Light transport matrices
Course announcements

• Homework 5 has been posted.
  - Due on Friday November 9th.

• Any problems with homework 4?

• No elevator pitch presentations for final projects.

• Extra office hours this week:
  - Monday 1:30-3:30 pm.
  - Tuesday noon-2:00 pm.
  - Friday’s office hours will be held by Alankar.

• Great talk this Thursday: Eric Fossum, inventor of the CMOS sensor, will talk about quantum (i.e., photon-counting) CMOS sensors.
Overview of today’s lecture

• Leftover from last time: Generalized bas-relief ambiguity.
• The light transport matrix.
• Image-based relighting.
• Photometric stereo revisited.
• Optical computing using the light transport matrix.
• Dual photography.
These slides were directly adapted from:

- Matt O’Toole (CMU).
The light transport matrix
How do these three images relate to each other?
How do these three images relate to each other?
the superposition principle

photo with lights 1 & 2 turned on

photo with light 1 turned on +
photo with light 2 turned on

photo taken under two light sources = sum of photos taken under each source individually
the superposition principle

photo taken under two light sources = sum of photos taken under each source individually
the superposition principle

why is the error not exactly zero?

Synthetic photo

Diff between synthetic and real photos

photo taken under two light sources = sum of photos taken under each source individually
image-based relighting

Relit photo

photo with light 1 turned on

photo with light 2 turned on
image-based relighting

Relit photo

photo with light 1 turned on

Weight 1

x 1

= +

photo with light 2 turned on

Weight 2

x 1
Image-based relighting

Relit photo

\[ \text{Relit photo} = \text{photo with light 1 turned on} \times 1 + \text{photo with light 2 turned on} \times 0 \]
image-based relighting

Relit photo

\[ \text{Relit photo} = \text{photo with light 1 turned on} \times \text{Weight 1} + \text{photo with light 2 turned on} \times \text{Weight 2} \]
Relit photo $p = \text{photo with light 1 turned on } T_1 \times l_1 + \text{photo with light 2 turned on } T_2 \times l_2$
Relit photo $p$ = photo with light 1 turned on $T_1$ Weight 1 $l_1$ + photo with light 2 turned on $T_2$ Weight 2 $l_2$

$p = \sum_{i=1}^{2} T_i l_i$
Relit photo $p = \sum_{i=1}^{2} T_i \times l_i$ with lighter 1 turned on $T_1$ and lighter 2 turned on $T_2$. Each lighter $l_i$ has a weight $l_i$. The pixel values are denoted by $p \in \mathbb{R}^n$. 
\[ p = \sum_{i=1}^{2} T_i \times l_i \]
\[ p = \sum_{i=1}^{2} T_i \times l_i \]
\[ p = \sum_{i=1}^{\text{Number of controllable sources}} T_i \times l_i \]

- Relit photo \( p \)
- Number of controllable sources
- \( T_1 \) and \( T_2 \)
- Weight 1 and Weight 2
- Contribution of each source
- Pixel values \( n \)

Photo with light 1 turned on: \( T_1 \times l_1 \)
Photo with light 2 turned on: \( T_2 \times l_2 \)
Relit photo $p$ = photo with light 1 turned on $T_1$ Weight 1 $x$ $l_1$ + photo with light 2 turned on $T_2$ Weight 2 $x$ $l_2$

$$p = \sum_{i=1}^{2} T_i \times l_i$$

Number of controllable sources

Contribution of each source

$n$ pixel values
Relit photo $p$ = Weight 1 $x$ $l_1$ + Weight 2 $x$ $l_2$

Number of controllable sources $n$

$p = \sum_{i=1}^{2} T_i \times l_i$

Contribution of each source

$n$ pixel values
Relit photo $p$ = photo with light 1 turned on $T_1$ \times \hat{I}_1 +$ photo with light 2 turned on $T_2$ \times \hat{I}_2$

Number of controllable sources

$\sum_{i=1}^{m} T_i \times I_i$

$\times \hat{I}_i$

$n$ pixel values
Can you think of a case where we have a very large $m$?
What does each row and column of $T$ correspond to here?
How would you go about measuring the light transport matrix?

Use a projector

$\mathbf{p} = \mathbf{T} \mathbf{p}'$

$\mathbf{c}' = \mathbf{T} \mathbf{p}'$

$n \times m$

$n$ pixel values

$m$

How would you go about measuring the light transport matrix?
Image-based relighting
Let’s say I have measured $T$.
- What does it mean to right-multiply it with some vector $l$?
Image-based relighting: Use the measurements I already have of the scene (the pictures I took when measuring $T$) to simulate new illuminations of the scene.

$n$ pixel values

$p = T$

$n \times m$

$m$ independent illumination degrees of freedom
Acquiring the Reflectance Field [Debevec et al. 2000]

image-based rendering & relighting

Light stage 1.0

Reflectance field
Acquiring the Reflectance Field

image-based rendering & relighting

Great demonstration: https://www.youtube.com/watch?v=mkzLLz1tXds

Debevec et al, SIG 2000
Acquiring the Reflectance Field

Light stage 6, Debevec et al., 2006
Photometric stereo revisited
the light transport matrix
Sloan et al 02, Ng et al 03, Seitz et al 05, Sen et al 05, ...

\[ p = T \]

n × m

transport matrix is a function of scene geometry, reflectance, etc.
Photometric Stereo [Woodham, 1980]

Diffuse reflections:

\[ t_{ij} = \rho_i (N_i \cdot L_j) l_j \]

\[ = \tilde{N}_i \cdot \tilde{L}_j \]

3x1 vector, unknown \hspace{1cm} 3x1 vector, known

simplifying assumptions:

camera pixel \( i \) and light source \( j \) produce image intensity \( t_{ij} \)
directional light source, convex object
\[ t_{ij} = \rho_i (N_i \cdot L_j) l_j \]

\[ = \tilde{N}_i \cdot \tilde{L}_j \]

3x1 vector, unknown \quad 3x1 vector, known

simplifying assumptions:
- directional light source,
- convex object
Photometric Stereo [Woodham, 1980]

\[ T_{n \times m} \text{ (rank 3)} = \tilde{N}_{n \times 3} \tilde{L}_{3 \times m} \]

recover surface normals + albedo by decomposing transport matrix \( T \)
Recovering Scene Geometry

“Mobile” Light Stage, Debevec et al., 2014

https://www.youtube.com/watch?v=4GiLAOtjHNo
Optical computing using the light transport matrix
main difficulties

question: what are the challenges with analyzing $T$?
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- matrix can be extremely large
main difficulties

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- matrix can be extremely large
- elements not directly accessible
main difficulties

question: what are the challenges with analyzing $T$?

- matrix can be extremely large
- elements not directly accessible
- global structure poorly understood
computing with light

numerical algorithms implemented directly in optics

numerical domain

transport matrix

\[ p = T1 \]

photo

illumination

vector

optical domain
computing with light

numerical algorithms implemented directly in optics

numerical domain

\[ p = T \, l \]

- transport matrix
- \( p \)
- \( T \)
- \( l \)

photo
illumination vector

optical domain

1. illuminate with \( l \)
2. capture \( p \)
computing with light

numerical algorithms implemented directly in optics

**numerical domain**

```python
function analyze(T)
...
for i = 1 to k {
    ...
    p_i = T l_i
    ...
    d_i = T r_i
    ...
}
...
return result
```

**optical domain**

1. illuminate with 1
2. capture p
computing with light

numerical algorithms implemented directly in optics

**numerical domain**

```
function analyze(T)
...  
for i = 1 to k {
  ...
  p_i = Tl_i
  ...
  d_i = Tr_i
  ...
}
...  
return result
```

**optical domain**

```
function analyze()
...  
for i = 1 to k {
  ...
  project l_i, capture p_i
  ...
  project r_i, capture d_i
  ...
}
...  
return result
```
find an illumination pattern that when projected onto scene, we get the same photo back (multiplied by a scalar)

What do we call these patterns?
computing transport eigenvectors

eigenvector of a square matrix $T$ when projected onto scene, we get the same photo back (multiplied by a scalar)

numerical goal
find $1, \lambda$ such that

$$T1 = \lambda 1$$

and $\lambda$ is maximal
optical power iteration

goal: find principal eigenvector of $T$
observation: it is a fixed point of the sequence $1, T1, T^21, T^31, \ldots$

numerical domain

function PowerIt($T$)

$$l_1 = \text{initial vector}$$

for $i = 1$ to $k$ {
$$p_i = Tl_i$$
$$l_{i+1} = p_i / \|p_i\|_2$$
}

return $l_{i+1}$

properties

- linear convergence [Trefethen and Bau 1997]
- eigenvalues must be distinct
- $l_1$ cannot be orthogonal to principal eigenvector
optical power iteration

goal: find principal eigenvector of $\mathbf{T}$
observation: it is a fixed point of the sequence $1, \mathbf{T}1, \mathbf{T}^21, \mathbf{T}^31, \ldots$

**numerical domain**

```plaintext
function PowerIt(T)

$l_1 = \text{initial vector}$

for $i = 1$ to $k$ {
    $p_i = Tl_i$
    $l_{i+1} = p_i / \|p_i\|_2$
}

return $l_{i+1}$
```

**optical domain**

```plaintext
function PowerIt()

$l_1 = \text{initial vector}$

for $i = 1$ to $k$ {
    project $l_i$, capture $p_i$
    $l_{i+1} = p_i / \|p_i\|_2$
}

return $l_{i+1}$
```
optical power iteration

goal: find principal eigenvector of $\mathbf{T}$
observation: it is a fixed point of the sequence $\mathbf{l}, \mathbf{Tl}, \mathbf{T^2l}, \mathbf{T^3l}, \ldots$

numerical domain

function PowerIt($\mathbf{T}$)

$l_1 = \text{initial vector}$

for $i = 1$ to $k$ {
    \[ \mathbf{p}_i = \mathbf{Tl}_i \]
    \[ l_{i+1} = \mathbf{p}_i / \|\mathbf{p}_i\|_2 \]
}

return $l_{i+1}$

optical domain

initialize $l_1$

project $l_i$

capture $\mathbf{Tl}_i$

normalize $l_i$
optical power iteration

goal: find principal eigenvector of $\mathbf{T}$

observation: it is a fixed point of the sequence $l, Tl, T^2l, T^3l, \ldots$
optical power iteration

goal: find principal eigenvector of $T$
observation: it is a fixed point of the sequence $1, Tl, T^2l, T^3l, \ldots$
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optical power iteration

goal: find principal eigenvector of $T$

observation: it is a fixed point of the sequence $1, T1, T^21, T^31, \ldots$

optical domain

initialize $l_1$

project

capture

normalize

$l_2$

$Tl_1$
optical power iteration

- **goal:** find principal eigenvector of $T$
- **observation:** it is a fixed point of the sequence $1, T1, T^21, T^31, \ldots$

optical domain

- initialize $l_1$
- project $l_2$
- capture $Tl_2$
- normalize
optical power iteration

**goal:** find principal eigenvector of $T$

**observation:** it is a fixed point of the sequence $l, Tl, T^2l, T^3l, \ldots$
optical power iteration

goal: find principal eigenvector of $\mathbf{T}$
observation: it is a fixed point of the sequence $1, \mathbf{T}l, \mathbf{T}^2l, \mathbf{T}^3l, \ldots$

optical domain
(approximate)
principal eigenvector
How would you measure the light transport matrix $T$?

- matrix can be extremely large
- elements not directly accessible
- global structure poorly understood
How would you measure the light transport matrix $T$?

Exhaustive/naïve approach: turn on projector pixels one at a time and take a photo for each of them.

- What does each photo correspond to in $T$?
How would you measure the light transport matrix \( T \)?

Exhaustive/naïve approach: turn on projector pixels one at a time and take a photo for each of them.

- How many photos do we need to capture?
Inverse transport
flashlight
How do you solve this problem if you know the light transport matrix $T$?
What do we do here?

$\mathbf{p} = \mathbf{T} + \mathbf{1}$

input photo

illumination

?
What if $T$ is not invertible?

$p = T$
How do you usually solve for $I$ when $T$ is large?

Numerical goal:
Given photo $p$, find illumination $I$ that minimizes

$$T I - p$$
Reminder from lecture 10: Gradient descent

Given the loss function:

\[ E(f) = \|Gf - v\|^2 \]

Minimize by iteratively computing:

\[ f^{i+1} = f^i - \eta^i r^i, \quad r^i = v - Af^i, \quad \eta^i = \frac{(r^i)^T r^i}{(r^i)^T Ar^i} \quad \text{for } i = 0, 1, ..., N \]

Is this cheaper than the pseudo-inverse approach?

- We never need to compute A, only its products with vectors f, r.
- Vectors f, r are images.
- Because A is the Laplacian matrix, these matrix-vector products can be efficiently computed using convolutions with the Laplacian kernel.
Gradient descent in this case

Given the loss function:

\[ E(f) = \| Gf - v \|^2 \]

Minimize by iteratively computing:

\[ f^{i+1} = f^i - \eta^i r^i, \quad r^i = v - Af^i, \quad \eta^i = \frac{(r^i)^T r^i}{(r^i)^T A r^i} \quad \text{for } i = 0, 1, ..., N \]

Is this cheaper than the pseudo-inverse approach?

• We never need to compute A, only its products with vectors f, r.
• Vectors f, r are images. What are f, r in this case?
• Because A is the Laplacian matrix, these matrix-vector products can be efficiently computed using convolutions with the Laplacian kernel. How do we compute matrix-vector products efficiently in this case?
inverting light transport

numerical goal

given photo $p$, find illumination $l$ that minimizes

\[
\begin{bmatrix}
T \\
1 \\
\end{bmatrix}
- p
\]

remarks

- $T$ low-rank or high-rank
- $T$ unknown & not acquired
- illumination sequence will be specific to input photo
inverting light transport

input photo

actual illumination
Dual photography
How do the light transport matrices for these two scenes relate to each other?

\[ \mathbf{c}^\prime = \mathbf{T} \mathbf{p}^\prime \]

\[ \mathbf{p}^\prime = \mathbf{T}^\top \mathbf{c}^\prime \]

\[ \mathbf{c}^\prime = \mathbf{T} \mathbf{p}^\prime \]

\[ \mathbf{p}^\prime = \mathbf{T}^\top \mathbf{c}^\prime \]
Helmholtz reciprocity: The two matrices are the transpose of each other.

Great demonstration: https://www.youtube.com/watch?v=eV58Ko3iFul
References

Basic reading:
• Sloan et al., “Precomputed radiance transfer for real-time rendering in dynamic, low-frequency lighting environments,” SIGGRAPH 2002.
• Ng et al., “All-frequency shadows using non-linear wavelet lighting approximation,” SIGGRAPH 2003.
  These three papers all discuss the concept of light transport matrix in detail.
• Debevec et al., “Acquiring the reflectance field of a human face,” SIGGRAPH 2000.
  The paper on image-based relighting.
  The original photometric stereo paper.
  The paper on eigenanalysis and optical computing using light transport matrices.
  The dual photography paper.

Additional reading:
  These two papers discuss alternative ways for efficient acquisition of the light transport matrix, using assumptions on its algebraic structure.