\boldsymbol{p}_3^\top \boldsymbol{X}} \qquad y' = \frac{\boldsymbol{p}_2^\top \boldsymbol{X}}{\boldsymbol{p}_3^\top \boldsymbol{X}}$$

Make them linear with algebraic manipulation…

$$\boldsymbol{p}_2^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} y' = 0$$

$$\boldsymbol{p}_1^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} x' = 0$$

Now we can setup a system of linear equations with multiple point correspondences

$$\boldsymbol{p}_2^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} y' = 0$$

$$\boldsymbol{p}_1^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} x' = 0$$

*How do we proceed?*

$$\boldsymbol{p}_2^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} y' = 0$$

$$\boldsymbol{p}_1^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} x' = 0$$

In matrix form …

$$\begin{bmatrix} \boldsymbol{X}^\top & \boldsymbol{0} & -x'\boldsymbol{X}^\top \\ \boldsymbol{0} & \boldsymbol{X}^\top & -y'\boldsymbol{X}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix} = \boldsymbol{0}$$

*How do we proceed?*

$$\boldsymbol{p}_2^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} y' = 0$$

$$\boldsymbol{p}_1^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} x' = 0$$

In matrix form …

$$\begin{bmatrix} \boldsymbol{X}^\top & \boldsymbol{0} & -x'\boldsymbol{X}^\top \\ \boldsymbol{0} & \boldsymbol{X}^\top & -y'\boldsymbol{X}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix} = \boldsymbol{0}$$

For N points …

$$\begin{bmatrix} \boldsymbol{X}_1^\top & \boldsymbol{0} & -x'\boldsymbol{X}_1^\top \\ \boldsymbol{0} & \boldsymbol{X}_1^\top & -y'\boldsymbol{X}_1^\top \\ \vdots & \vdots & \vdots \\ \boldsymbol{X}_N^\top & \boldsymbol{0} & -x'\boldsymbol{X}_N^\top \\ \boldsymbol{0} & \boldsymbol{X}_N^\top & -y'\boldsymbol{X}_N^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix} = \boldsymbol{0}$$

*How do we solve this system?*

Solve for camera matrix by

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \|\mathbf{A}\boldsymbol{x}\|^2 \text{ subject to } \|\boldsymbol{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{X}_1^\top & \mathbf{0} & -x'\boldsymbol{X}_1^\top \\ \mathbf{0} & \boldsymbol{X}_1^\top & -y'\boldsymbol{X}_1^\top \\ \vdots & \vdots & \vdots \\ \boldsymbol{X}_N^\top & \mathbf{0} & -x'\boldsymbol{X}_N^\top \\ \mathbf{0} & \boldsymbol{X}_N^\top & -y'\boldsymbol{X}_N^\top \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}$$

**SVD!**

# Solve for camera matrix by

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \|\mathbf{A}\boldsymbol{x}\|^2 \text{ subject to } \|\boldsymbol{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{X}_1^\top & \mathbf{0} & -x'\boldsymbol{X}_1^\top \\ \mathbf{0} & \boldsymbol{X}_1^\top & -y'\boldsymbol{X}_1^\top \\ \vdots & \vdots & \vdots \\ \boldsymbol{X}_N^\top & \mathbf{0} & -x'\boldsymbol{X}_N^\top \\ \mathbf{0} & \boldsymbol{X}_N^\top & -y'\boldsymbol{X}_N^\top \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}$$

Solution **x** is the column of **V** corresponding to smallest singular value of

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$$

Solve for camera matrix by

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \|\mathbf{A}\boldsymbol{x}\|^2 \text{ subject to } \|\boldsymbol{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{X}_1^\top & \mathbf{0} & -x'\boldsymbol{X}_1^\top \\ \mathbf{0} & \boldsymbol{X}_1^\top & -y'\boldsymbol{X}_1^\top \\ \vdots & \vdots & \vdots \\ \boldsymbol{X}_N^\top & \mathbf{0} & -x'\boldsymbol{X}_N^\top \\ \mathbf{0} & \boldsymbol{X}_N^\top & -y'\boldsymbol{X}_N^\top \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}$$

Equivalently, solution ***x*** is the Eigenvector corresponding to smallest Eigenvalue of

$$\mathbf{A}^\top\mathbf{A}$$

Now we have:

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix}$$

*Are we done?*

Almost there …

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix}$$

*How do you get the intrinsic and extrinsic parameters from the projection matrix?*

# Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

# Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

# Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}|-\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M}|-\mathbf{M}\mathbf{c}] \end{aligned}$$

# Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}|-\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M}|-\mathbf{M}\mathbf{c}] \end{aligned}$$

Find the camera center **C**

*What is the projection of the camera center?*

Find intrinsic **K** and rotation **R**

# Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}|-\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M}|-\mathbf{M}\mathbf{c}] \end{aligned}$$

Find the camera center **C**

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

*How do we compute the camera center from this?*

Find intrinsic **K** and rotation **R**

# Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}|-\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M}|-\mathbf{M}\mathbf{c}] \end{aligned}$$

Find the camera center **C**

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

**SVD** of P!

***c** is the Eigenvector corresponding to smallest Eigenvalue*

Find intrinsic **K** and rotation **R**

# Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}| - \mathbf{R}\mathbf{c}] \\ &= [\mathbf{M}| - \mathbf{M}\mathbf{c}] \end{aligned}$$

Find the camera center **C**

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

SVD of P!

***c** is the Eigenvector corresponding to smallest Eigenvalue*

Find intrinsic **K** and rotation **R**

$$\mathbf{M} = \mathbf{K}\mathbf{R}$$

*Any useful properties of K and R we can use?*

# Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}|-\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M}|-\mathbf{M}\mathbf{c}] \end{aligned}$$

Find the camera center **C**

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

SVD of P!

***c** is the Eigenvector corresponding to smallest Eigenvalue*

Find intrinsic **K** and rotation **R**

$$\mathbf{M} = \mathbf{K}\mathbf{R}$$

right upper triangle

orthogonal

*How do we find K and R?*

# Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}|-\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M}|-\mathbf{M}\mathbf{c}] \end{aligned}$$

Find the camera center **C**

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

SVD of P!

***c** is the Eigenvector corresponding to smallest Eigenvalue*

Find intrinsic **K** and rotation **R**

$$\mathbf{M} = \mathbf{K}\mathbf{R}$$

QR decomposition

# Geometric camera calibration

Given a set of matched points

$$\{\mathbf{X}_i, \boldsymbol{x}_i\}$$

point in 3D space — point in the image

*Where do we get these matched points from?*

and camera model

$$\boldsymbol{x} = \boldsymbol{f}(\mathbf{X}; \boldsymbol{p}) = \mathbf{P}\mathbf{X}$$

projection model — parameters — Camera matrix

Find the (pose) estimate of

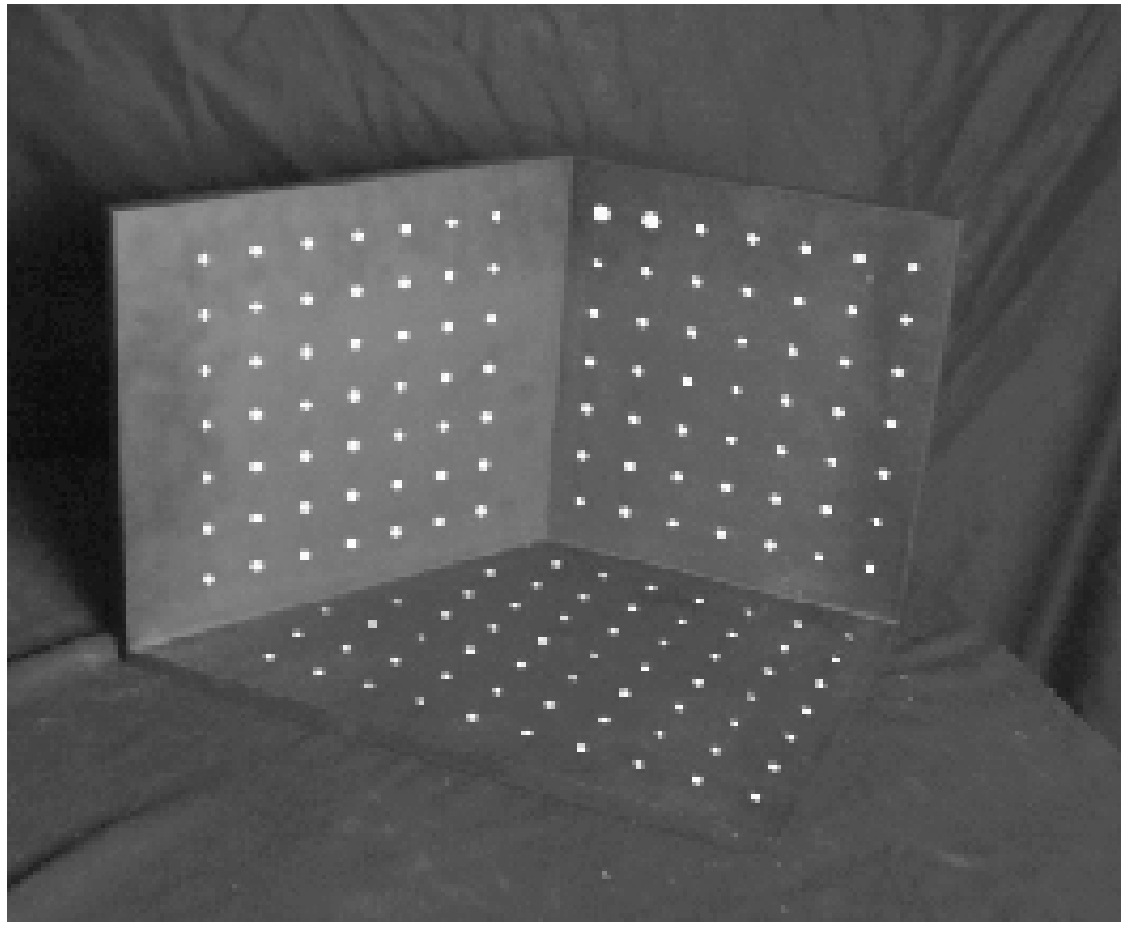
$$\mathbf{P}$$

We'll use a **perspective** camera model for pose estimation

# Calibration using a reference object

Place a known object in the scene:

- identify correspondences between image and scene
- compute mapping from scene to image



Issues:

- must know geometry very accurately
- must know 3D->2D correspondence

# Geometric camera calibration

Advantages:

- Very simple to formulate.
- Analytical solution.

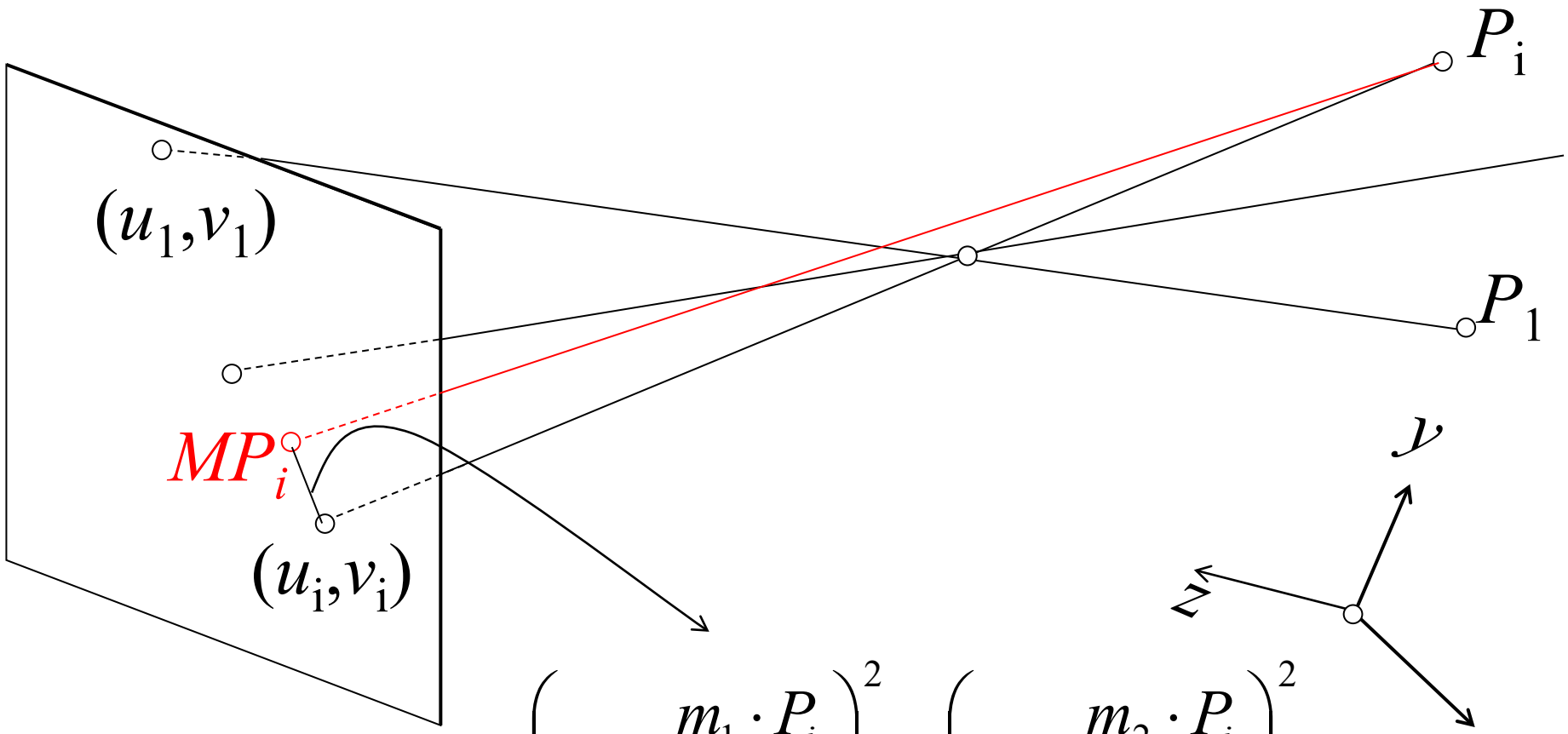
Disadvantages:

- Doesn't model radial distortion.
- Hard to impose constraints (e.g., known *f*).
- Doesn't minimize the correct error function.

For these reasons, *nonlinear methods* are preferred

- Define error function E between projected 3D points and image positions
  - E is nonlinear function of intrinsics, extrinsics, radial distortion
- Minimize E using nonlinear optimization techniques

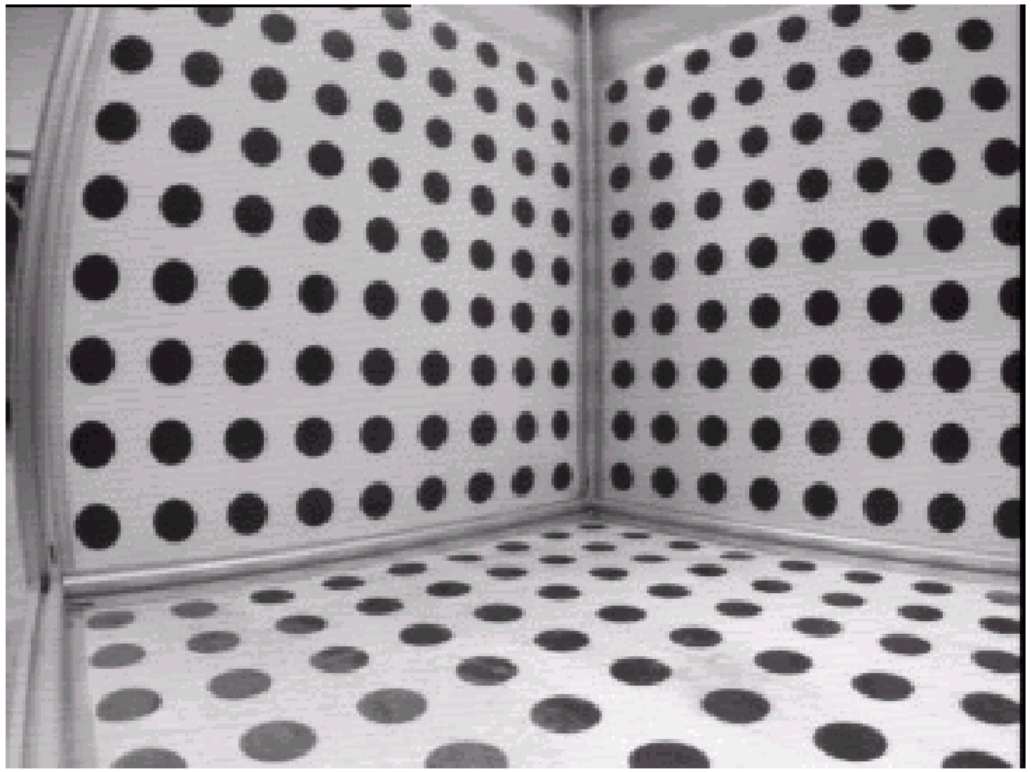
# Minimizing reprojection error



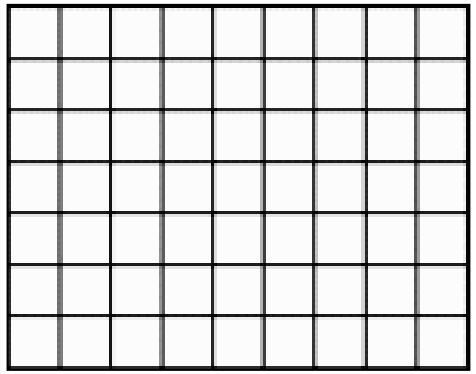

$$\left(u_i - \frac{m_1 \cdot P_i}{m_3 \cdot P_i}\right)^2 + \left(v_i - \frac{m_2 \cdot P_i}{m_3 \cdot P_i}\right)^2$$

Is this equivalent to what we were doing previously?

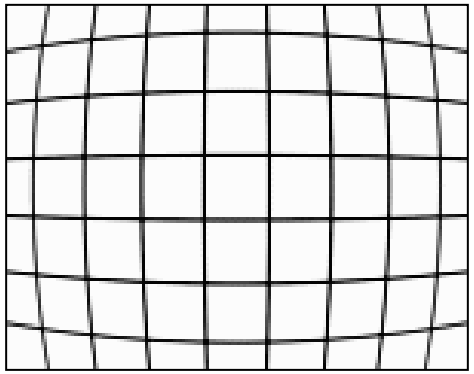
# Radial distortion



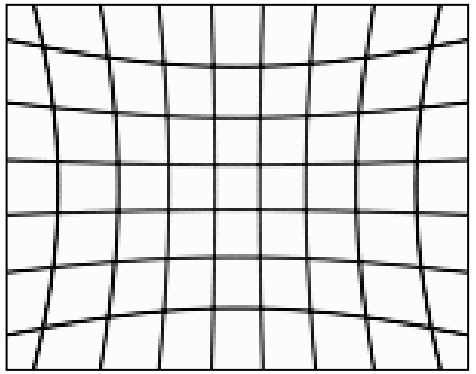
What causes this distortion?



no distortion

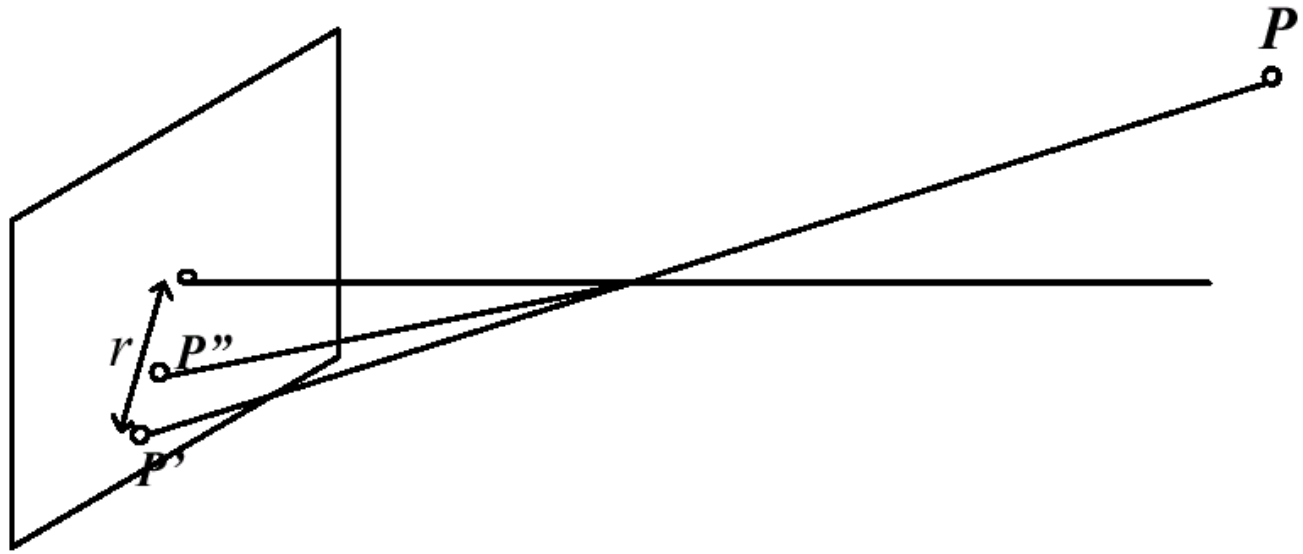


barrel distortion



pincushion distortion

# Radial distortion model




Ideal:

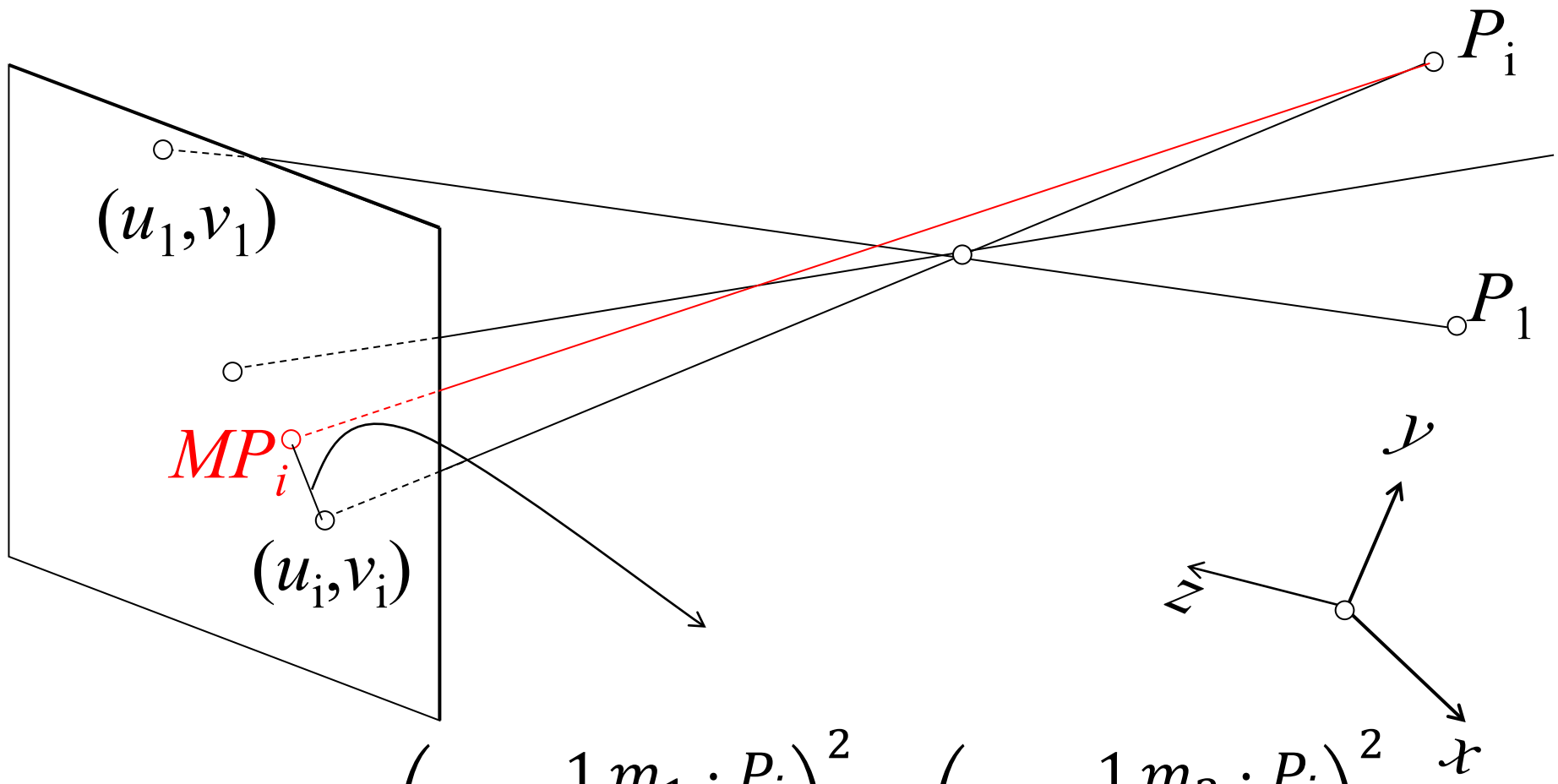
$$x' = f\frac{x}{z} \qquad y' = f\frac{y}{z}$$

Distorted:

$$x'' = \frac{1}{\lambda}x' \qquad y'' = \frac{1}{\lambda}y'$$

$$\lambda = 1 + k_1 r^2 + k_2 r^4 + \cdots$$

# Minimizing reprojection error with radial distortion




Add distortions to reprojection error:

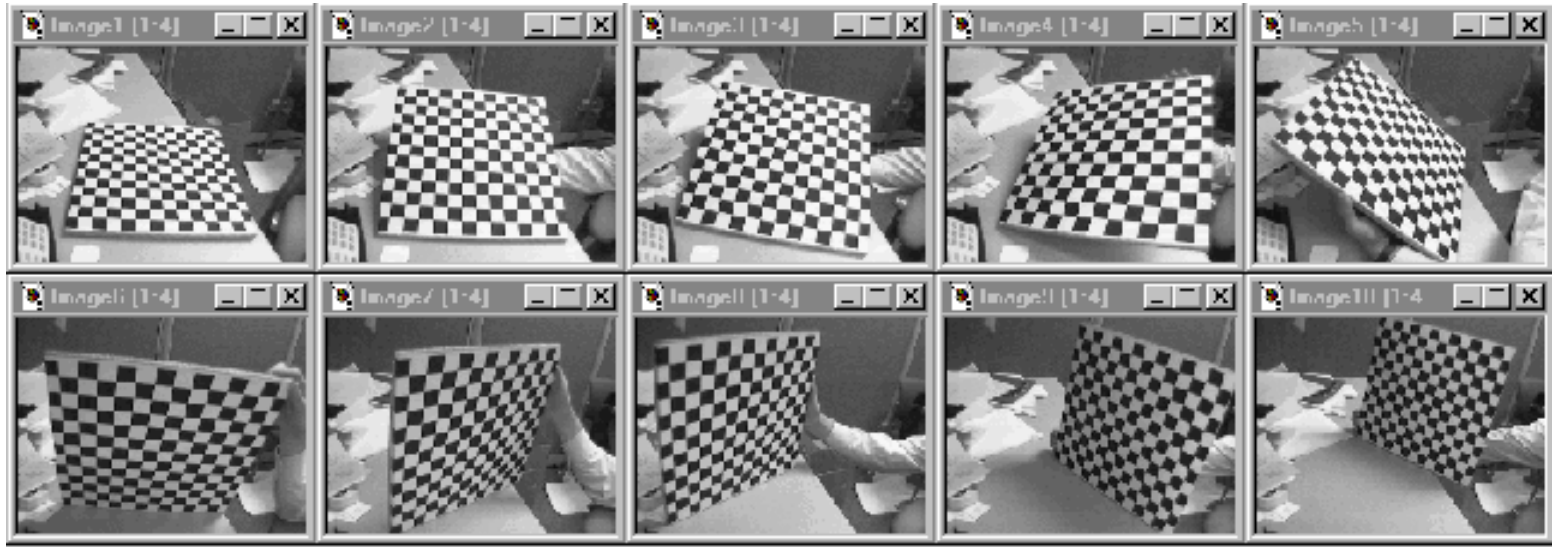
$$\left(u_i - \frac{1}{\lambda}\frac{m_1 \cdot P_i}{m_3 \cdot P_i}\right)^2 + \left(v_i - \frac{1}{\lambda}\frac{m_2 \cdot P_i}{m_3 \cdot P_i}\right)^2$$

# Correcting radial distortion

before

after

# Alternative: Multi-plane calibration

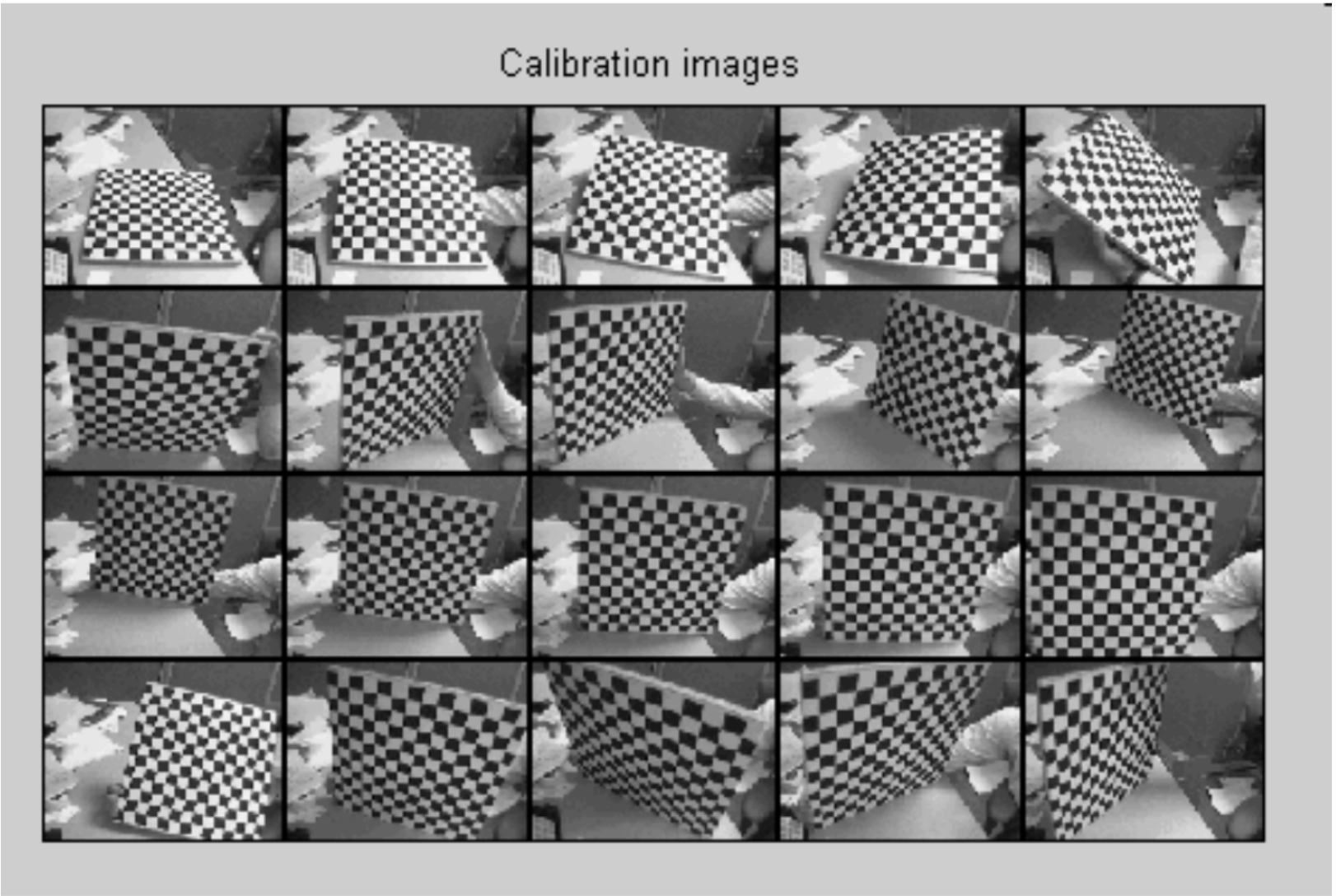


Advantages:

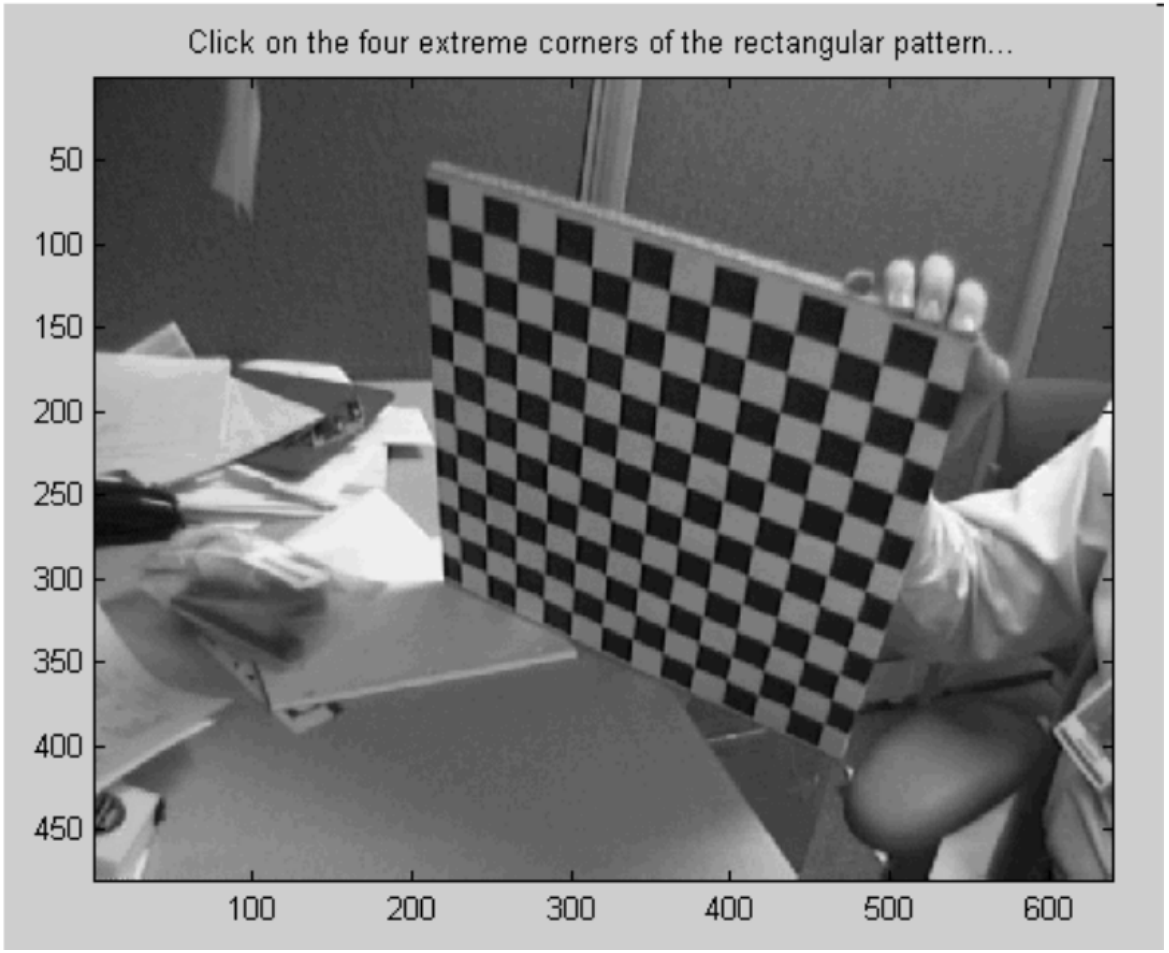
- Only requires a plane
- Don’t have to know positions/orientations
- Great code available online!
  - Matlab version: http://www.vision.caltech.edu/bouguetj/calib_doc/index.html
  - Also available on OpenCV.

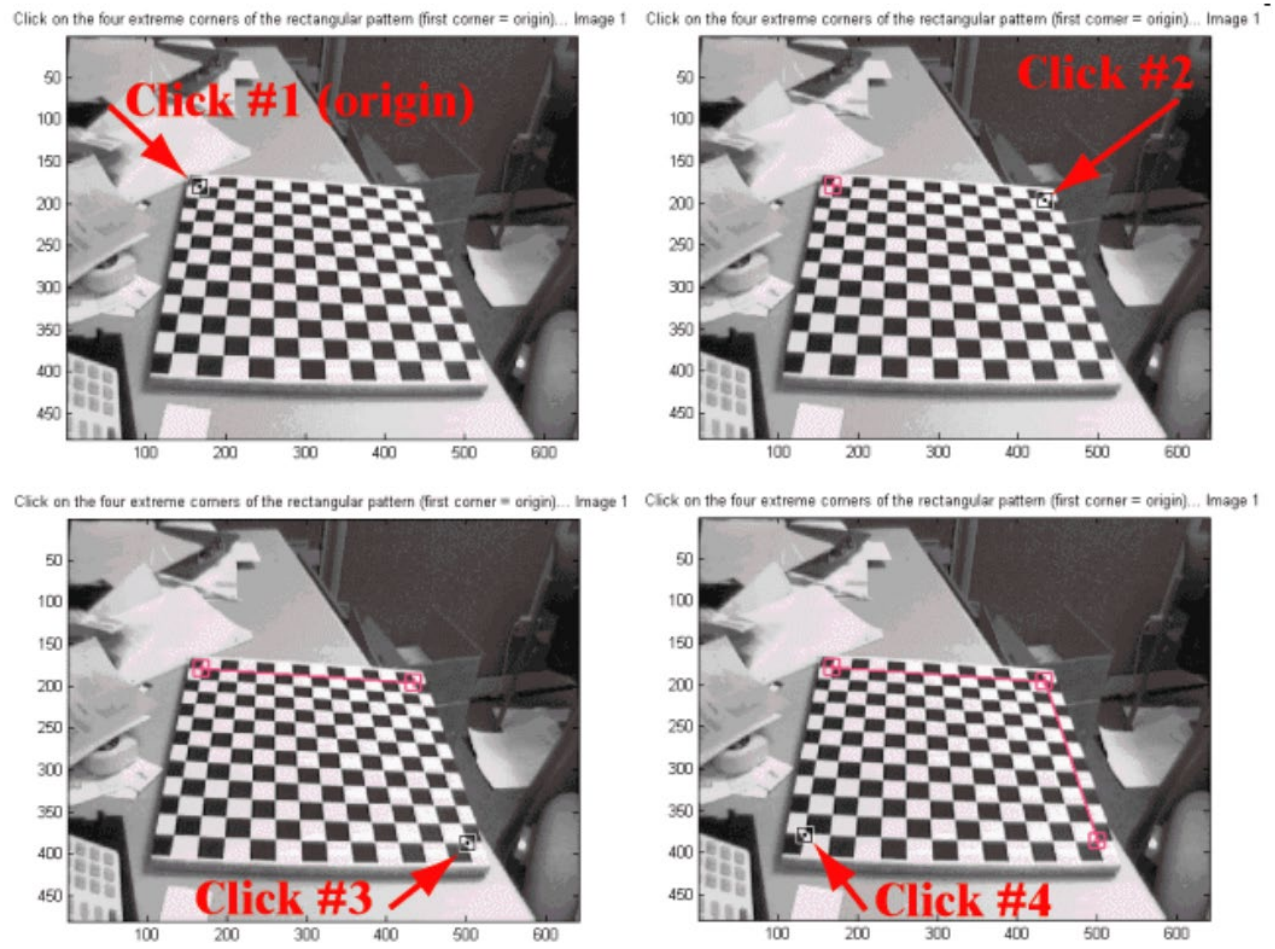
Disadvantage: Need to solve non-linear optimization problem.

# Step-by-step demonstration

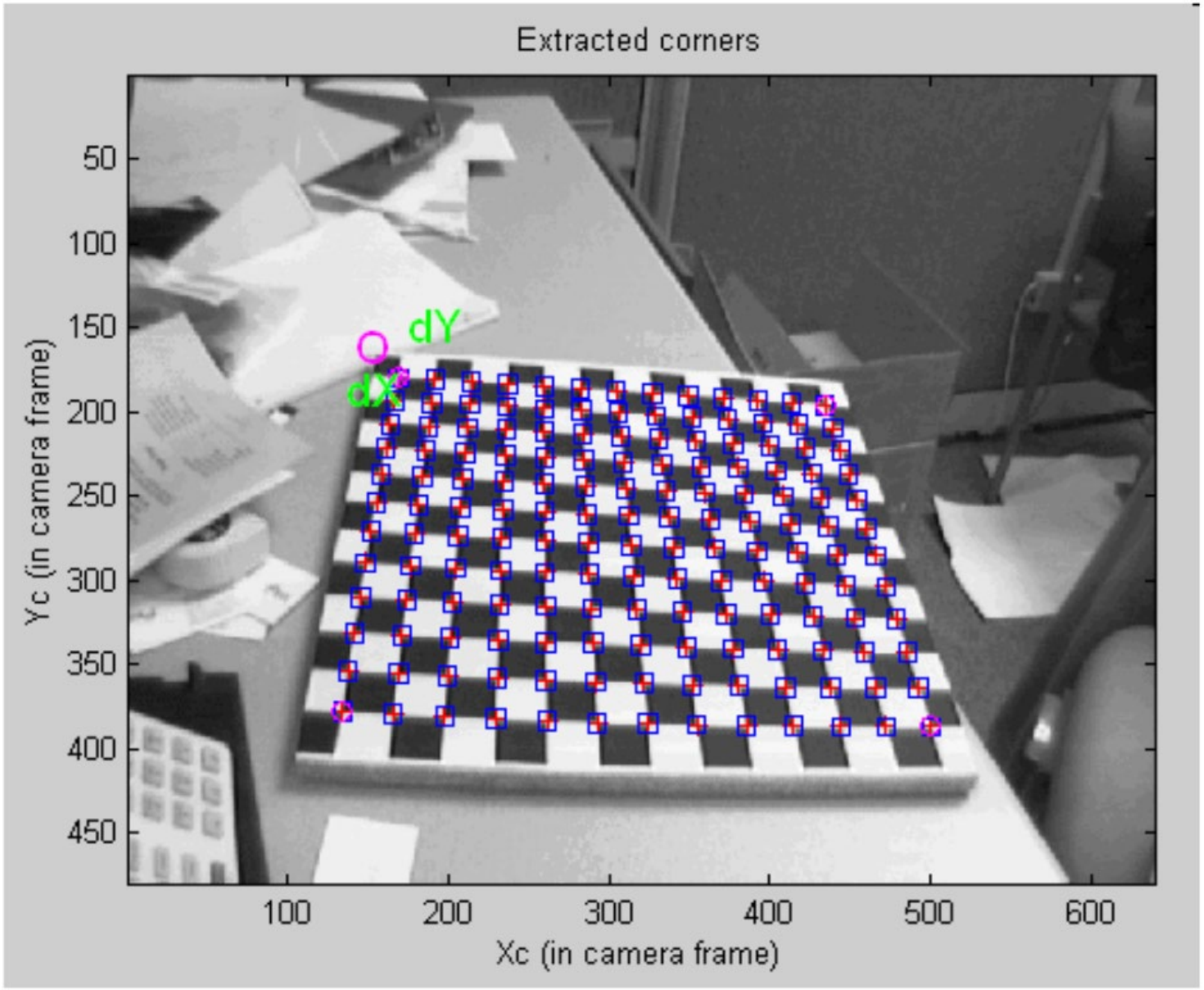


# Step-by-step demonstration

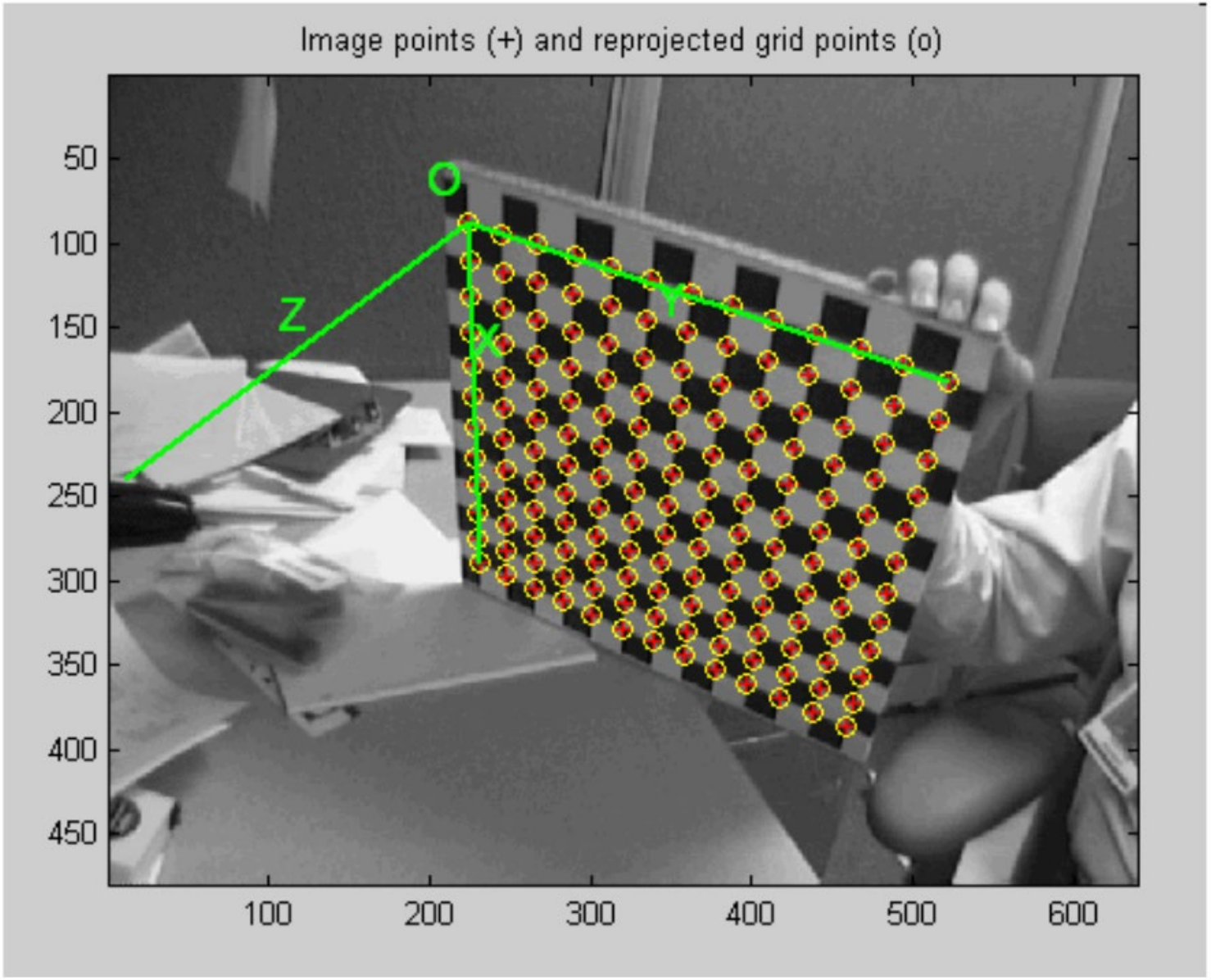


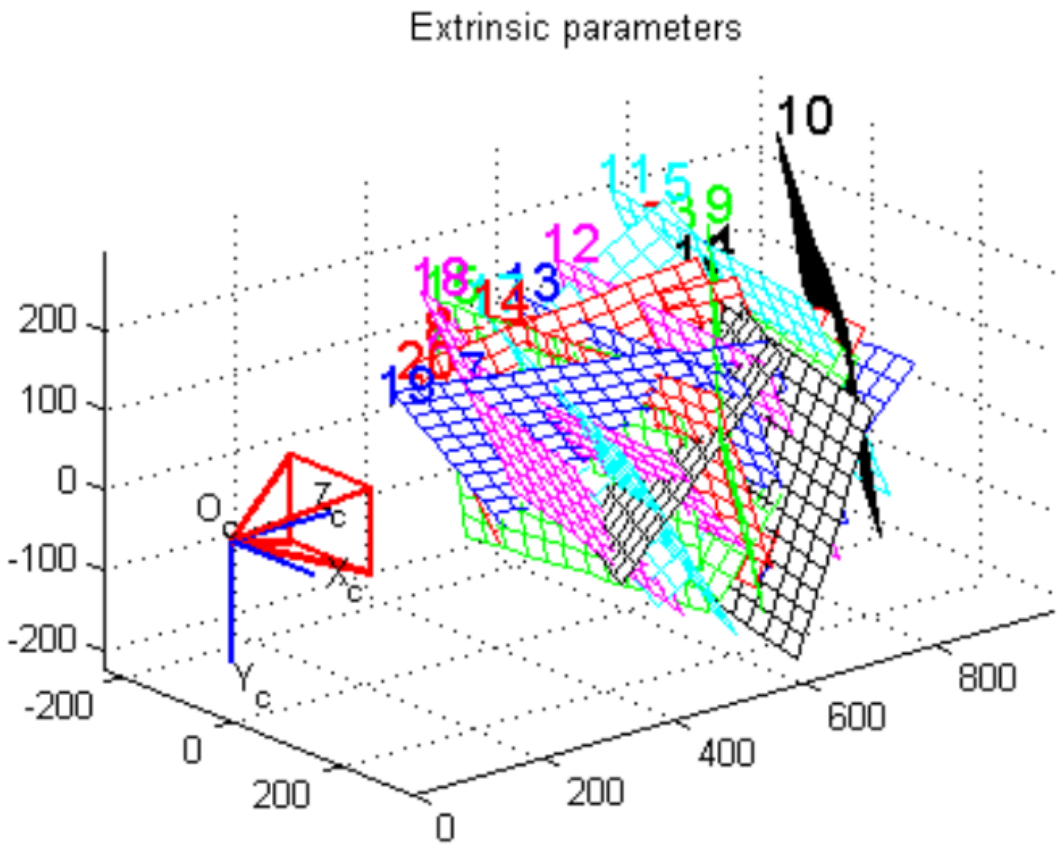
# Step-by-step demonstration

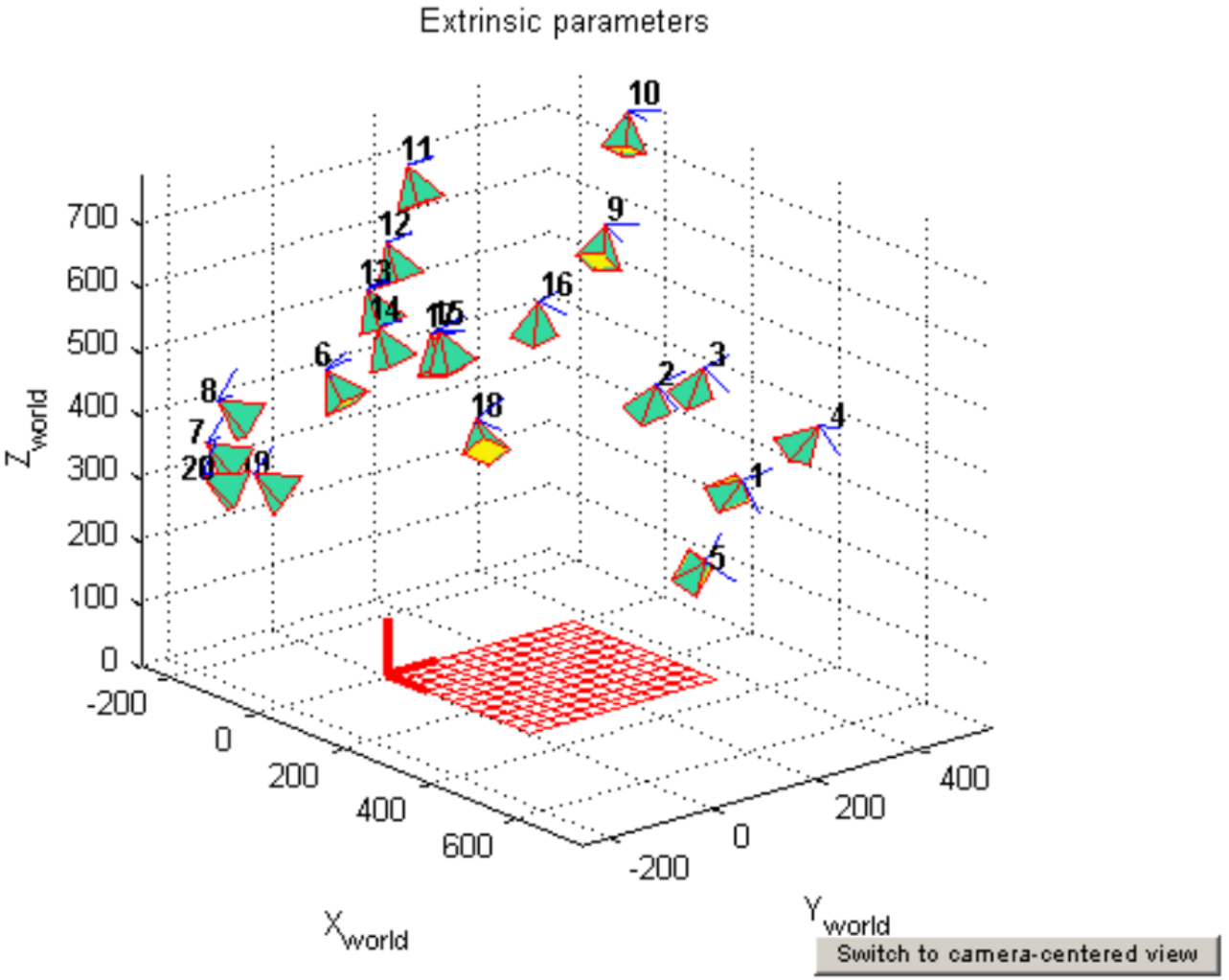


# Step-by-step demonstration



# Step-by-step demonstration



# What does it mean to “calibrate a camera”?

# What does it mean to “calibrate a camera”?

Many different ways to calibrate a camera:

- Radiometric calibration. ← lecture 5-ish
- Color calibration. ← lecture 7-ish
- Geometric calibration. ← lecture 19 (this lecture)
- Noise calibration. ← lecture 6-ish
- Lens (or aberration) calibration. ← lecture 12-ish, (maybe) later lecture

# References

Basic reading:

- Szeliski textbook, Section 2.1.5, 6.2.
- Bouguet, “Camera calibration toolbox for Matlab,” available at http://www.vision.caltech.edu/bouguetj/calib_doc/

  The main resource for camera calibration in Matlab, where the screenshots in this lecture were taken from. It also has a detailed of the camera calibration algorithm and an extensive reference section.

Additional reading:

- Hartley and Zisserman, “Multiple View Geometry in Computer Vision,” Cambridge University Press 2004.

  Chapter 6 of this book has a very thorough treatment of camera models.
- Richter-Gebert, “Perspectives on Projective Geometry,” Springer 2011.

  A great math textbook on everything to do with projective geometry.
- Gortler, “Foundations of 3D Computer Graphics,” MIT Press 2012.

  Chapter 10 of this book has a nice discussion of pinhole cameras from a graphics point of view.
- Zhang, “A flexible new technique for camera calibration,” PAMI 2000.

  The paper that introduced camera calibration from multiple views of a planar target.
- Yu and McMillan, “General Linear Cameras,” ECCV 2004.

  This paper presents a very general model and classification of linear cameras.