Introduction to Compressive Sensing
Aswin Sankaranarayanan
Is this system linear?

\[ y = \Phi x \]
Is this system linear?

Given $y$, can we recovery $x$?
Under-determined problems

If $M < N$, then the system is information lossy
Can we increase the resolution of this image?

(Link: Depixelizing pixel art)
Under-determined problems

\[ \begin{align*}
\begin{bmatrix}
M 	imes 1 \\
\text{measurements}
\end{bmatrix} 
&= 
\begin{bmatrix}
\Phi \\
\text{measurement matrix}
\end{bmatrix} 
\begin{bmatrix}
N 	imes 1 \\
\text{signal}
\end{bmatrix}
\end{align*} \]

Fewer knowns than unknowns!
Under-determined problems

Fewer knowns than unknowns!

An infinite number of solutions to such problems
Credit: Rob Fergus and Antonio Torralba
Is there anything we can do about this?
Complete the sentences

I cnt blv I m bl t rd ths sntnc.

Wntr s cmng, n .. Wntr s hr

Hy, I m slvng n ndr-dtrmnd lnr systm.

how: ?
Complete the matrix

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</tbody>
</table>
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how: ?
Complete the image

Model ?
Dictionary of visual words

I cnt blv I m bl t rd ths sntnc.

Shrlck s th vc f th drgn

Hy, I m slvng n ndr-dtrmnd lnr systm.
Dictionary of visual words
Compressive Sensing

A toolset to solve under-determined systems by exploiting additional structure/models on the signal we are trying to recover.
modern sensors are linear systems!!!
Sampling

Can we recover the analog signal from its discrete time samples?
An analog signal can be reconstructed perfectly from discrete samples *provided you sample it densely.*
The Nyquist Recipe

sample faster

sample denser

the more you sample,
the more detail is preserved
The Nyquist Recipe

sample faster

sample denser

the more you sample, the more detail is preserved

But what happens if you do not follow the Nyquist recipe?
The Nyquist Recipe

sample faster

sample denser

the more you sample,
the more detail is preserved

But what happens if you do not follow the Nyquist recipe?
What you must learn is that these rules are no different than the rules of a computer system. Some of them can be bent. Others can be broken.
Breaking resolution barriers

- Observing a 40 fps spinning tool with a 25 fps camera

Normal Video:
25fps

Compressively obtained video:
25fps

Recovered Video:
2000fps

Slide/Image credit: Reddy et al. 2011
Compressive Sensing

Use of **motion flow-models** in the context of compressive video recovery

**128x128 images sensed at 61x comp.**

- Naïve frame-to-frame recovery
- CS-MUVI at 61x compression

Sankaranarayanan et al. ICCP 2012, SIAM J. Imaging Sciences, 2015*
Compressive Imaging Architectures

Scalable imaging architectures that deliver videos at **mega-pixel resolutions** in infrared

A mega-pixel image obtained from a 64x64 pixel array sensor

Chen et al. CVPR 2015, Wang et al. ICCP 2015
Advances in Compressive Imaging
Linear Inverse Problems

• Many classic problems in computer can be posed as linear inverse problems

• Notation
  – **Signal** of interest \( x \in \mathbb{R}^N \)
  – **Observations** \( y \in \mathbb{R}^M \)
  – Measurement model \( y = \Phi x + e \)

• Problem definition: given \( y \), recover \( x \)
Linear Inverse Problems

\[ y = \Phi x \]

Measurement matrix has a \((N-M)\) dimensional null-space

Solution is no longer unique
Sparse Signals

\[ y = \Phi x \]

- \( y \) : \( M \times 1 \) measurements
- \( \Phi \) : \( M \times N \) matrix
- \( x \) : \( N \times 1 \) sparse signal

\[ K < M \leq N \]

nonzero entries
How Can It Work?

- Matrix $\Phi$ not full rank...
  
  $M < N$

  ... and so loses information in general

- But we are only interested in \textit{sparse} vectors
Restricted Isometry Property (RIP)

- Preserve the structure of sparse/compressible signals
Restricted Isometry Property (RIP)

- RIP of order $2K$ implies: for all $K$-sparse $x_1$ and $x_2$

\[
(1 - \delta_{2K}) \leq \frac{\|\Phi x_1 - \Phi x_2\|_2^2}{\|x_1 - x_2\|_2^2} \leq (1 + \delta_{2K})
\]
How Can It Work?

- Matrix $\Phi$ not full rank...
  
  $M < N$
  
  ... and so loses information in general

- Design $\Phi$ so that each of its $Mx2K$ submatrices are full rank (RIP)
How Can It Work?

• Matrix $\Phi$ not full rank...

\[ M < N \]

... and so loses information in general

• **Design** $\Phi$ so that each of its $M \times 2K$ submatrices are full rank (RIP)

• Random measurements provide RIP with

\[ M = O(K \log(N/K)) \]
CS Signal Recovery

• Random projection \( \Phi \) not full rank

• Recovery problem:
given \( y = \Phi x \)
find \( x \)

• Null space

• Search in null space for the “sparsest” \( x \)

\( (N-M)\text{-dim hyperplane at random angle} \)
$\ell_1$ Signal Recovery

- **Recovery:**
  - given $y = \Phi x$
  - find $x$ (sparse)

- **Optimization:**
  - $\hat{x} = \arg \min_{y=\Phi x} \|x\|_1$

- **Convexify** the $\ell_0$ optimization

Candes  Romberg  Tao  Donoho
Signal Recovery

• **Recovery:**
  (ill-posed inverse problem)
  \[ y = \Phi x \]
  find \( \hat{x} \) (sparse)

• **Optimization:**

  \[ \hat{x} = \arg \min_{y=\Phi x} \| x \|_1 \]

• **Convexify** the \( l_0 \) optimization

• **Polynomial time alg**
  (linear programming)
Compressive Sensing

Let. \( y = \Phi x_0 + e \)

\[
\hat{x} = \arg \min_x \| x \|_1 \quad s.t. \quad \| y - \Phi x \|_2 \leq \| e \|
\]

If \( \Phi \) satisfies RIP with \( \delta_{2K} \leq \sqrt{2} - 1 \),

Then

\[
\| \hat{x} - x_0 \|_1 \leq C_1 \| e \|_2 + C_2 \| x_0 - x_{0,K} \|_2 / \sqrt{K}
\]

Best K-sparse approximation