Fourier Transform and Frequency Domain

Hi, Dr. Elizabeth?
Yeah, uh... I accidentally took the Fourier transform of my cat...

Meow!
Course announcements

• Last call for responses to Doodle about rescheduling the September 27th lecture!
  - Link available on Piazza.
  - Currently 17 responses. I’ll pick a date on Tuesday evening.

• Homework 1 is being graded.
  - Grades with comments will be uploaded on Canvas hopefully by Wednesday.
  - How was it?

• Homework 2 has been posted.
  - Much larger than homework 1.
  - Start early! Experiments take a long time to run.
  - How many have read/started/finished it?
Overview of today’s lecture

• Some history.
• Fourier series.
• Frequency domain.
• Fourier transform.
• Frequency-domain filtering.
• Revisiting sampling.
Slide credits

Most of these slides were adapted from:

• Kris Kitani (15-463, Fall 2016).

Some slides were inspired or taken from:

• Fredo Durand (MIT).
• James Hays (Georgia Tech).
Some history
Who is this guy?
What is he famous for?

Jean Baptiste Joseph Fourier
(1768-1830)
What is he famous for?

The Fourier series claim (1807):
‘Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.’

Jean Baptiste Joseph Fourier
(1768-1830)
What is he famous for?

Jean Baptiste Joseph Fourier (1768-1830)

The Fourier series claim (1807):
‘Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.’

... and apparently also for the discovery of the greenhouse effect
Is this claim true?

The Fourier series claim (1807):

‘Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.’

Jean Baptiste Joseph Fourier (1768-1830)
Is this claim true?

The Fourier series claim (1807):
‘Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.’

Well, almost.
• The theorem requires additional conditions.
• Close enough to be named after him.
• Very surprising result at the time.
Is this claim true?

The Fourier series claim (1807):
‘Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.’

Well, almost.
• The theorem requires additional conditions.
• Close enough to be named after him.
• Very surprising result at the time.

Jean Baptiste Joseph Fourier (1768-1830)

The committee examining his paper had expressed skepticism, in part due to not so rigorous proofs.
Amusing aside

Only known portrait of Adrien-Marie Legendre

1820 watercolor caricatures of French mathematicians Adrien-Marie Legendre (left) and Joseph Fourier (right) by French artist Julien-Leopold Boilly.

For two hundred years, people were misidentifying this portrait as him.

Louis Legendre (same last name, different person)
Fourier series
Basic building block

\[ A \sin(\omega x + \phi) \]

Fourier’s claim: Add enough of these to get *any periodic* signal you want!
Fourier’s claim: Add enough of these to get any periodic signal you want!
How would you generate this function?

Examples

= ? + ?
Examples

How would you generate this function?

\[
\text{[Graph 1]} = \text{[Graph 2]} + \sin(2\pi x)
\]

?
Examples

How would you generate this function?

\[ f(x) = \sin(2\pi x) + \frac{1}{3} \sin(2\pi 3x) \]
Examples

How would you generate this function?

square wave

= ? + ?
Examples

How would you generate this function?

\[ \text{square wave} \approx \text{function} + \text{function} = \text{result} \]
Examples

How would you generate this function?

square wave ≈ + =
Examples

How would you generate this function?

\[ \text{square wave} \approx f(t) + g(t) = h(t) \]
Examples

How would you generate this function?

square wave
How would you generate this function?

square wave

≈

How would you express this mathematically?
Examples

$\text{square wave} = A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi k x)$

How would you visualize this in the frequency domain?
Examples

\[
\text{square wave} = A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi k x)
\]

infinite sum of sine waves
Frequency domain
Visualizing the frequency spectrum

![Graph showing a frequency spectrum with amplitude on the y-axis and frequency on the x-axis, with labels $k$, $2k$, $3k$, and $4k$.]
Visualizing the frequency spectrum

Recall the temporal domain visualization

\[ f(x) = \sin(2\pi kx) + \frac{1}{3} \sin(2\pi 3kx) \]

![Waveform diagram]
Visualizing the frequency spectrum

Recall the temporal domain visualization

\[ f(x) = \sin(2\pi kx) + \frac{1}{3} \sin(2\pi 3kx) \]

How do we plot ... \( \sin(2\pi kx) \)
Visualizing the frequency spectrum

Recall the temporal domain visualization

\[ f(x) = \sin(2\pi kx) + \frac{1}{3} \sin(2\pi 3kx) \]
Visualizing the frequency spectrum

Recall the temporal domain visualization

\[ f(x) = \sin(2\pi kx) + \frac{1}{3} \sin(2\pi 3kx) \]
Visualizing the frequency spectrum

Recall the temporal domain visualization

\[ f(x) = \sin(2\pi kx) + \frac{1}{3} \sin(2\pi 3kx) \]

What is at zero frequency?

Need to understand this to understand the 2D version!
Visualizing the frequency spectrum

Recall the temporal domain visualization

\[ f(x) = \sin(2\pi k x) + \frac{1}{3} \sin(2\pi 3k x) \]

- not visualizing the symmetric negative part
- signal average (zero for a sine wave with no offset)

Need to understand this to understand the 2D version!
Examples

Spatial domain visualization

Frequency domain visualization

1D

2D

$|F(k)|$
Examples

Spatial domain visualization
1D

Frequency domain visualization
2D

What do the three dots correspond to?
Examples

Spatial domain visualization

Frequency domain visualization

$k_x$ $k_y$
Examples

Spatial domain visualization

Frequency domain visualization

$k_x$

$k_y$
Examples

How would you generate this image with sine waves?
Examples

How would you generate this image with sine waves?

Has both an x and y components
Examples

\[ \begin{array}{c}
\text{[Image]} \\
+ \\
\text{[Image]} \\
\text{=} \\
? 
\end{array} \]
Examples

\[ \begin{align*}
\text{\includegraphics{example1.png}} & + \quad \text{\includegraphics{example2.png}} \\
\text{\includegraphics{example3.png}} & + \quad \text{\includegraphics{example4.png}} \\
\end{align*} \]
Examples

\[
\begin{array}{c}
\text{+} \\
\text{=} \\
\end{array}
\]
Basic building block

$A \sin(\omega x + \phi)$

amplitude  sinusoid  angular variable  phase
frequency

What about non-periodic signals?

Fourier’s claim: Add enough of these to get any periodic signal you want!
Fourier transform
Recalling some basics

Complex numbers have two parts:

rectangular coordinates

$$R + jI$$

what's this? what's this?
Recalling some basics

Complex numbers have two parts:

**Rectangular coordinates**

\[ R + jI \]

- real
- imaginary
Recalling some basics

Complex numbers have two parts:

- **Rectangular coordinates**
  \[ R + jI \]
  real \hspace{1cm} imaginary

- **Polar coordinates**
  \[ r (\cos \theta + j \sin \theta) \]
  how do we compute these?

Alternative reparameterization:

- **Polar transform**
Recalling some basics

Complex numbers have two parts:

Rectangular coordinates: $R + jI$

real imaginary

Alternative reparameterization:

Polar coordinates: $r(\cos\theta + j\sin\theta)$

polar transform

$r = \sqrt{R^2 + I^2}$

$\theta = \tan^{-1}\left(\frac{I}{R}\right)$
Recalling some basics

Complex numbers have two parts:

- **Rectangular coordinates**: \( R + jI \)
  - real
  - imaginary

Alternative reparameterization:

- **Polar coordinates**: \( r(\cos \theta + j \sin \theta) \)
  - polar transform
  - polar transform
  - \( \theta = \tan^{-1}\left(\frac{I}{R}\right) \)
  - \( r = \sqrt{R^2 + I^2} \)

How do you write these in exponential form?
Recalling some basics

Complex numbers have two parts:

**Rectangular coordinates**

\[ R + jI \]
real imaginary

Alternative reparameterization:

**Polar coordinates**

\[ r(\cos \theta + j \sin \theta) \]

polar transform

\[ \theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2} \]

or equivalently

\[ re^{j\theta} \]

how did we get this?

**Exponential form**
Recalling some basics

Complex numbers have two parts:

rectangular coordinates

\[ R + j I \]
real \hspace{1cm} imaginary

Alternative reparameterization:

polar coordinates

\[ r (\cos \theta + j \sin \theta) \]
polar transform
\[ \theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2} \]

or equivalently

\[ re^{j\theta} \]
Euler’s formula
\[ e^{j\theta} = \cos \theta + j \sin \theta \]

This will help us understand the Fourier transform equations.
Fourier transform

Fourier transform
\[
F(k) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi k x} \, dx
\]
\[
f(x) = \int_{-\infty}^{\infty} F(k) e^{j2\pi k x} \, dk
\]

inverse Fourier transform

Where is the connection to the ‘summation of sine waves’ idea?
Where is the connection to the ‘summation of sine waves’ idea?
Where is the connection to the ‘summation of sine waves’ idea?

\[ f(x) = \sum_{k=0}^{N-1} F(k) e^{i2\pi k x / N} \]

- Euler’s formula: \( e^{i\theta} = \cos \theta + j \sin \theta \)
- Sum over frequencies
- Scaling parameter
- Wave components
Fourier transform pairs

spatial domain

frequency domain

Note the symmetry: duality property of Fourier transform
Computing the discrete Fourier transform (DFT)
Computing the discrete Fourier transform (DFT)

\[ F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x)e^{-j2\pi k x/N} \]
is just a matrix multiplication:

\[ F = Wf \]

\[
\begin{bmatrix}
F(0) \\
F(1) \\
F(2) \\
F(3) \\
\vdots \\
F(N - 1)
\end{bmatrix} =
\begin{bmatrix}
W^0 & W^0 & W^0 & W^0 & \cdots & W^0 \\
W^0 & W^1 & W^2 & W^3 & \cdots & W^{N-1} \\
W^0 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\
W^0 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
W^0 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W^1
\end{bmatrix}
\begin{bmatrix}
f(0) \\
f(1) \\
f(2) \\
f(3) \\
\vdots \\
f(N - 1)
\end{bmatrix}
\]

\[ W = e^{-j2\pi/N} \]

In practice this is implemented using the fast Fourier transform (FFT) algorithm.
Another way to compute the Fourier transform

Use a lens!

An ideal thin lens is an optical Fourier transform engine.
Fourier transforms of natural images

original

amplitude

phase
Fourier transforms of natural images

Image phase matters!

cheetah phase with zebra amplitude  
zebra phase with cheetah amplitude
Frequency-domain filtering
Why do we care about all this?
The convolution theorem

The Fourier transform of the convolution of two functions is the product of their Fourier transforms:

$$\mathcal{F}\{g \ast h\} = \mathcal{F}\{g\}\mathcal{F}\{h\}$$

The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms:

$$\mathcal{F}^{-1}\{gh\} = \mathcal{F}^{-1}\{g\} \ast \mathcal{F}^{-1}\{h\}$$

Convolution in spatial domain is equivalent to multiplication in frequency domain!
What do we use convolution for?
Convolution for 1D continuous signals

Definition of linear shift-invariant filtering as convolution:

\[(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy\]

filtered signal \(\rightarrow\) filter \(\rightarrow\) input signal

Using the convolution theorem, we can interpret and implement all types of linear shift-invariant filtering as multiplication in frequency domain.

Why implement convolution in frequency domain?
Frequency-domain filtering in Matlab

Filtering with \texttt{fft}:

\begin{verbatim}
im = double(imread('...'))/255;
im = rgb2gray(im); % "im" should be a gray-scale floating point image
[imh, imw] = size(im);

hs = 50; % filter half-size
fil = fspecial('gaussian', hs*2+1, 10);

fftsize = 1024; % should be order of 2 (for speed) and include padding
im_fft = fft2(im, fftsize, fftsize); % 1) fft im with padding
fil_fft = fft2(fil, fftsize, fftsize); % 2) fft fil, pad to same size as image
im_fil_fft = im_fft .* fil_fft; % 3) multiply fft images
im_fil = ifft2(im_fil_fft); % 4) inverse fft2
im_fil = im_fil(1+hs:size(im,1)+hs, 1+hs:size(im, 2)+hs); % 5) remove padding
\end{verbatim}

Displaying with \texttt{fft}:

\begin{verbatim}
figure(1), imagesc(log(abs(fftshift(im_fft)))), axis image, colormap jet
\end{verbatim}
Spatial domain filtering

Fourier transform

Frequency domain filtering

Filter kernel

Inverse Fourier transform
Revisiting blurring

Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?
Gaussian blur
Box blur
A lens’ kernel is its aperture

This is (one of the reasons) why we try to make lens apertures as circular as possible.

An ideal thin lens is an optical Fourier transform engine.
More filtering examples

filters shown in frequency-domain
More filtering examples

low-pass

band-pass

filters shown in frequency-domain
More filtering examples

? high-pass
More filtering examples

high-pass
More filtering examples

original image

low-pass filter

frequency magnitude
More filtering examples

original image

low-pass filter

frequency magnitude
More filtering examples

original image

high-pass filter

frequency magnitude
More filtering examples

original image

high-pass filter

frequency magnitude
More filtering examples

original image

band-pass filter

frequency magnitude
More filtering examples

original image

band-pass filter

frequency magnitude
More filtering examples

original image

frequency magnitude

band-pass filter
More filtering examples

original image

band-pass filter

frequency magnitude
Revisiting sampling
The Nyquist-Shannon sampling theorem

A continuous signal can be perfectly reconstructed from its discrete version if sampling occurred with frequency:

\[ f_s \geq 2f_{\text{max}} \]

This is called the Nyquist frequency

Equivalent reformulation: When downsampling, aliasing does not occur if samples are taken at the Nyquist frequency or higher.
The Nyquist-Shannon sampling theorem

A continuous signal can be perfectly reconstructed from its discrete version if sampling occurred with frequency:

\[ f_s \geq 2f_{\text{max}} \]

This is called the Nyquist frequency.
How does the Nyquist-Shannon theorem relate to the Gaussian pyramid?
How does the Nyquist-Shannon theorem relate to the Gaussian pyramid?

- Gaussian blurring is low-pass filtering.
- By blurring we try to sufficiently decrease the Nyquist frequency to avoid aliasing.

How large should the Gauss blur we use be?
Frequency-domain filtering in human vision

Gala Contemplating the Mediterranean Sea Which at Twenty Meters Becomes the Portrait of Abraham Lincoln (Homage to Rothko)

Salvador Dali, 1976
Frequency-domain filtering in human vision

Low-pass filtered version
Frequency-domain filtering in human vision

High-pass filtered version
Variable frequency sensitivity

Experiment: Where do you see the stripes?
Variable frequency sensitivity

Campbell-Robson contrast sensitivity curve

- Early processing in humans filters for various orientations and scales of frequency
- Perceptual cues in the mid frequencies dominate perception

Our eyes are sensitive to mid-range frequencies
References

Basic reading:
• Szeliski textbook, Sections 3.4.

Additional reading:
  the standard reference on Fourier optics
• Hubel and Wiesel, “Receptive fields, binocular interaction and functional architecture in the cat’s visual
cortex,” The Journal of Physiology 1962
  a foundational paper describing information processing in the visual system, including the different
types of filtering it performs; Hubel and Wiesel won the Nobel Prize in Medicine in 1981 for the
discoveries described in this paper