Two-view geometry
Course announcements

• Homework 4 is out.
  - Due October 26th.
  - Start early: part 3 (lightfield capture) takes a lot of time to get right.
  - Any questions?

• Due October 21st: Project ideas posted on Piazza.

• Extra office hours this afternoon, 5-7 pm.

• ECE Seminar tomorrow: Rajiv Laroia, “Is Computational Imaging the future of Photography?”
  - New time and date: noon – 1:30 pm, Scaife Hall 125
Light camera L16

- Use multiple views (i.e., lightfield) to refocus.
- Use deconvolution to keep thin (i.e., skip compound lens).
Overview of today’s lecture

• Leftover from lecture 13
• Reminder about pinhole and lens cameras
• Camera matrix.
• Other camera models.
• Camera calibration.
Slide credits

Many of these slides were adapted from:

- Srinivasa Narasimhan (16-720, Fall 2017).
Overview of today’s lecture

- Leftover from lecture 13: camera calibration.
- Triangulation.
- Epipolar geometry.
- Essential matrix.
- Fundamental matrix.
- 8-point algorithm.
Triangulation
Triangulation

Given camera 1 with matrix $\mathbf{P}$, and camera 2 with matrix $\mathbf{P'}$, we obtain $\mathbf{x}$ and $\mathbf{x'}$. Image 1 and Image 2.
Triangulation

Which 3D points map to $\mathbf{x}$?

image 1

$\mathbf{x}$

camera 1 with matrix $\mathbf{P}$

image 2

$\mathbf{x}'$

camera 2 with matrix $\mathbf{P}'$
Triangulation

How can you compute this ray?

camera 1 with matrix $P$

camera 2 with matrix $P'$
Create two points on the ray:
1) find the camera center; and
2) apply the pseudo-inverse of $P$ on $x$. Then connect the two points.
How do we find the exact point on the ray? $P^+ x$
Triangulation

Find 3D object point

Will the lines intersect?

camera 1 with matrix $P$
camera 2 with matrix $P'$
Triangulation

Find 3D object point (no single solution due to noise)
Triangulation

Given a set of (noisy) matched points

\[ \{ x_i, x'_i \} \]

and camera matrices

\[ P, P' \]

Estimate the 3D point

\[ X \]
Can we compute $X$ from a single correspondence $x$?
Can we compute $X$ from two correspondences $x$ and $x'$?
\[ x = PX \]

Can we compute \( X \) from two correspondences \( x \) and \( x' \)?

yes if perfect measurements
There will not be a point that satisfies both constraints because the measurements are usually noisy.

Can we compute $X$ from two correspondences $x$ and $x'$?

Yes if perfect measurements.

$x = PX$  
$k$nown  
$k$nown

$x' = P'X$  
$x = PX$

Need to find the best fit.
\[ \mathbf{x} = \mathbf{P} \mathbf{X} \]

(homogeneous coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

\[ \mathbf{x} = \alpha \mathbf{P} \mathbf{X} \]

(homogeneous coordinate)

Same ray direction but differs by a scale factor

\[
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix} = \alpha \begin{bmatrix}
    p_1 & p_2 & p_3 & p_4 \\
    p_5 & p_6 & p_7 & p_8 \\
    p_9 & p_{10} & p_{11} & p_{12}
\end{bmatrix} \begin{bmatrix}
    X \\
    Y \\
    Z \\
    1
\end{bmatrix}
\]

How do we solve for unknowns in a similarity relation?
How do we solve for unknowns in a similarity relation?

Remove scale factor, convert to linear system and solve with SVD.
\[ x = \mathbf{P} \mathbf{X} \]
\[ \text{(homogeneous coordinate)} \]

Also, this is a similarity relation because it involves homogeneous coordinates

\[ x = \alpha \mathbf{P} \mathbf{X} \]
\[ \text{(inhomogeneous coordinate)} \]

Same ray direction but differs by a scale factor

\[
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
= \alpha
\begin{bmatrix}
    p_1 & p_2 & p_3 & p_4 \\
    p_5 & p_6 & p_7 & p_8 \\
    p_9 & p_{10} & p_{11} & p_{12}
\end{bmatrix}
\begin{bmatrix}
    X \\
    Y \\
    Z \\
    1
\end{bmatrix}
\]

*How do we solve for unknowns in a similarity relation?*

Remove scale factor, convert to linear system and solve with SVD!
Recall: Cross Product

Vector (cross) product
takes two vectors and returns a vector perpendicular to both

c = a \times b

a \times b = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}

cross product of two vectors in the same direction is zero

a \times a = 0

remember this!!!
\[ x = \alpha P X \]
Same direction but differs by a scale factor

\[ x \times P X = 0 \]
Cross product of two vectors of same direction is zero
(this equality removes the scale factor)
\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \alpha \begin{bmatrix}
p_1 & p_2 & p_3 & p_4 \\
p_5 & p_6 & p_7 & p_8 \\
p_9 & p_{10} & p_{11} & p_{12}
\end{bmatrix} \begin{bmatrix}
X \\
Y \\
Z \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \alpha \begin{bmatrix}
p_1^\top & \quad & \quad & \quad \\
p_2^\top & \quad & \quad & \quad \\
p_3^\top & \quad & \quad & \quad
\end{bmatrix} \begin{bmatrix}
X
\end{bmatrix}
\]

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \alpha \begin{bmatrix}
p_{1X} \\
p_{2X} \\
p_{3X}
\end{bmatrix}
\]
\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \alpha
\begin{bmatrix}
p_1 & p_2 & p_3 & p_4 \\
p_5 & p_6 & p_7 & p_8 \\
p_9 & p_{10} & p_{11} & p_{12}
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \alpha
\begin{bmatrix}
p_{1}^{\top}X \\
p_{2}^{\top}X \\
p_{3}^{\top}X
\end{bmatrix}
\]

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\times
\begin{bmatrix}
p_{1}^{\top}X \\
p_{2}^{\top}X \\
p_{3}^{\top}X
\end{bmatrix}
= \begin{bmatrix}
y p_{3}^{\top}X - p_{2}^{\top}X \\
p_{1}^{\top}X - x p_{3}^{\top}X \\
x p_{2}^{\top}X - y p_{1}^{\top}X
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]
Using the fact that the cross product should be zero

\[ \mathbf{x} \times \mathbf{P} \mathbf{X} = 0 \]

\[
\begin{bmatrix}
  x \\
  y \\
  1
\end{bmatrix} \times \begin{bmatrix}
  p_1^\top \mathbf{X} \\
  p_2^\top \mathbf{X} \\
  p_3^\top \mathbf{X}
\end{bmatrix} = \begin{bmatrix}
  yp_3^\top \mathbf{X} - p_2^\top \mathbf{X} \\
  p_1^\top \mathbf{X} - xp_3^\top \mathbf{X} \\
  xp_2^\top \mathbf{X} - yp_1^\top \mathbf{X}
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}
\]

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you 3 equations.
Using the fact that the cross product should be zero

\[ \mathbf{x} \times \mathbf{PX} = 0 \]

\[
\begin{bmatrix}
  x \\
  y \\
  1
\end{bmatrix} \times \begin{bmatrix}
  \mathbf{p}_1^\top \mathbf{X} \\
  \mathbf{p}_2^\top \mathbf{X} \\
  \mathbf{p}_3^\top \mathbf{X}
\end{bmatrix} = \begin{bmatrix}
  y\mathbf{p}_3^\top \mathbf{X} - \mathbf{p}_2^\top \mathbf{X} \\
  \mathbf{p}_1^\top \mathbf{X} - x\mathbf{p}_3^\top \mathbf{X} \\
  x\mathbf{p}_2^\top \mathbf{X} - y\mathbf{p}_1^\top \mathbf{X}
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}
\]

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you 2 equations
Now we can make a system of linear equations
(two lines for each 2D point correspondence)

\[
\begin{bmatrix}
y p_3^\top X - p_2^\top X \\
p_1^\top X - x p_3^\top X
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix}
y p_3^\top - p_2^\top \\
p_1^\top - x p_3^\top
\end{bmatrix}
X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[ A_i X = 0 \]
How do we solve homogeneous linear system?

Concatenate the 2D points from both images

\[
\begin{bmatrix}
yp_3 - p_2 \\
p_1 - xp_3 \\
y'p_3' - p_2' \\
p_1' - x'p_3'
\end{bmatrix}
\]

\[
X = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Is this a sanity check? Dimensions?

\[
AX = 0
\]

How do we solve homogeneous linear system?
How do we solve homogeneous linear system?

Concatenate the 2D points from both images

\[
\begin{bmatrix}
yp_3^T - p_2^T \\
p_1^T - xp_3^T \\
y'p_3'^T - p_2'^T \\
p_1'^T - x'p_3'^T
\end{bmatrix}
\begin{bmatrix}
x \\
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
A X = 0
\end{bmatrix}
\]

How do we solve homogeneous linear system?

S V D !
Recall: Total least squares

(Warning: change of notation. $x$ is a vector of parameters!)

$$E_{TLS} = \sum_i (a_i x)^2$$

$$= \| A x \|^2$$  \hspace{1cm} \text{(matrix form)}

$$\| x \|^2 = 1$$  \hspace{1cm} \text{constraint}

minimize $\| A x \|^2$
subject to $\| x \|^2 = 1$

minimize $\frac{\| A x \|^2}{\| x \|^2}$
\hspace{1cm} \text{(Rayleigh quotient)}

Solution is the eigenvector corresponding to smallest eigenvalue of $A^T A$
Epipolar geometry
Epipolar geometry

Image plane
Epipolar geometry

Image plane

Baseline
Epipolar geometry

- Image plane
- Baseline
- Epipole (projection of o' on the image plane)
Epipolar geometry

Epipolar plane

Baseline

Epipole (projection of o’ on the image plane)

Image plane
Epipolar geometry

- Epipolar line (intersection of Epipolar plane and image plane)
- Epipole (projection of o’ on the image plane)
- Baseline
- Image plane
Quiz

What is this?
Quiz

Epipolar plane
Quiz

What is this?
Quiz

Epipolar plane

Epipolar line
(intersection of Epipolar plane and image plane)
What is this?
Quiz

Epipolar plane

Epipolar line
(intersection of Epipolar plane and image plane)

Epipole
(projection of o’ on the image plane)
Quiz

Epipole
(projection of o’ on the image plane)

Epipolar line
(intersection of Epipolar plane and image plane)

Epipolar plane

What is this?
Quiz

Epipolar plane

Epipolar line
(intersection of Epipolar plane and image plane)

Epipole
(projection of o' on the image plane)

Baseline
Epipolar constraint

Potential matches for $x$ lie on the epipolar line $l'$.
Epipolar constraint

Potential matches for \( x \) lie on the epipolar line \( l' \)
The point $x$ (left image) maps to a _________ in the right image.

The baseline connects the _________ and _________.

An epipolar line (left image) maps to a _________ in the right image.

An epipole $e$ is a projection of the _____________ on the image plane.

All epipolar lines in an image intersect at the _____________.
Converging cameras

Where is the epipole in this image?
Converging cameras

Where is the epipole in this image? It’s not always in the image
Parallel cameras

Where is the epipole?
Parallel cameras

epipole at infinity
The epipolar constraint is an important concept for stereo vision

**Task:** Match point in left image to point in right image

*How would you do it?*
Recall: Epipolar constraint

Potential matches for $x$ lie on the epipolar line $l'$. 
The epipolar constraint is an important concept for stereo vision.

**Task:** Match point in left image to point in right image

Want to avoid search over entire image

Epipolar constraint reduces search to a single line.
The epipolar constraint is an important concept for stereo vision.

**Task:** Match point in left image to point in right image.

Want to avoid search over entire image.
Epipolar constraint reduces search to a single line.

*How do you compute the epipolar line?*
The essential matrix
Recall: Epipolar constraint

Potential matches for \( x \) lie on the epipolar line \( l' \).
Given a point in one image, multiplying by the **essential matrix** will tell us the **epipolar line** in the second view.

\[ \mathbf{Ex} = \ell' \]
Motivation

The Essential Matrix is a 3 x 3 matrix that encodes \textit{epipolar geometry}.

Given a point in one image, multiplying by the \textbf{essential matrix} will tell us the \textbf{epipolar line} in the second view.
Epipolar Line

\[ ax + by + c = 0 \]

in vector form

\[ \mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \]

If the point \( \mathbf{x} \) is on the epipolar line \( \mathbf{l} \) then

\[ \mathbf{x}^\top \mathbf{l} = ? \]
Epipolar Line

\[ ax + by + c = 0 \]

in vector form

\[
\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}
\]

If the point \( \mathbf{x} \) is on the epipolar line \( \mathbf{l} \) then

\[
\mathbf{x}^\top \mathbf{l} = 0
\]
Recall: Dot Product

\[ c = a \times b \]

\[ c \cdot a = 0 \quad c \cdot b = 0 \]

dot product of two orthogonal vectors is zero
vector representing the line is normal (orthogonal) to the plane

\[ l = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \]

vector representing the point \( x \) is inside the plane

Therefore:

\[ x^\top l = 0 \]
So if \( x^\top l = 0 \) and \( Ex = l' \) then

\[ x'^\top Ex = ? \]
So if $x^\top l = 0$ and $Ex = l'$ then

$$x'^\top Ex = 0$$
Essential Matrix vs Homography

What’s the difference between the essential matrix and a homography?
Essential Matrix vs Homography

What’s the difference between the essential matrix and a homography?

They are both 3 x 3 matrices but …

\[ l' = Ex \]

Essential matrix maps a point to a line

\[ x' = Hx \]

Homography maps a point to a point
Where does the Essential matrix come from?
\[ x' = R(x - t) \]
$x' = R(x - t)$

Does this look familiar?
Camera-camera transform just like world-camera transform
These three vectors are coplanar

\( x, t, x' \)
If these three vectors are coplanar \( x, t, x' \) then

\[
x^\top (t \times x) = ?
\]
If these three vectors are coplanar \( x, t, x' \) then

\[
x^\top (t \times x) = 0
\]
Recall: Cross Product

Vector (cross) product
takes two vectors and returns a vector perpendicular to both

\[ c = a \times b \]

\[ c \cdot a = 0 \quad c \cdot b = 0 \]
If these three vectors are coplanar \( \mathbf{x}, \mathbf{t}, \mathbf{x}' \) then

\[
(\mathbf{x} - \mathbf{t})^\top (\mathbf{t} \times \mathbf{x}) = ?
\]
If these three vectors are coplanar $x, t, x'$ then

$$(x - t)^\top (t \times x) = 0$$
putting it together

rigid motion

\[ x' = R(x - t) \]

coplanarity

\[ (x - t)^\top (t \times x) = 0 \]

\[ (x'^\top R)(t \times x) = 0 \]
Cross product

\[ \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \]

Can also be written as a matrix multiplication

\[ \mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \]

Skew symmetric
putting it together

rigid motion

\[ x' = R(x - t) \]

coplanarity

\[ (x - t)^\top (t \times x) = 0 \]

\[ (x'^\top R)(t \times x) = 0 \]

\[ (x'^\top R)([t \times] x) = 0 \]
putting it together

rigid motion

\[ x' = R(x - t) \]

coplanarity

\[ (x - t)^\top (t \times x) = 0 \]

\[ (x'\top R)(t \times x) = 0 \]

\[ (x'\top R)([t_\times]x) = 0 \]

\[ x'\top (R[t_\times])x = 0 \]
putting it together

rigid motion

\[ x' = R(x - t) \]

coplanarity

\[ (x - t)^\top (t \times x) = 0 \]

\[ (x'\top R)(t \times x) = 0 \]

\[ (x'\top R)([t \times] x) = 0 \]

\[ x'\top (R[t \times]) x = 0 \]

\[ x'\top E x = 0 \]
putting it together

rigid motion

\[ x' = R(x - t) \]

coplanarity

\[ (x - t)^\top (t \times x) = 0 \]

\[ (x'^\top R)(t \times x) = 0 \]

\[ (x'^\top R)([t \times] x) = 0 \]

\[ x'^\top (R[t \times]) x = 0 \]

\[ x'^\top E x = 0 \]

Essential Matrix

[Longuet-Higgins 1981]
properties of the E matrix

Longuet-Higgins equation

\[ x' ^ \top Ex = 0 \]

(points in normalized coordinates)
properties of the E matrix

Longuet-Higgins equation

\[ x'^\top E x = 0 \]

Epipolar lines

\[ x^\top l = 0 \quad x'^\top l' = 0 \]

\[ l' = Ex \quad l = E^T x' \]

(points in normalized coordinates)
properties of the E matrix

Longuet-Higgins equation

\[ \mathbf{x}'^\top \mathbf{E} \mathbf{x} = 0 \]

Epipolar lines

\[ \mathbf{x}^\top \mathbf{l} = 0 \quad \mathbf{x}'^\top \mathbf{l}' = 0 \]
\[ \mathbf{l}' = \mathbf{E} \mathbf{x} \quad \mathbf{l} = \mathbf{E}^T \mathbf{x}' \]

Epipoles

\[ \mathbf{e}'^\top \mathbf{E} = 0 \quad \mathbf{E} \mathbf{e} = 0 \]

(points in normalized camera coordinates)
Recall: Epipolar constraint

Potential matches for $x$ lie on the epipolar line $l'$.
Given a point in one image, multiplying by the **essential matrix** will tell us the **epipolar line** in the second view.

\[
Ex = l'
\]

**Assumption:**
points aligned to camera coordinate axis (calibrated camera)
How do you generalize to uncalibrated cameras?
The fundamental matrix
The **Fundamental matrix** is a *generalization* of the **Essential matrix**, where the assumption of calibrated cameras is removed.
\[ \hat{x}'^\top E \hat{x} = 0 \]

The Essential matrix operates on image points expressed in **normalized coordinates**
(points have been aligned (normalized) to camera coordinates)

\[ \hat{x}' = K^{-1} x' \]
\[ \hat{x} = K^{-1} x \]

camera point
image point
The Essential matrix operates on image points expressed in **normalized coordinates** (points have been aligned (normalized) to camera coordinates)

\[
\hat{x}'^\top E \hat{x} = 0
\]

Writing out the epipolar constraint in terms of image coordinates

\[
\hat{x}' = K^{-1} x' \\
\hat{x} = K^{-1} x
\]

camera point  
image point

\[
x'^\top K'^-^\top E K^{-1} x = 0
\]

\[
x'^\top (K'^-^\top E K^{-1}) x = 0
\]

\[
x'^\top F x = 0
\]
Same equation works in image coordinates!

\[ x'\top Fx = 0 \]

it maps pixels to epipolar lines
properties of the $E$ matrix

Longuet-Higgins equation

$\mathbf{x}'^\top \mathbf{E} \mathbf{x} = 0$

Epipolar lines

$\mathbf{x}^\top \mathbf{l} = 0$

$\mathbf{l}' = \mathbf{E} \mathbf{x}$

$\mathbf{x}'^\top \mathbf{l}' = 0$

$\mathbf{l} = \mathbf{E}^\top \mathbf{x}'$

Epipoles

$\mathbf{e}'^\top \mathbf{E} = 0$

$\mathbf{E} \mathbf{e} = 0$

(points in image coordinates)
Breaking down the fundamental matrix

\[
F = K'^{-T} E K^{-1}
\]

\[
F = K'^{-T} [t_x] R K^{-1}
\]

Depends on both intrinsic and extrinsic parameters
Breaking down the fundamental matrix

\[
F = K'^{-\top} EK^{-1}
\]

\[
F = K'^{-\top} [t_x]RK^{-1}
\]

 Depends on both intrinsic and extrinsic parameters

*How would you solve for \( F \)?*

\[
x'_m \top F x_m = 0
\]
The 8-point algorithm
Assume you have $M$ matched image points

$$\begin{align*}
\{x_m, x'_m\} \quad m = 1, \ldots, M
\end{align*}$$

Each correspondence should satisfy

$$x'_m^T F x_m = 0$$

How would you solve for the 3 x 3 $F$ matrix?
Assume you have $M$ matched image points

$$\{x_m, x'_m\} \quad m = 1, \ldots, M$$

Each correspondence should satisfy

$$x'_m^\top F x_m = 0$$

*How would you solve for the $3 \times 3 F$ matrix?*

\[ S \quad V \quad D \]
Assume you have $M$ matched image points

$$\{x_m, x'_m\} \quad m = 1, \ldots, M$$

Each correspondence should satisfy

$$x'_m^\top F x_m = 0$$

How would you solve for the 3 x 3 $F$ matrix?

Set up a homogeneous linear system with 9 unknowns
How many equation do you get from one correspondence?
\[
\begin{bmatrix}
    x'_m & y'_m & 1
\end{bmatrix}
\begin{bmatrix}
    f_1 & f_2 & f_3 \\
    f_4 & f_5 & f_6 \\
    f_7 & f_8 & f_9
\end{bmatrix}
\begin{bmatrix}
    x_m \\
    y_m \\
    1
\end{bmatrix} = 0
\]

ONE correspondence gives you ONE equation

\[
x_m x'_m f_1 + x_m y'_m f_2 + x_m f_3 +
\]

\[
y_m x'_m f_4 + y_m y'_m f_5 + y_m f_6 +
\]

\[
x'_m f_7 + y'_m f_8 + f_9 = 0
\]
Set up a homogeneous linear system with 9 unknowns

\[
\begin{bmatrix}
    x'_m & y'_m & 1
\end{bmatrix}
\begin{bmatrix}
    f_1 & f_2 & f_3 \\
    f_4 & f_5 & f_6 \\
    f_7 & f_8 & f_9
\end{bmatrix}
\begin{bmatrix}
    x_m \\
    y_m \\
    1
\end{bmatrix} = 0
\]

How many equations do you need?
Each point pair (according to epipolar constraint) contributes only one scalar equation

$$x_m'^T F x_m = 0$$

**Note:** This is different from the Homography estimation where each point pair contributes 2 equations.

We need at least 8 points

**Hence, the 8 point algorithm!**
How do you solve a homogeneous linear system?

$$\mathbf{A}\mathbf{X} = 0$$
How do you solve a homogeneous linear system?

\[ AX = 0 \]

Total Least Squares

minimize \( \|Ax\|^2 \)

subject to \( \|x\|^2 = 1 \)
How do you solve a homogeneous linear system?

\[ AX = 0 \]

Total Least Squares

minimize \[ \| Ax \|^2 \]

subject to \[ \| x \|^2 = 1 \]

S V D !
Eight-Point Algorithm

0. (Normalize points)
1. Construct the M x 9 matrix $\mathbf{A}$
2. Find the SVD of $\mathbf{A}$
3. Entries of $\mathbf{F}$ are the elements of column of $\mathbf{V}$ corresponding to the least singular value
4. (Enforce rank 2 constraint on $\mathbf{F}$)
5. (Un-normalize $\mathbf{F}$)
Eight-Point Algorithm

0. (Normalize points)

1. Construct the M x 9 matrix \( A \)

2. Find the SVD of \( A \)

3. Entries of \( F \) are the elements of column of \( V \) corresponding to the least singular value

4. (Enforce rank 2 constraint on \( F \))

5. (Un-normalize \( F \))

See Hartley-Zisserman for why we do this
Eight-Point Algorithm

0. (Normalize points)
1. Construct the M x 9 matrix $A$
2. Find the SVD of $A$
3. Entries of $F$ are the elements of column of $V$
   corresponding to the least singular value
4. (Enforce rank 2 constraint on $F$)
5. (Un-normalize $F$)
Eight-Point Algorithm

0. (Normalize points)
1. Construct the M x 9 matrix $\mathbf{A}$
2. Find the SVD of $\mathbf{A}$
3. Entries of $\mathbf{F}$ are the elements of column of $\mathbf{V}$ corresponding to the least singular value
4. (Enforce rank 2 constraint on $\mathbf{F}$)
5. (Un-normalize $\mathbf{F}$)

How do we do this? $\text{SVD}$!
Enforcing rank constraints

Problem: Given a matrix $F$, find the matrix $F'$ of rank $k$ that is closest to $F$,

$$\min_{F'} \|F - F'\|^2 \quad \text{rank}(F') = k$$

Solution: Compute the singular value decomposition of $F$,

$$F = U\Sigma V^T$$

Form a matrix $\Sigma'$ by replacing all but the $k$ largest singular values in $\Sigma$ with 0.

Then the problem solution is the matrix $F'$ formed as,

$$F' = U\Sigma' V^T$$
Eight-Point Algorithm

0. (Normalize points)
1. Construct the M x 9 matrix \( A \)
2. Find the SVD of \( A \)
3. Entries of \( F \) are the elements of column of \( V \) corresponding to the least singular value
4. (Enforce rank 2 constraint on \( F \))
5. (Un-normalize \( F \))
Example
epipolar lines
\[
F = \begin{bmatrix}
-0.00310695 & -0.0025646 & 2.96584 \\
-0.028094 & -0.00771621 & 56.3813 \\
13.1905 & -29.2007 & -9999.79
\end{bmatrix}
\]

\[
x = \begin{bmatrix}
343.53 \\
221.70 \\
1.0
\end{bmatrix}
\]

\[
l' = Fx = \begin{bmatrix}
0.0295 \\
0.9996 \\
-265.1531
\end{bmatrix}
\]
\[ l' = Fx \]

\[
\begin{bmatrix}
0.0295 \\
0.9996 \\
-265.1531
\end{bmatrix}
\]
Where is the epipole?

How would you compute it?
The epipole is in the right null space of $F$

$$Fe = 0$$

*How would you solve for the epipole?*
The epipole is in the right null space of $\mathbf{F}$

$\mathbf{F} \mathbf{e} = 0$

How would you solve for the epipole?

S V D!
References

Basic reading:
• Szeliski textbook, Section 8.1 (not 8.1.1-8.1.3), Chapter 11, Section 12.2.
• Hartley and Zisserman, Section 11.12.