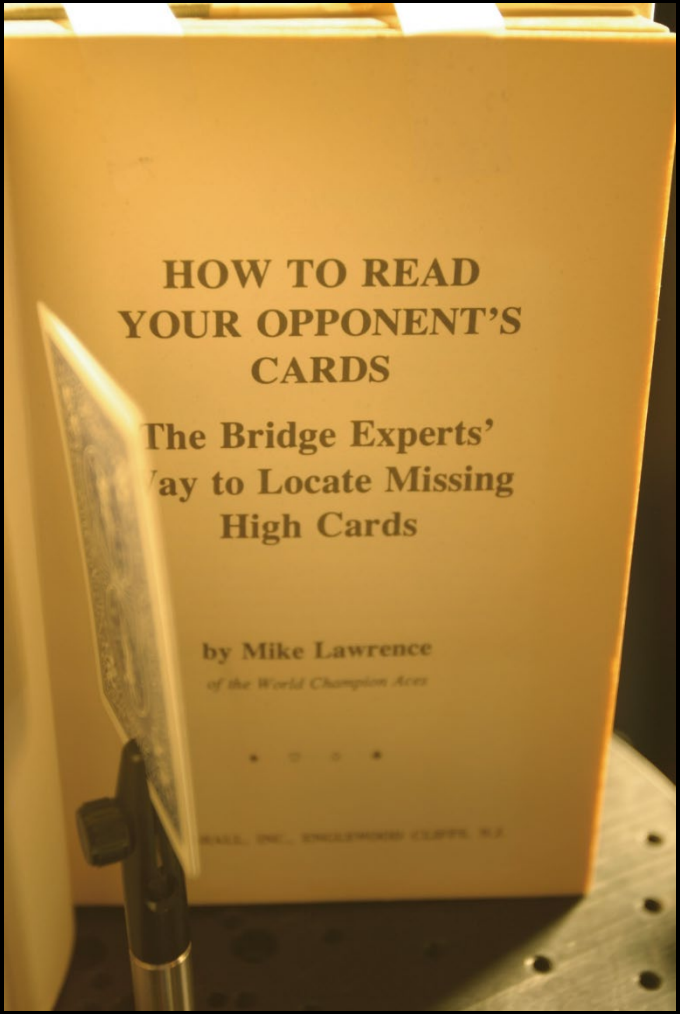
Light transport matrices



15-463, 15-663, 15-862
Computational Photography
Fall 2023, Lecture 21

http://graphics.cs.cmu.edu/courses/15-463

Course announcements

- Homework assignment 6 is tomorrow.
    - Any questions?

- Final project logistics posted on course website.
    - Make sure to read the details.

Overview of today’s lecture

- The light transport matrix.
- Image-based relighting.
- Optical computing using the light transport matrix.
- Dual photography.

Slide credits

These slides were directly adapted from:

- Matt O’Toole (CMU).

The light transport matrix

photo with lights 1 & 2 turned on

photo with light 1 turned on

photo with light 2 turned on

How do these three images relate to each other?

the superposition principle

photo with lights 1 & 2 turned on

=

photo with light 1 turned on

+

photo with light 2 turned on

photo taken under two light sources =
sum of photos taken under each source individually

the superposition principle


Synthetic photo

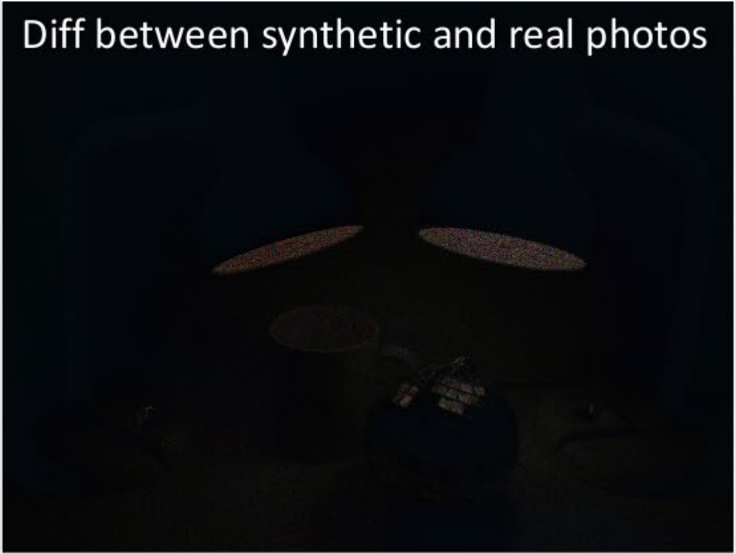

Real photo

photo taken under two light sources =
sum of photos taken under each source individually

the superposition principle

why is the error not exactly zero?

Synthetic photo



Diff between synthetic and real photos

photo taken under two light sources =
sum of photos taken under each source individually

image-based relighting

Relit photo

=

photo with light 1 turned on

photo with light 2 turned on

image-based relighting

Relit photo

=

photo with light 1 turned on

Weight 1

x 1

+

photo with light 2 turned on

Weight 2

x 1

image-based relighting

Relit photo

=

photo with light 1 turned on

Weight 1

x $\boxed{1}$

+

photo with light 2 turned on

Weight 2

x $\boxed{0}$

Relit photo $\mathbf{p}$

$=$

photo with light 1 turned on $\mathbf{T_1}$

Weight 1

x $\boxed{\mathbf{l_1}}$

$+$

photo with light 2 turned on $\mathbf{T_2}$

Weight 2

x $\boxed{\mathbf{l_2}}$

Relit photo $\mathbf{p}$

$=$

photo with light 1 turned on $\mathbf{T_1}$

Weight 1

x $\boxed{\mathbf{l_1}}$ $+$

photo with light 2 turned on $\mathbf{T_2}$

Weight 2

x $\boxed{\mathbf{l_2}}$

$$\boxed{\mathbf{p}} = \sum_{i=1}^{2} \boxed{\mathbf{T}_i} \times \boxed{\mathbf{l}_i}$$


Relit photo $\mathbf{p}$

=


photo with light 1 turned on $\mathbf{T_1}$

Weight 1

x $\boxed{\mathbf{l_1}}$

+


photo with light 2 turned on $\mathbf{T_2}$

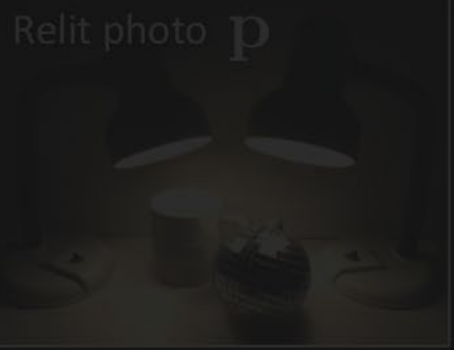
Weight 2

x $\boxed{\mathbf{l_2}}$

n

$$\boxed{\mathbf{p}} = \sum_{i=1}^{2} \boxed{\mathbf{T}_i} \times \boxed{\mathbf{l}_i}$$

n pixel values



Relit photo $\mathbf{p}$

=

photo with light 1 turned on $\mathbf{T_1}$

Weight 1

x $\boxed{\mathbf{l}_1}$ +

photo with light 2 turned on $\mathbf{T_2}$

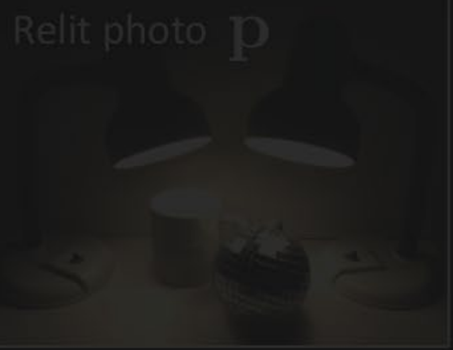
Weight 2

x $\boxed{\mathbf{l}_2}$

$$\overset{\uparrow}{\underset{\downarrow}{n}}\ \boxed{\mathbf{p}} = \sum_{i=1}^{2} \boxed{\mathbf{T}_i} \times \boxed{\mathbf{l}_i}$$

n pixel values

Relit photo $\mathbf{p}$



=

photo with light 1 turned on $\mathbf{T_1}$

Weight 1

x $\boxed{\mathbf{l_1}}$

+

photo with light 2 turned on $\mathbf{T_2}$

Weight 2

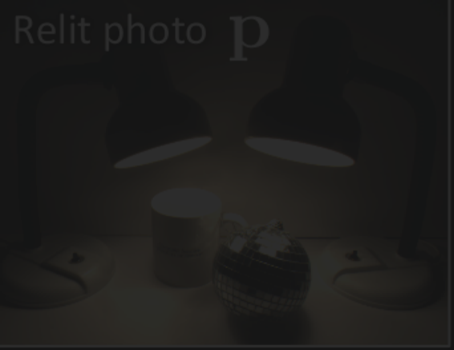
x $\boxed{\mathbf{l_2}}$

$$\mathbf{p} = \sum_{i=1}^{2} \mathbf{T}_i \times \mathbf{l}_i$$

n

n pixel values

Contribution of the source



Relit photo $\mathbf{p}$

$=$

photo with light 1 turned on $\mathbf{T_1}$

Weight 1

x $\boxed{\mathbf{l_1}}$ $+$

photo with light 2 turned on $\mathbf{T_2}$

Weight 2

x $\boxed{\mathbf{l_2}}$

Number of controllable sources

Contribution of each source

$$n \quad \mathbf{p} = \sum_{i=1}^{2} \mathbf{T}_i \times \boxed{\mathbf{l}_i}$$

n pixel values

Relit photo $\mathbf{p}$

=

photo with light 1 turned on $\mathbf{T_1}$

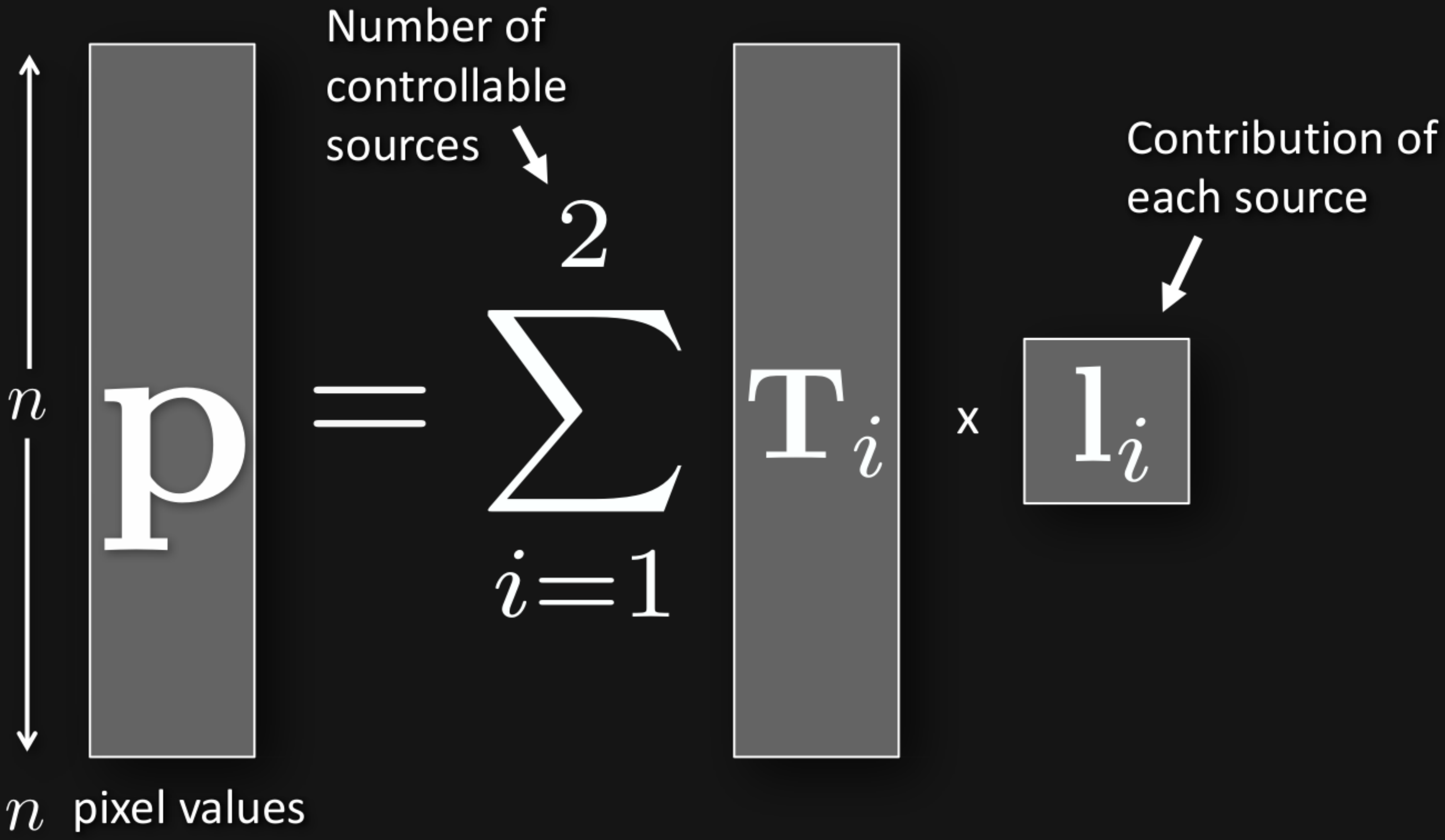
Weight 1

x $\boxed{\mathbf{l_1}}$ +

photo with light 2 turned on $\mathbf{T_2}$

Weight 2

x $\boxed{\mathbf{l_2}}$



Number of controllable sources

Contribution of each source

$$\mathbf{p} = \sum_{i=1}^{2} \mathbf{T}_i \times \mathbf{l}_i$$

n pixel values

Relit photo $\mathbf{p}$

$=$

photo with light 1 turned on $\mathbf{T}_1$

Weight 1 x $\mathbf{l}_1$ $+$

photo with light 2 turned on $\mathbf{T}_2$

Weight 2 x $\mathbf{l}_2$

Number of controllable sources

Contribution of each source

$$\overset{\uparrow}{\underset{\downarrow}{n}}\ \boxed{\mathbf{p}} = \sum_{i=1}^{②} \boxed{\mathbf{T}_i} \times \boxed{\mathbf{l}_i}$$

n pixel values

Relit photo $\mathbf{p}$

$=$

photo with light 1 turned on $\mathbf{T_1}$

Weight 1

x $\mathbf{l_1}$ $+$

photo with light 2 turned on $\mathbf{T_2}$

Weight 2

x $\mathbf{l_2}$

Number of controllable sources

Contribution of each source

$$n \quad \mathbf{p} = \sum_{i=1}^{m} \mathbf{T}_i \times \mathbf{l}_i$$

n pixel values

Relit photo $\mathbf{p}$

$=$

photo with light 1 turned on $\mathbf{T_1}$

Weight 1

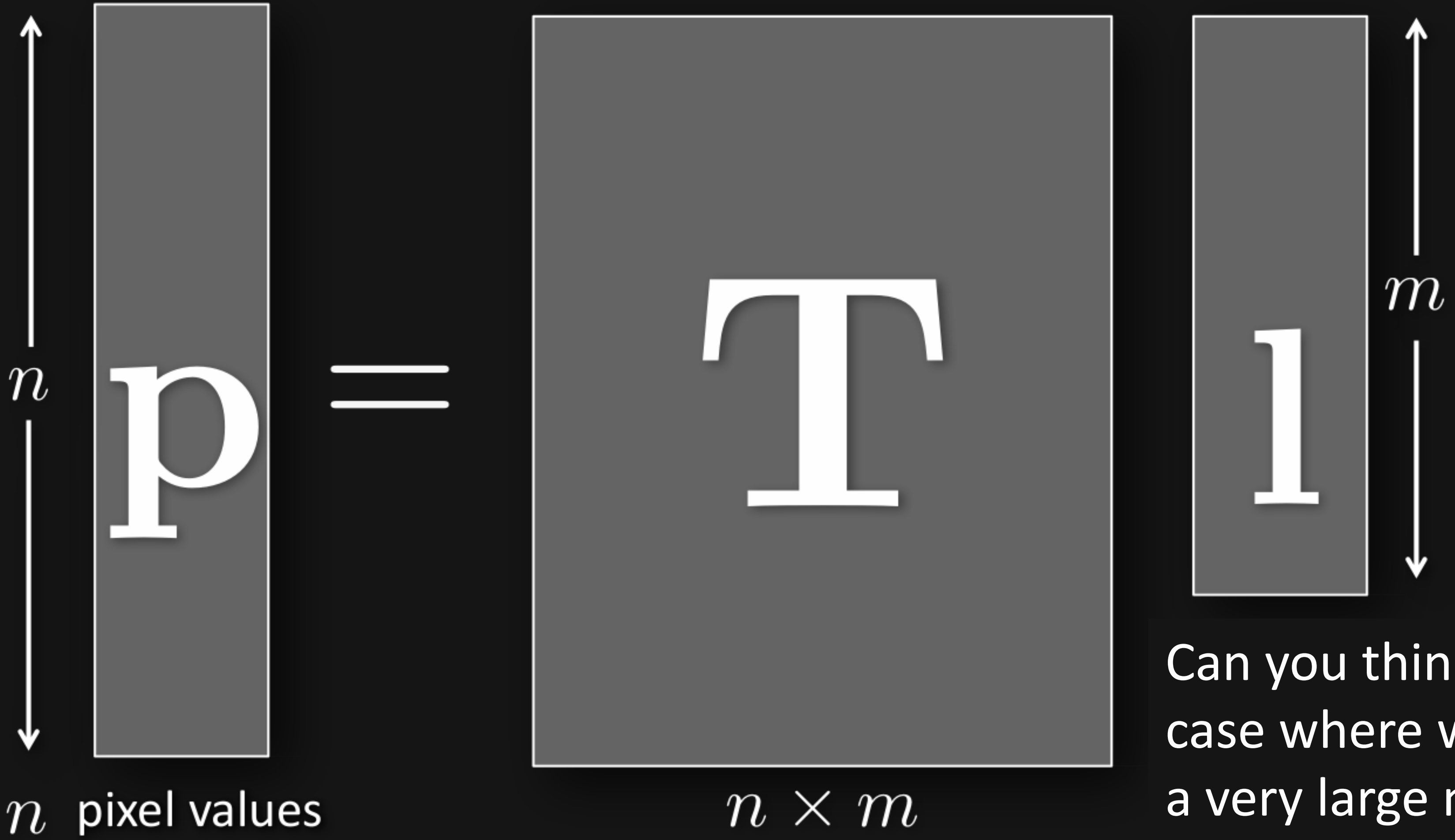
x $\boxed{\mathbf{l_1}}$ $+$

photo with light 2 turned on $\mathbf{T_2}$

Weight 2

x $\boxed{\mathbf{l_2}}$

light transport matrix



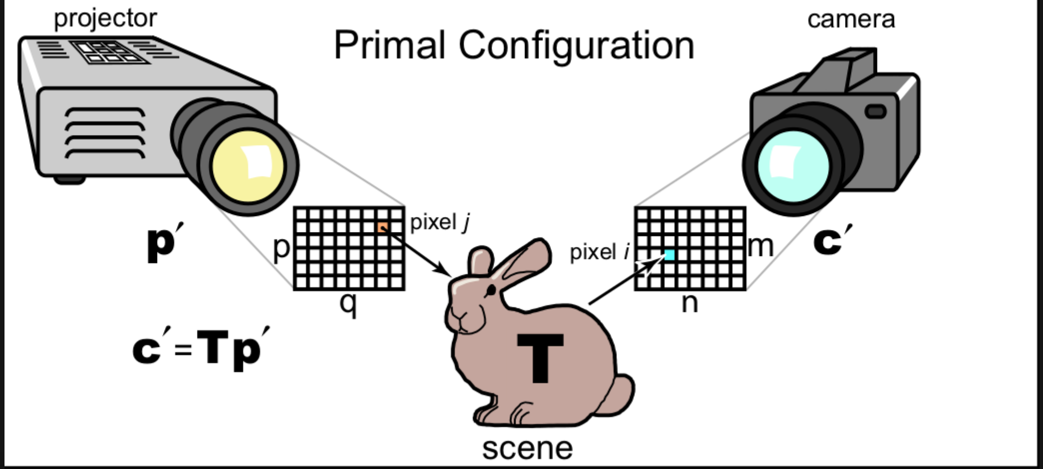
$$\mathbf{p} = \mathbf{T} \, \mathbf{l}$$

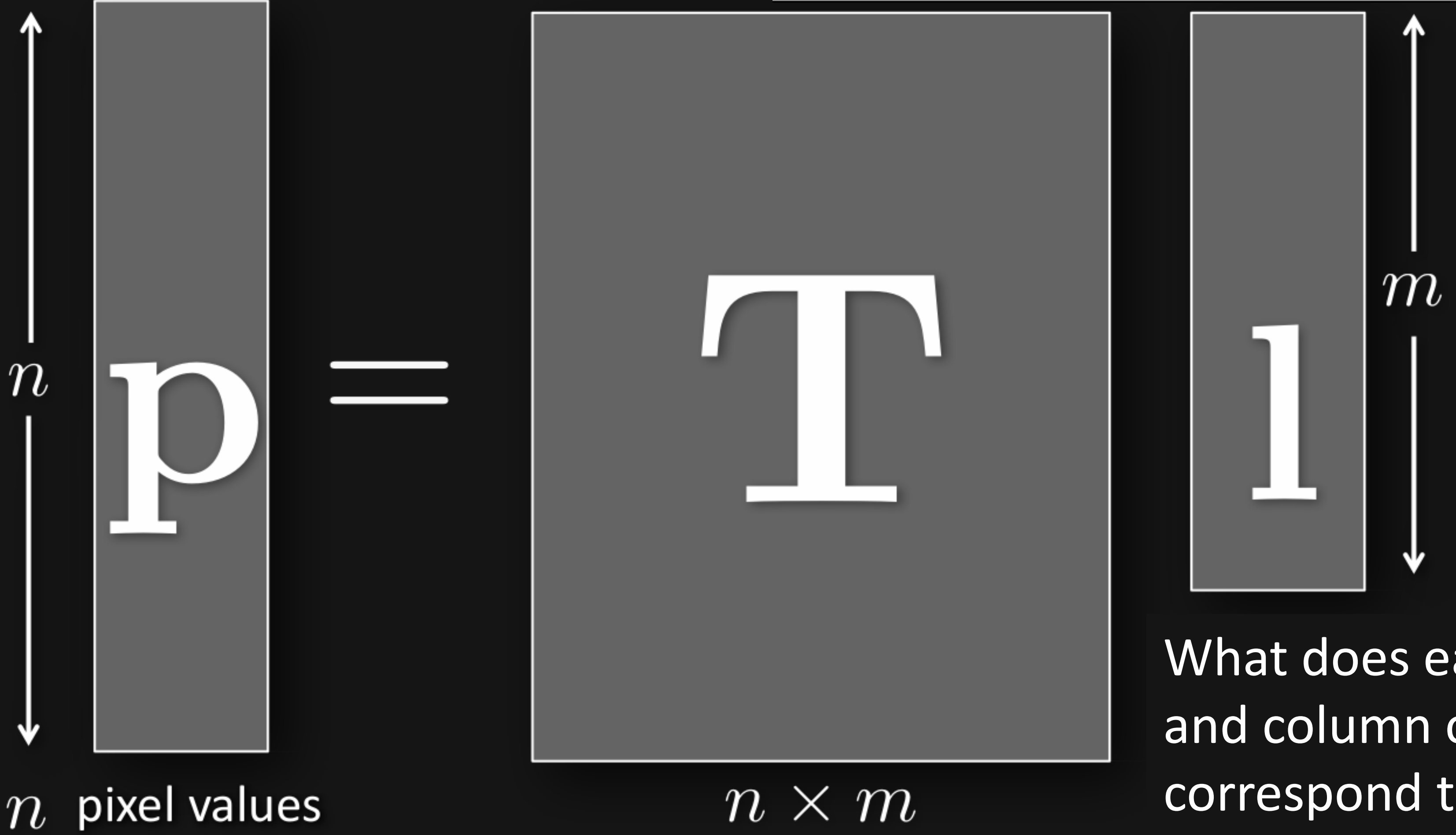
n pixel values

$n \times m$

Can you think of a case where we have a very large m?

Use a projector





What does each row and column of T correspond to here?

Image-based relighting

Let’s say I have measured T.

- What does it mean to right-multiply it with some vector l?

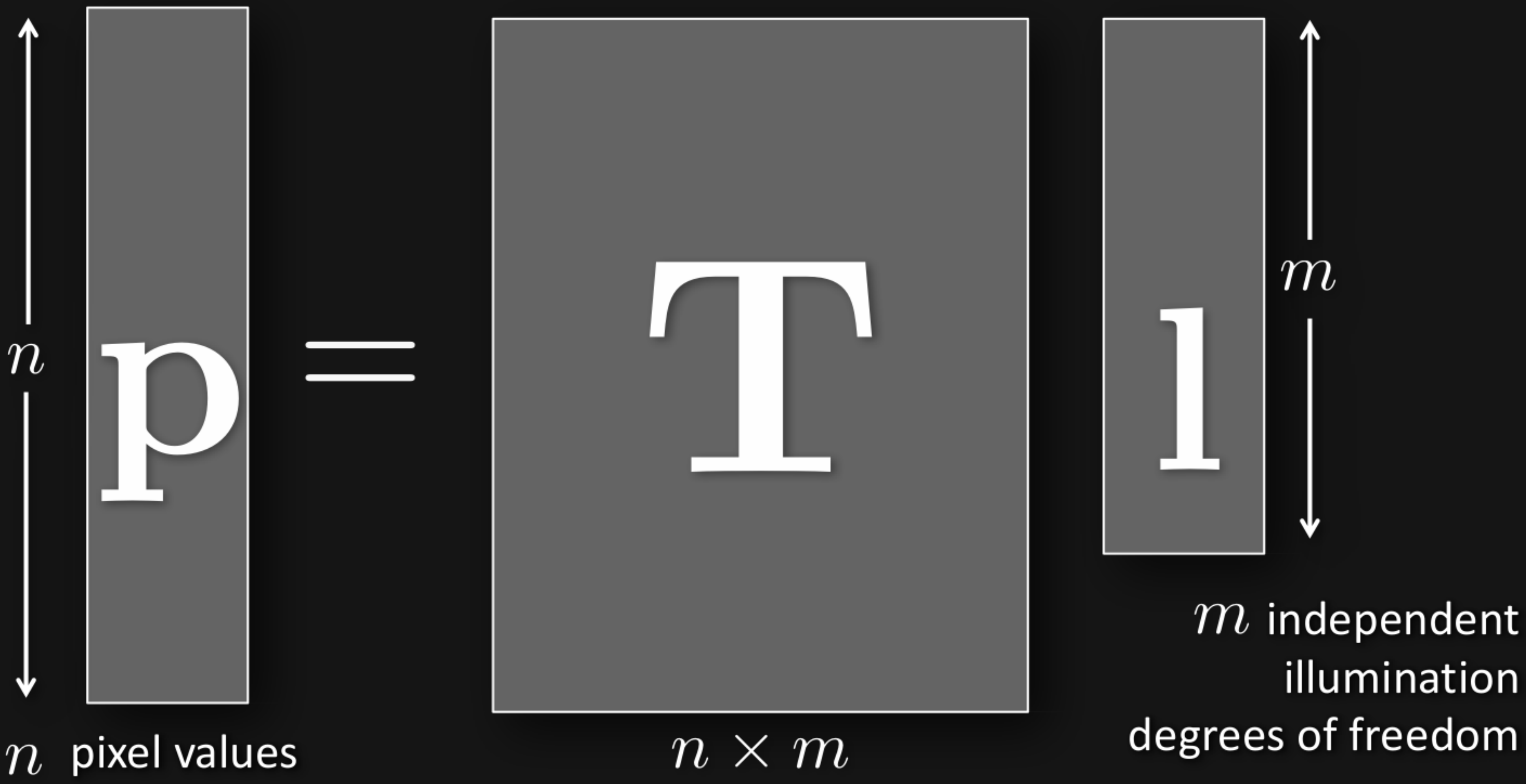
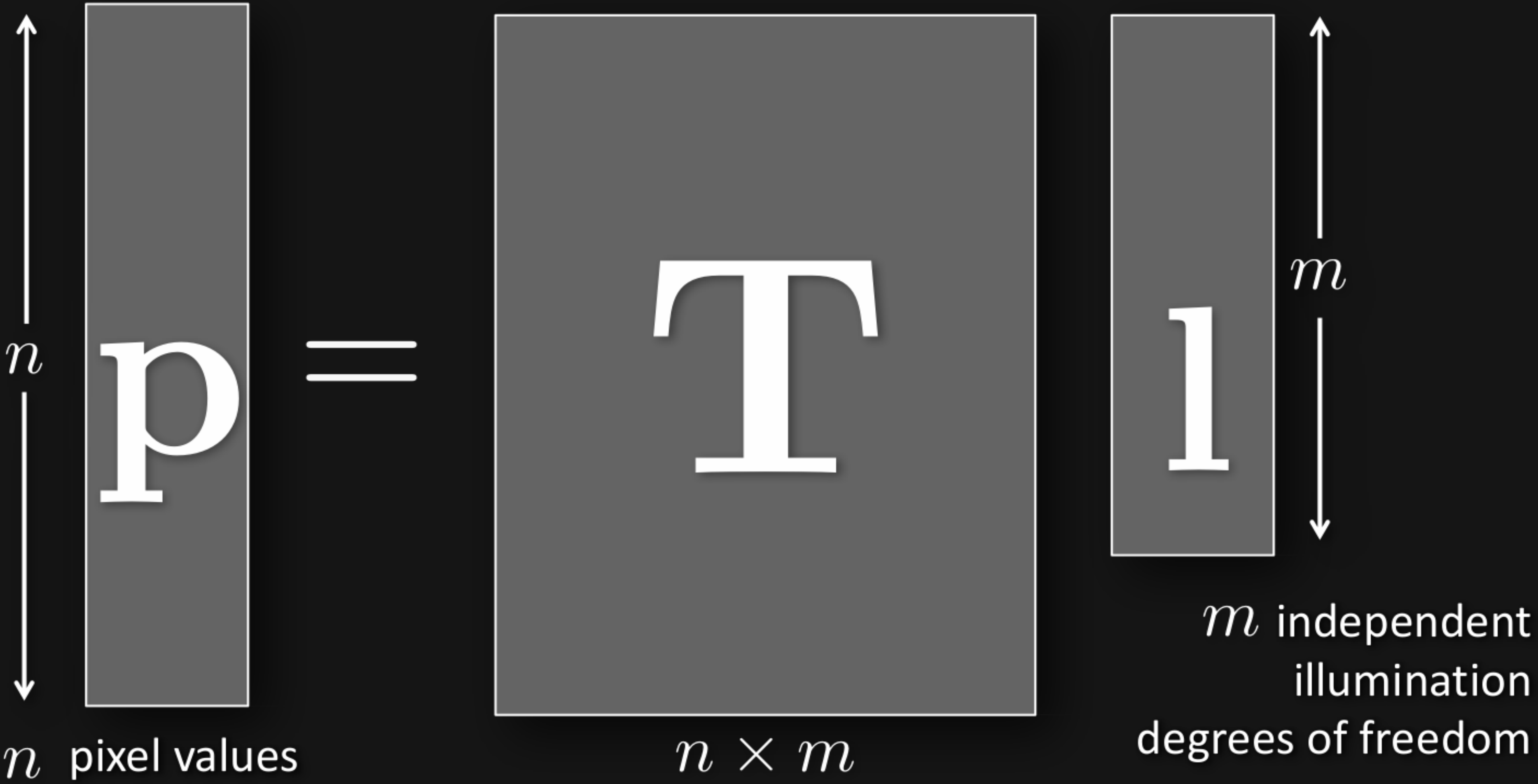


Image-based relighting: Use the measurements we already have of the scene (the pictures I took when measuring T) to simulate new illuminations of the scene.



$$\mathbf{p} = \mathbf{T}\,\mathbf{l}$$

n pixel values

$n \times m$

m independent illumination degrees of freedom

Acquiring the Reflectance Field [Debevec et al. 2000]

image-based rendering & relighting

Light stage 1.0

Reflectance field

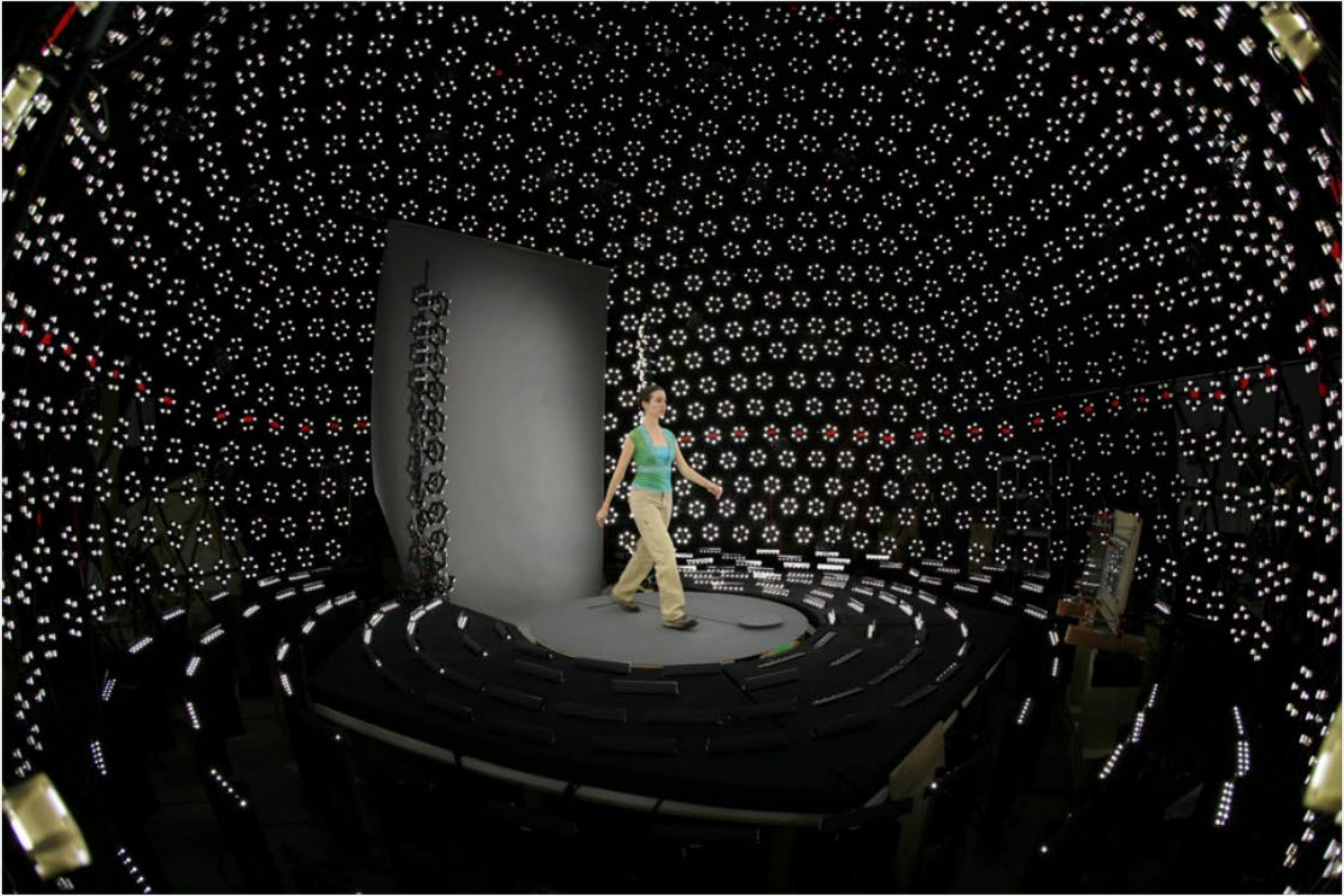
Acquiring the Reflectance Field

image-based rendering & relighting

Debevec et al, SIG 2000

Great demonstration: https://www.youtube.com/watch?v=mkzLLz1tXds

Acquiring the Reflectance Field

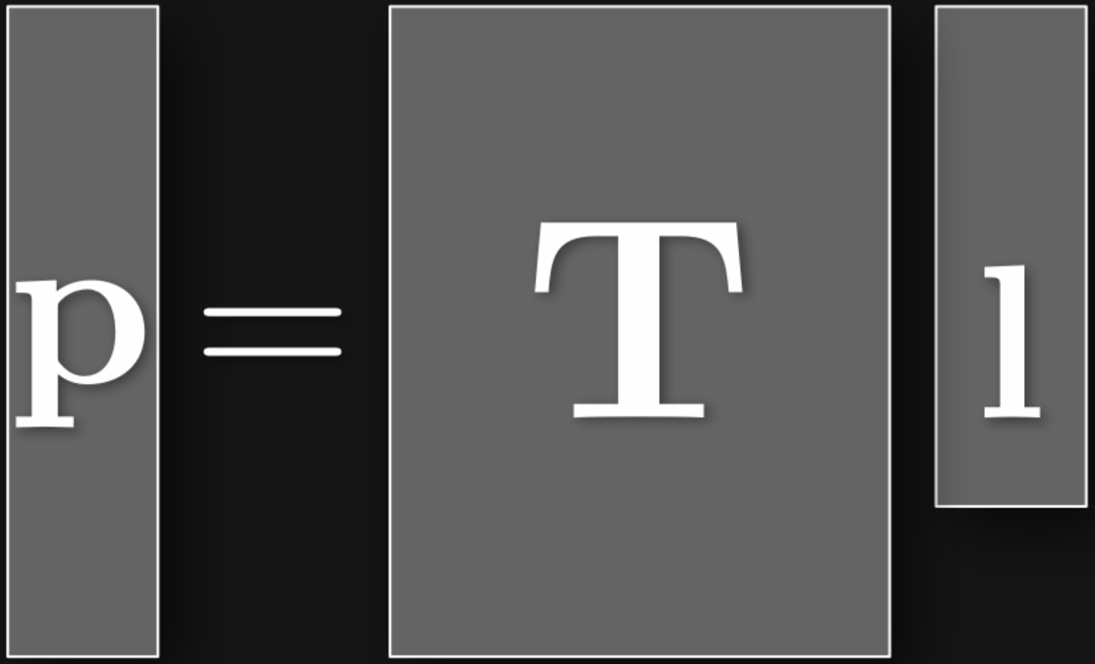


Light stage 6, Debevec et al., 2006

Optical computing using the light transport matrix

main difficulties

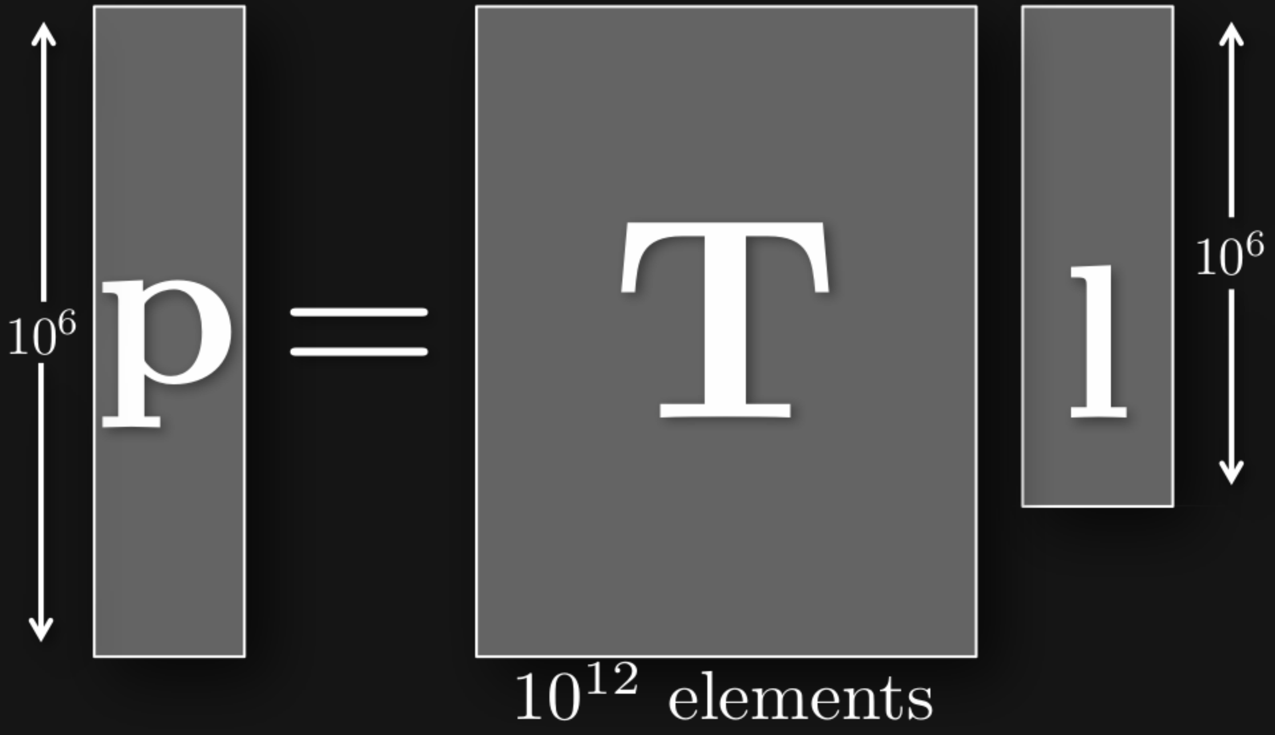
question: what are the challenges with analyzing $\mathbf{T}$?



$$\mathbf{p} = \mathbf{T}\mathbf{1}$$

main difficulties

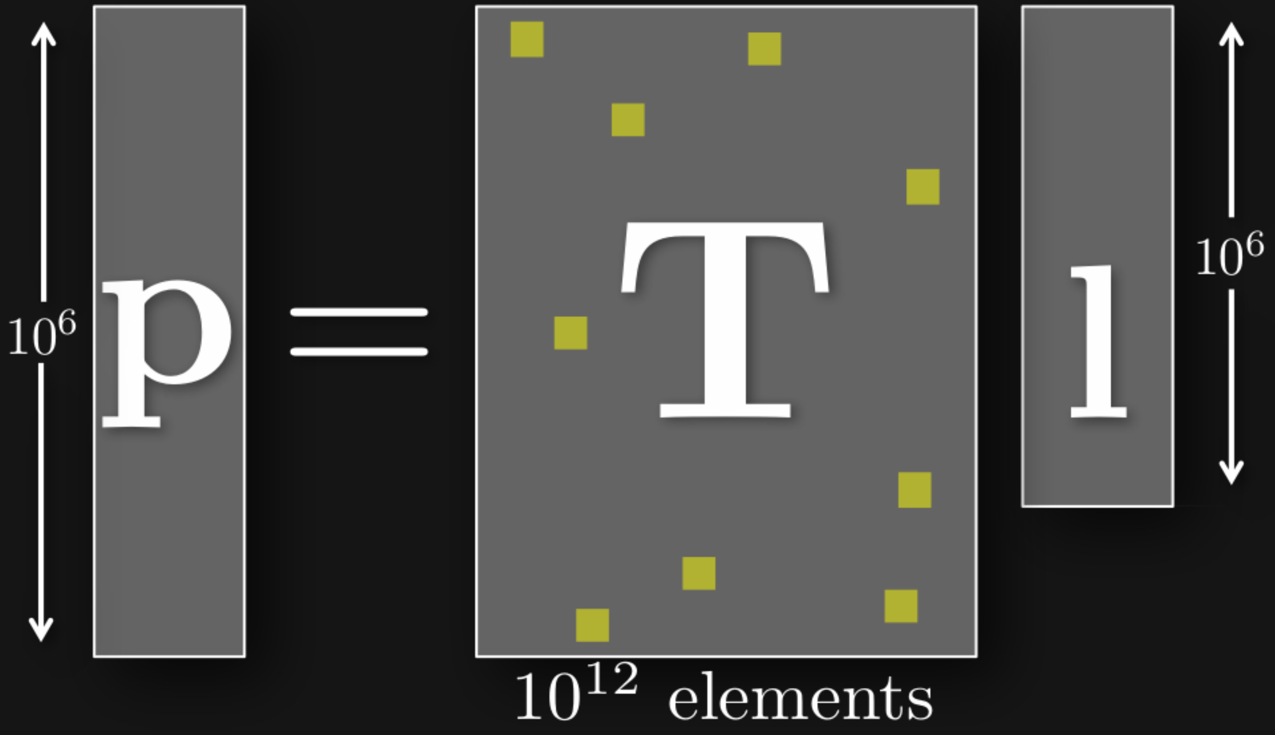
question: what are the challenges with analyzing $\mathbf{T}$?



- matrix can be extremely large

main difficulties

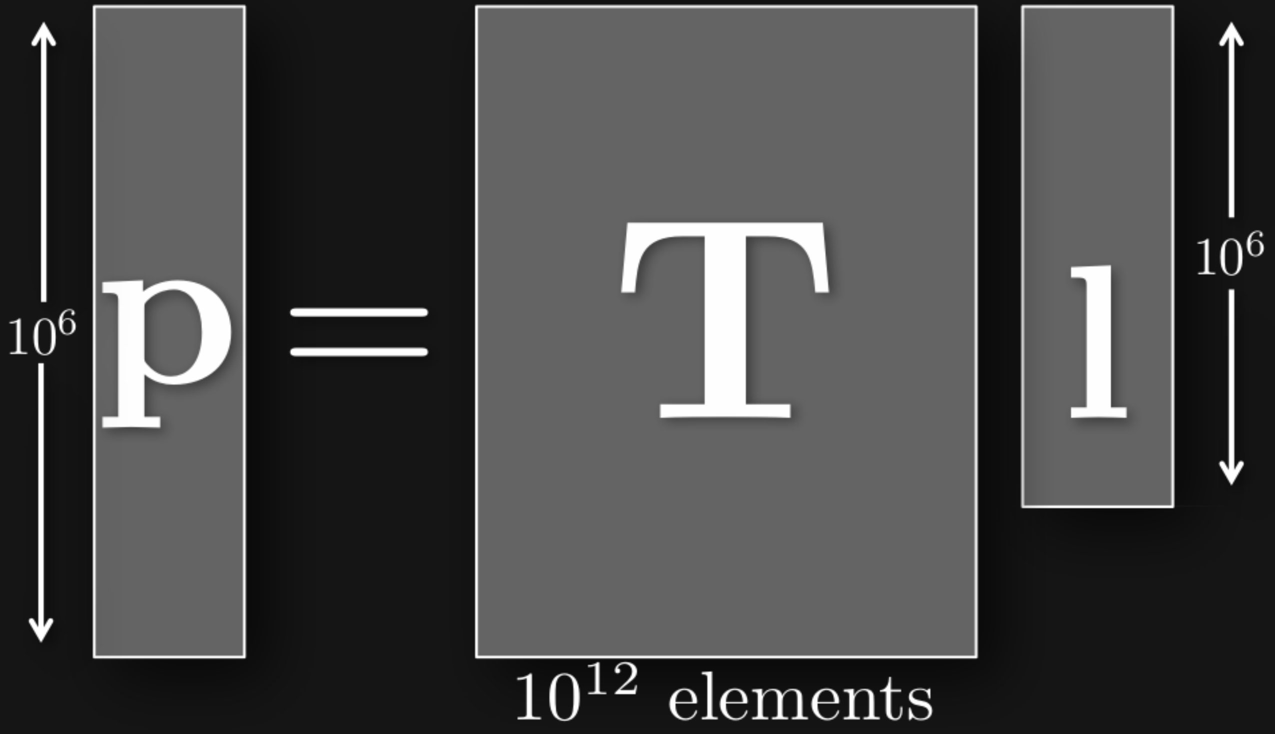
question: what are the challenges with analyzing $\mathbf{T}$?



- matrix can be extremely large
- elements not directly accessible

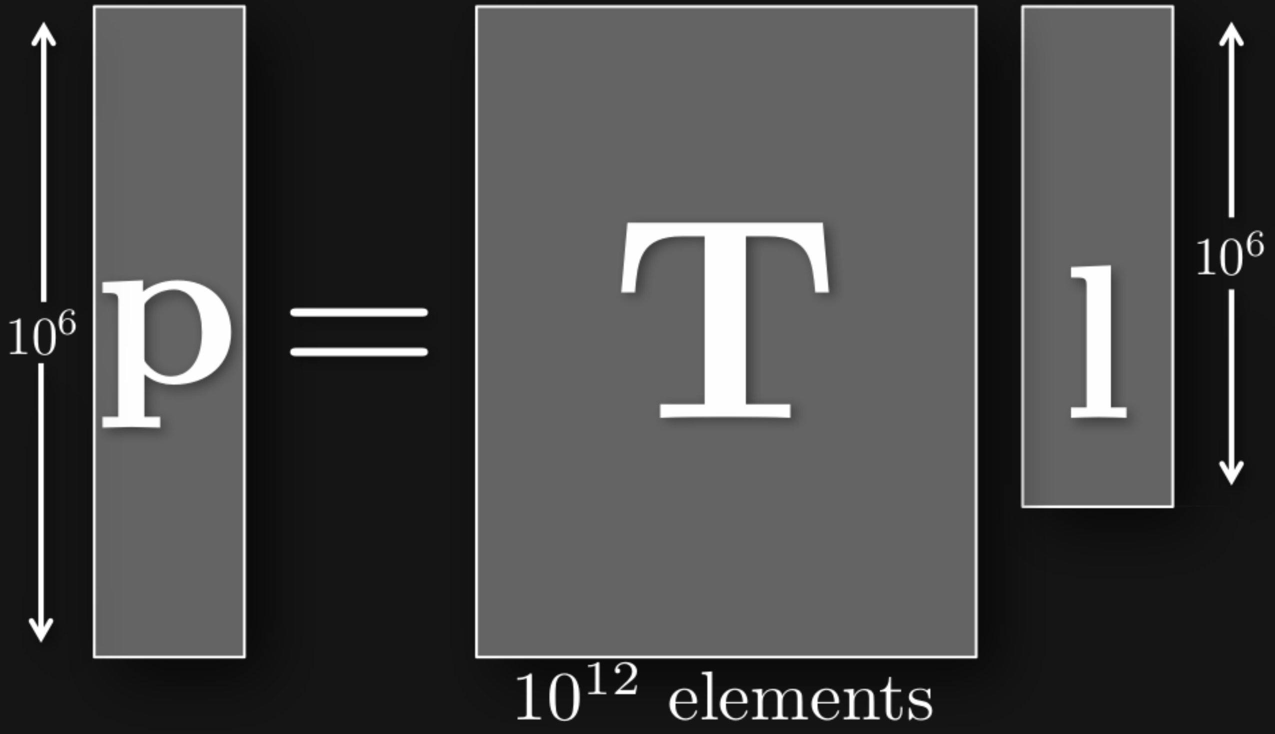
main difficulties

question: what are the challenges with analyzing $\mathbf{T}$?



- matrix can be extremely large
- elements not directly accessible
- global structure poorly understood

How would you measure the light transport matrix T?



10^6

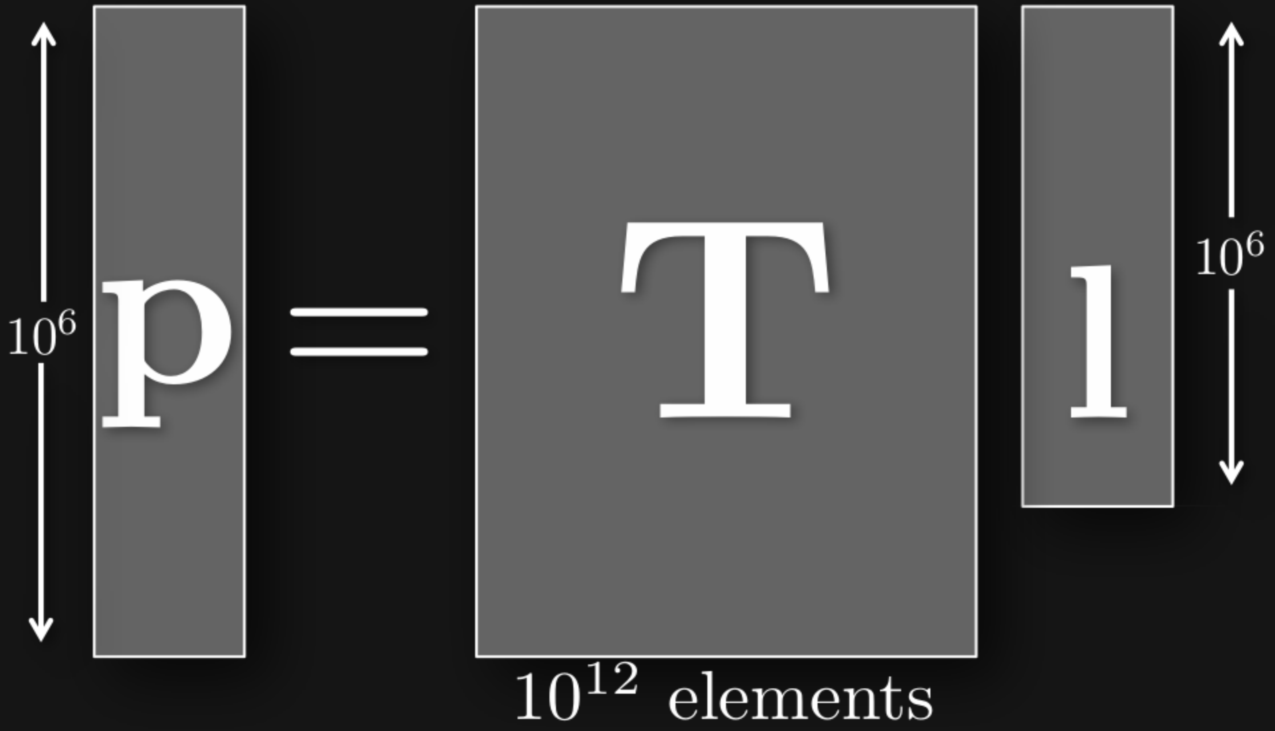
$$\mathbf{p} = \mathbf{T}\,\mathbf{l}$$

10^6

10^{12} elements

- matrix can be extremely large
- elements not directly accessible
- global structure poorly understood

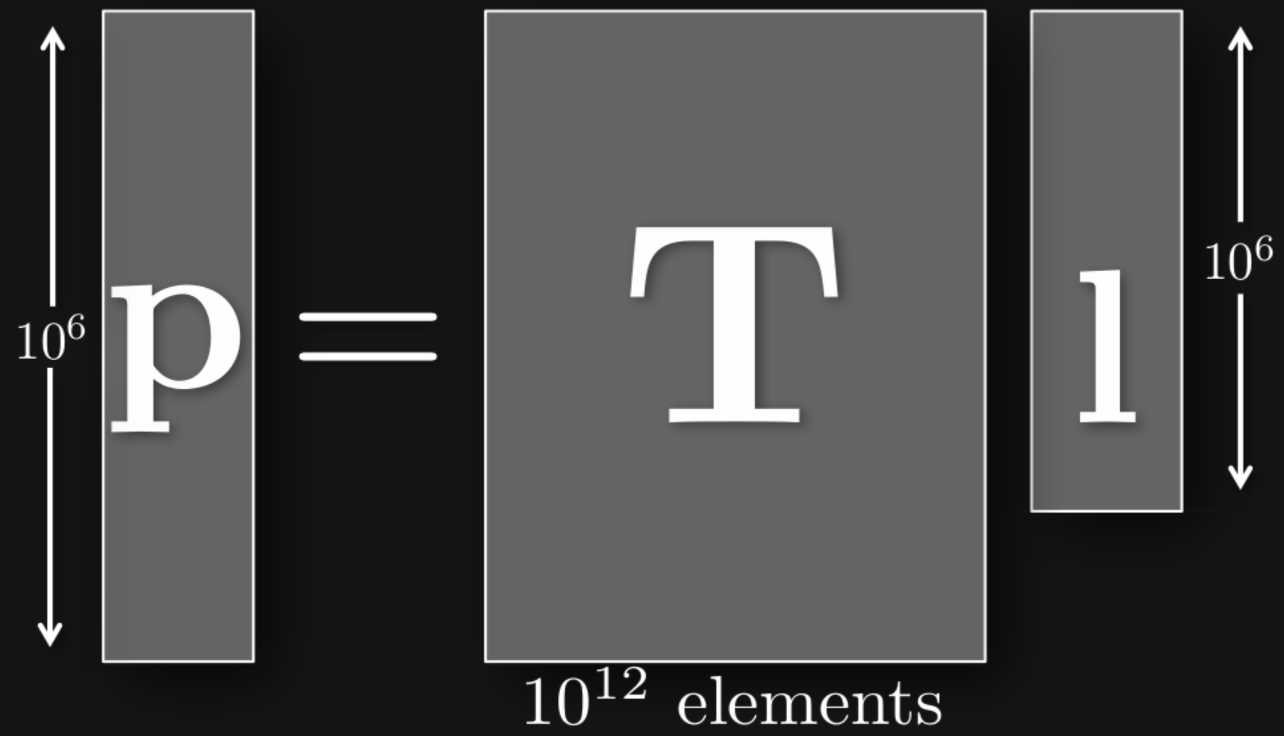
How would you measure the light transport matrix T?



Exhaustive/naïve approach: turn on projector pixels one at a time and take a photo for each of them.

- What does each photo correspond to in T?

How would you measure the light transport matrix T?



Exhaustive/naïve approach: turn on projector pixels one at a time and take a photo for each of them.

- How many photos do we need to capture?

computing with light

numerical algorithms implemented directly in optics

numerical domain

$$\underset{\substack{\uparrow\\\text{photo}}}{\mathbf{p}} = \overset{\substack{\text{transport}\\\text{matrix}\\\downarrow}}{\mathbf{T}}\;\underset{\substack{\uparrow\\\text{illumination}\\\text{vector}}}{\mathbf{l}}$$

⟺

optical domain

computing with light

numerical algorithms implemented directly in optics

numerical domain

transport matrix

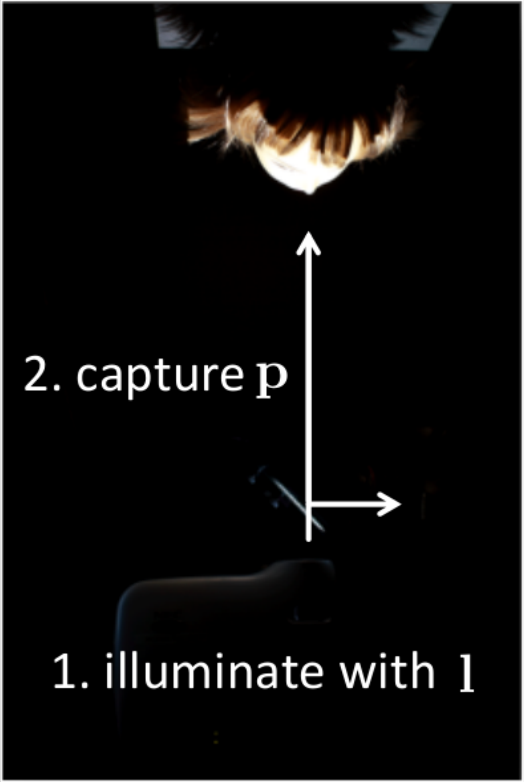
$$\mathbf{p} = \mathbf{T}\,\mathbf{l}$$

photo

illumination vector

$\Longleftrightarrow$

optical domain



computing with light

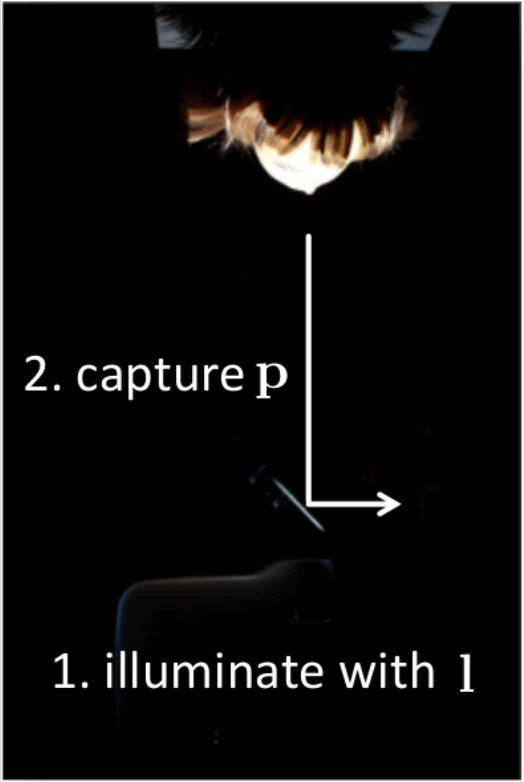
numerical algorithms implemented directly in optics

numerical domain

```
function analyze($\mathbf{T}$)
. . .
for $i = 1$ to $k$ {
    . . .
    $\boxed{\mathbf{p}_i = \mathbf{T}\mathbf{l}_i}$
    . . .
    $\boxed{\mathbf{d}_i = \mathbf{T}\mathbf{r}_i}$
    . . .
}
. . .
return result
```

⟺

optical domain



computing with light

numerical algorithms implemented directly in optics

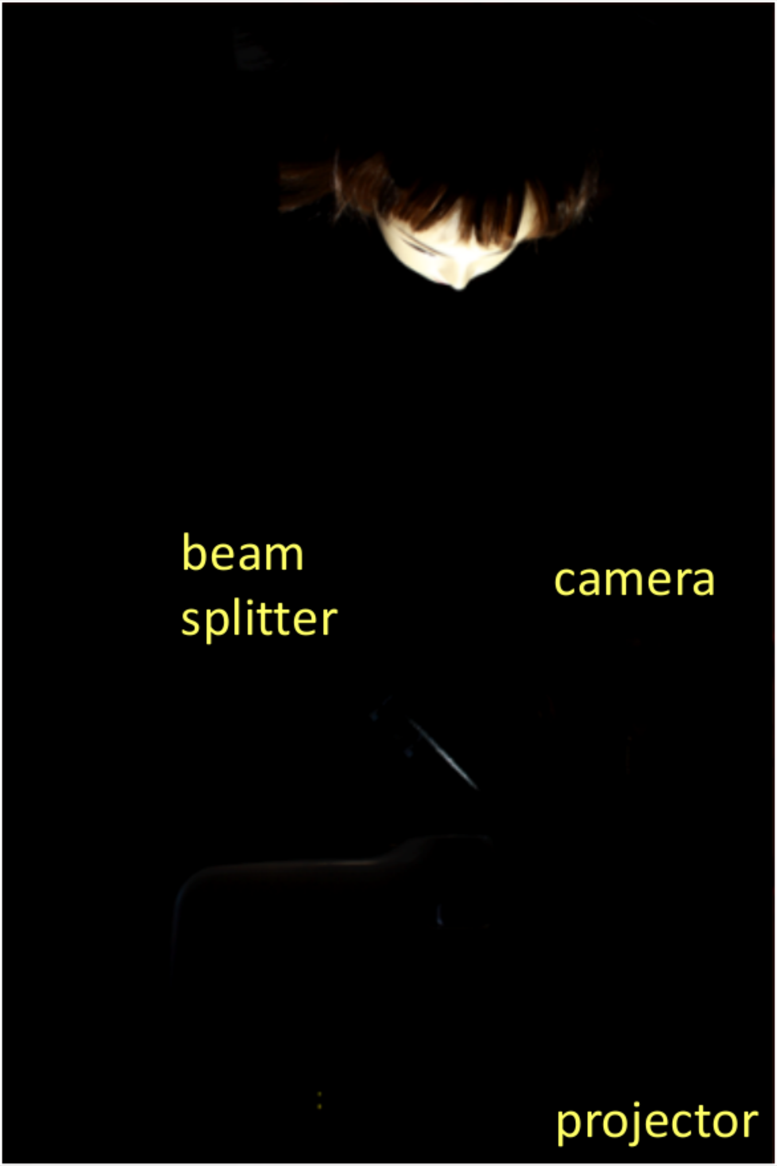
numerical domain

```
function analyze($\boxed{(\mathbf{T})}$)
. . .
for $i = 1$ to $k$ {
    . . .
    $\boxed{\mathbf{p}_i = \mathbf{T}\mathbf{l}_i}$
    . . .
    $\boxed{\mathbf{d}_i = \mathbf{T}\mathbf{r}_i}$
    . . .
}
. . .
return result
```

$$\Longleftrightarrow$$

optical domain

```
function analyze($\boxed{()}$)
. . .
for $i = 1$ to $k$ {
    . . .
    $\boxed{\text{project } \mathbf{l}_i\text{, capture } \mathbf{p}_i}$
    . . .
    $\boxed{\text{project } \mathbf{r}_i\text{, capture } \mathbf{d}_i}$
    . . .
}
. . .
return result
```



find an illumination pattern that

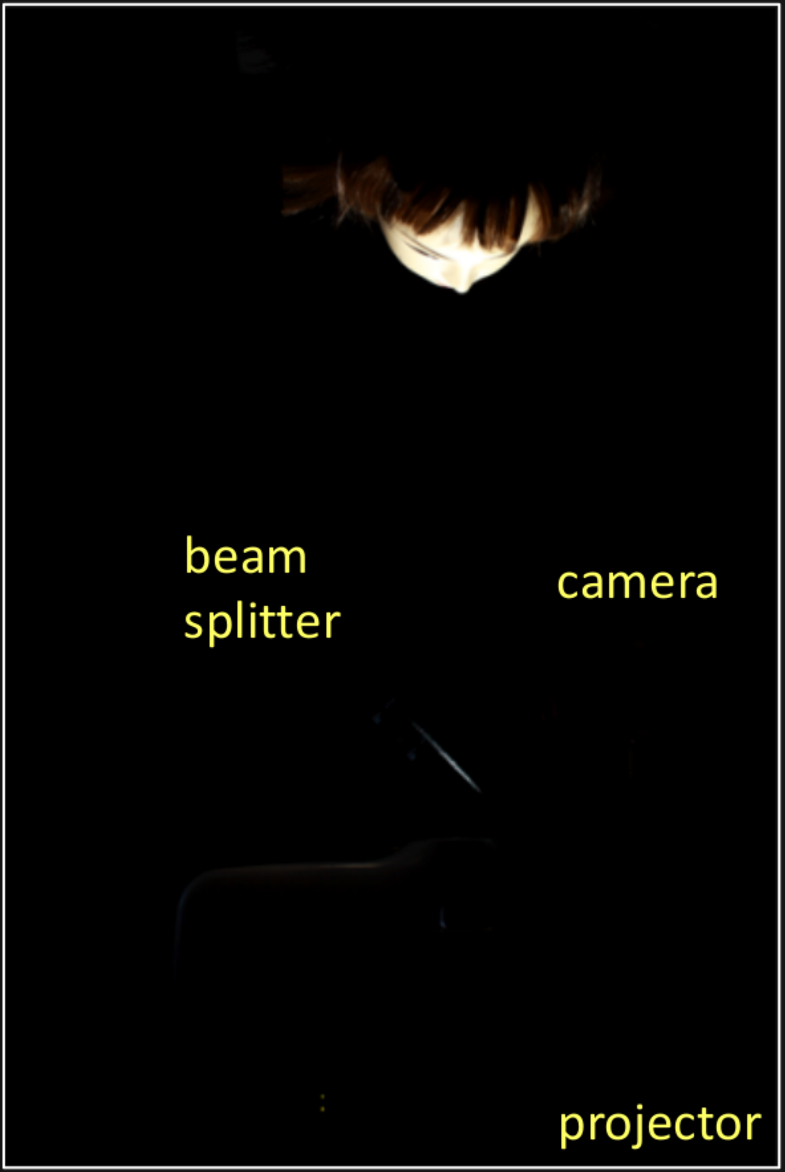
when projected onto scene,
we get the same photo back
(multiplied by a scalar)

project

capture

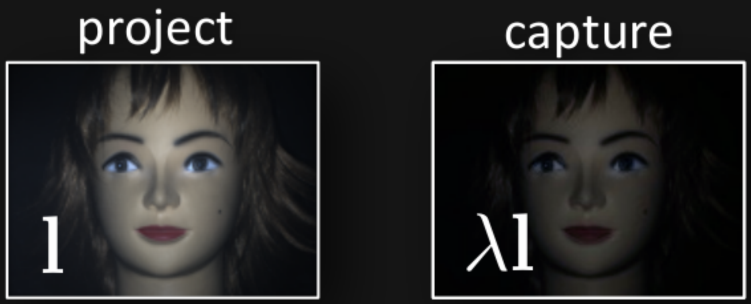
What do we call these patterns?

computing transport eigenvectors



eigenvector of a square matrix T

when projected onto scene, we get the same photo back (multiplied by a scalar)



numerical goal

find $\mathbf{l}, \lambda$ such that

$$\mathbf{T}\mathbf{l} = \lambda\mathbf{l}$$

and λ is maximal

optical power iteration

goal: find principal eigenvector of $\mathbf{T}$

observation: it is a fixed point of the sequence $\mathbf{l}, \mathbf{T}\mathbf{l}, \mathbf{T}^2\mathbf{l}, \mathbf{T}^3\mathbf{l}, \ldots$

numerical domain

```
function PowerIt($\mathbf{T}$)

$\boxed{\mathbf{l}_1 = \text{initial vector}}$

for $i = 1$ to $k$ {
    $\boxed{\mathbf{p}_i = \mathbf{T}\mathbf{l}_i}$

    $\boxed{\mathbf{l}_{i+1} = \mathbf{p}_i / \|\mathbf{p}_i\|_2}$
}

return $\mathbf{l}_{i+1}$
```

properties

- linear convergence [Trefethen and Bau 1997]
- eigenvalues must be distinct
- $\mathbf{l}_1$ cannot be orthogonal to principal eigenvector

optical power iteration

goal: find principal eigenvector of $\mathbf{T}$

observation: it is a fixed point of the sequence $\mathbf{l}, \mathbf{T}\mathbf{l}, \mathbf{T}^2\mathbf{l}, \mathbf{T}^3\mathbf{l}, \ldots$

numerical domain

```
function PowerIt($\mathbf{T}$)

$\mathbf{l}_1$ = initial vector

for $i = 1$ to $k$ {
    $\boxed{\mathbf{p}_i = \mathbf{T}\mathbf{l}_i}$

    $\mathbf{l}_{i+1} = \mathbf{p}_i / \|\mathbf{p}_i\|_2$
}

return $\mathbf{l}_{i+1}$
```

$\Longleftrightarrow$

optical domain

```
function PowerIt()

$\mathbf{l}_1$ = initial vector

for $i = 1$ to $k$ {
    $\boxed{\text{project } \mathbf{l}_i\text{, capture } \mathbf{p}_i}$

    $\mathbf{l}_{i+1} = \mathbf{p}_i / \|\mathbf{p}_i\|_2$
}

return $\mathbf{l}_{i+1}$
```

optical power iteration

goal: find principal eigenvector of $\mathbf{T}$

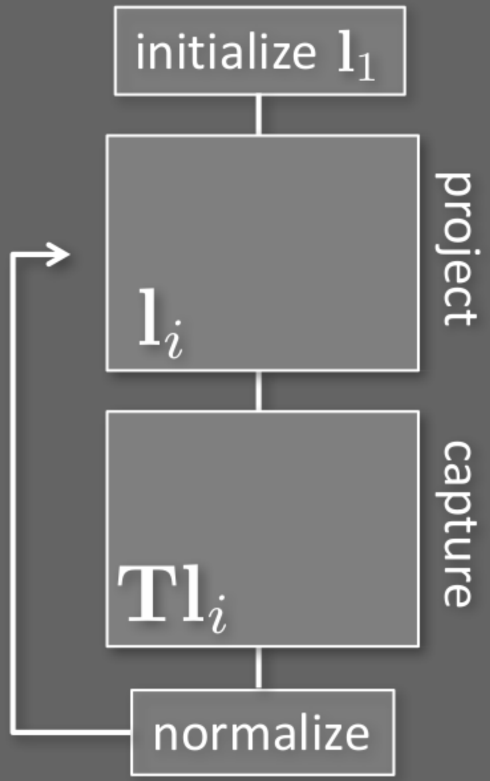
observation: it is a fixed point of the sequence $\mathbf{l}, \mathbf{T}\mathbf{l}, \mathbf{T}^2\mathbf{l}, \mathbf{T}^3\mathbf{l}, \ldots$

numerical domain

```
function PowerIt($\mathbf{T}$)

$\mathbf{l}_1$ = initial vector

for $i = 1$ to $k$ {
    $\boxed{\mathbf{p}_i = \mathbf{T}\mathbf{l}_i}$

    $\mathbf{l}_{i+1} = \mathbf{p}_i / \|\mathbf{p}_i\|_2$
}

return $\mathbf{l}_{i+1}$
```

$\Longleftrightarrow$

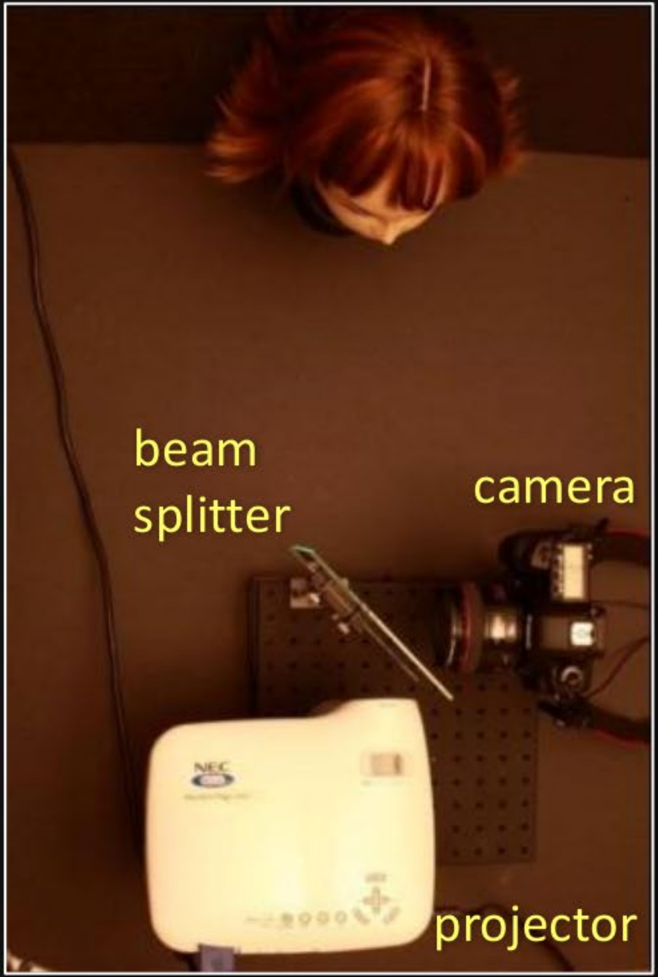
optical domain

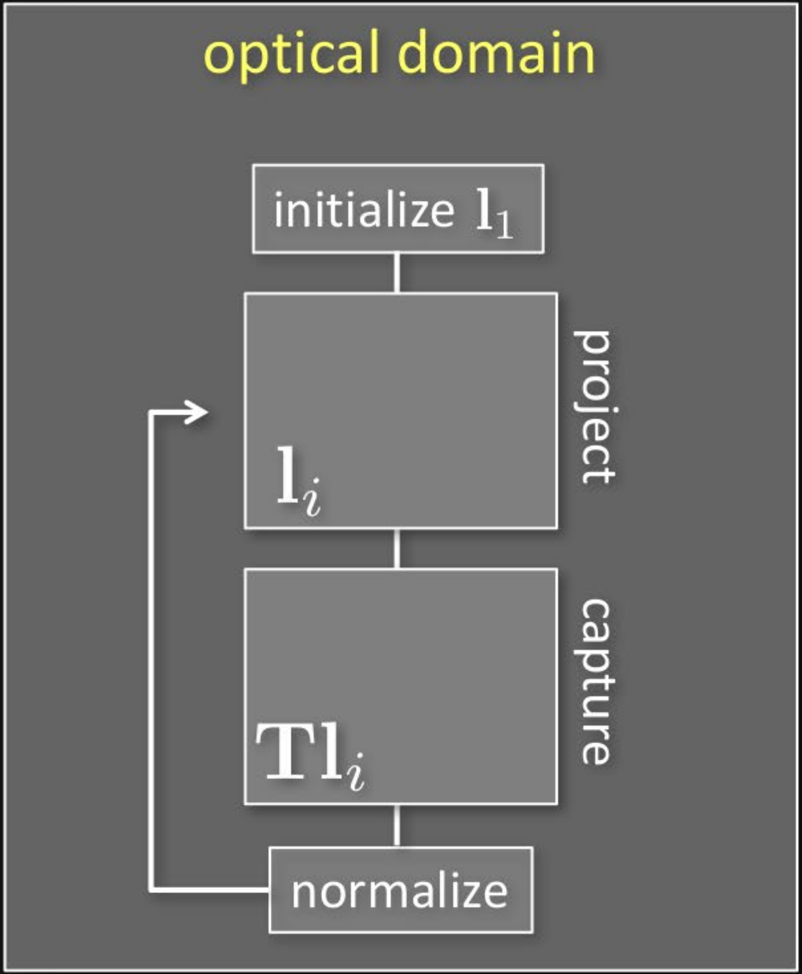


optical power iteration

goal: find principal eigenvector of $\mathbf{T}$

observation: it is a fixed point of the sequence $\mathbf{l}, \mathbf{T}\mathbf{l}, \mathbf{T}^2\mathbf{l}, \mathbf{T}^3\mathbf{l}, \ldots$

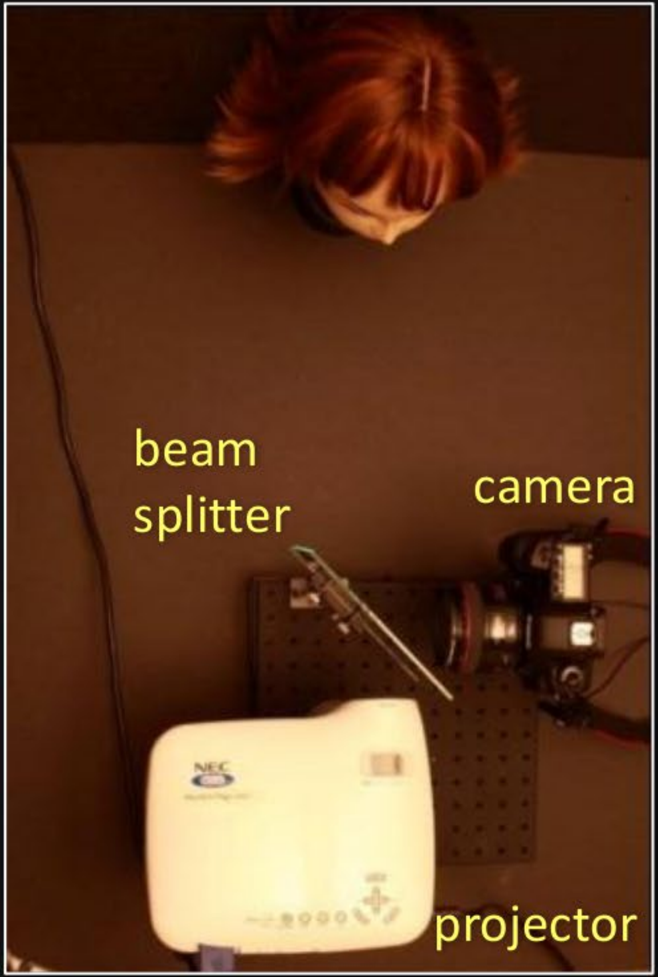


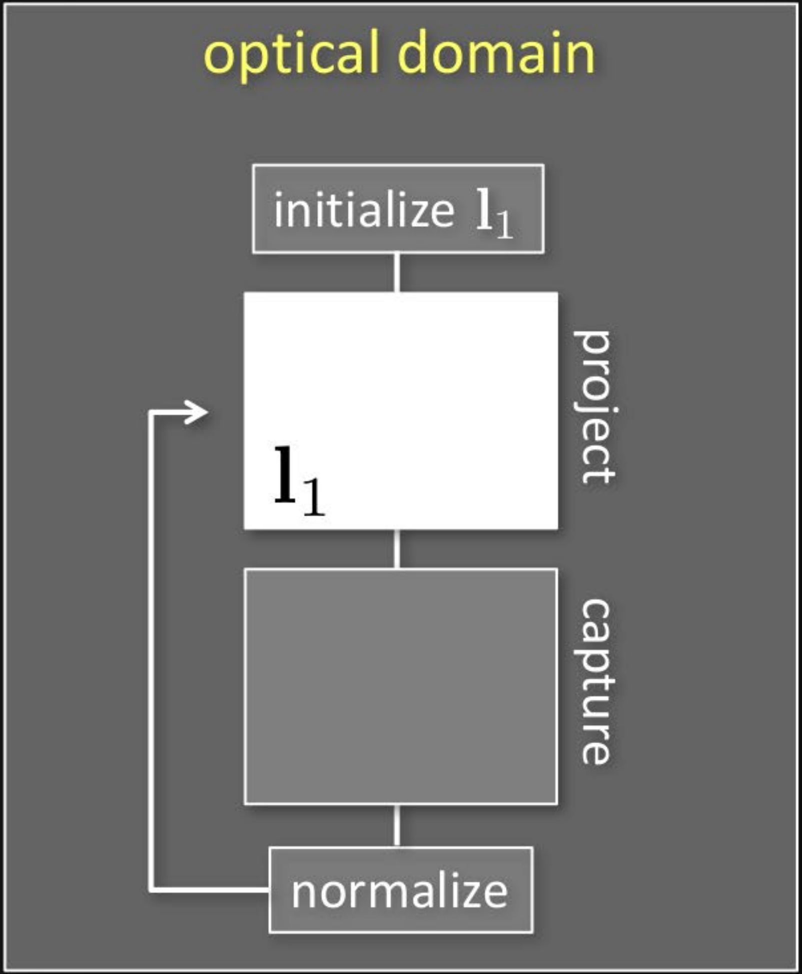


optical power iteration

goal: find principal eigenvector of $\mathbf{T}$

observation: it is a fixed point of the sequence $\mathbf{l}, \mathbf{T}\mathbf{l}, \mathbf{T}^2\mathbf{l}, \mathbf{T}^3\mathbf{l}, \ldots$

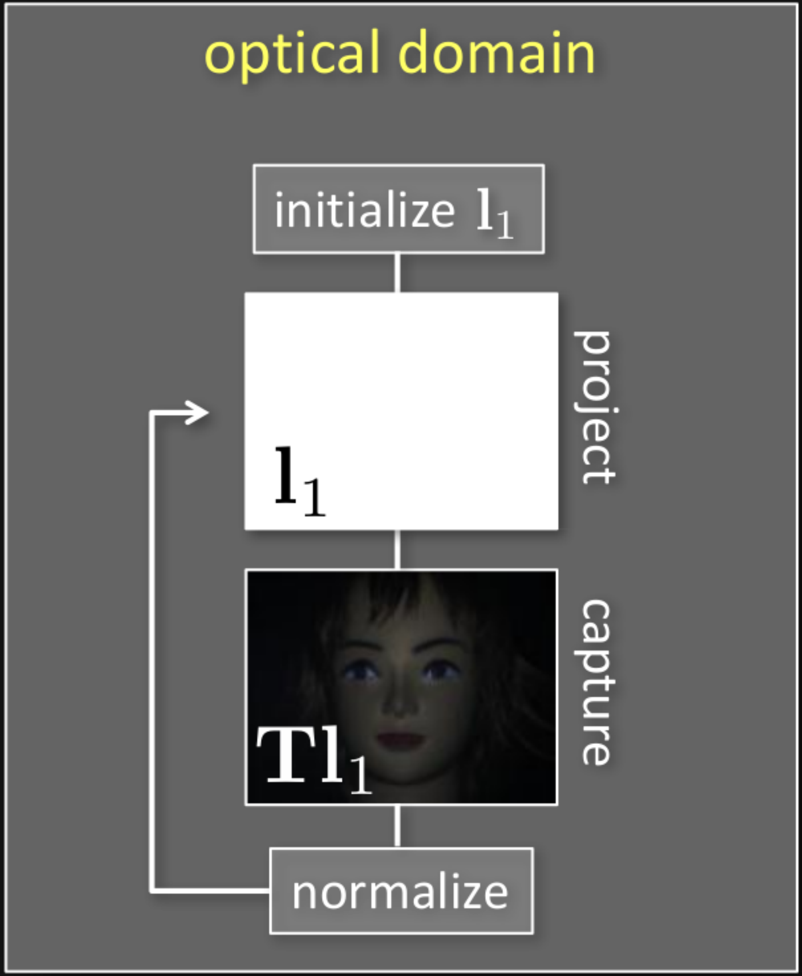




optical power iteration

goal: find principal eigenvector of $\mathbf{T}$

observation: it is a fixed point of the sequence $\mathbf{l}, \mathbf{T}\mathbf{l}, \mathbf{T}^2\mathbf{l}, \mathbf{T}^3\mathbf{l}, \ldots$



optical domain

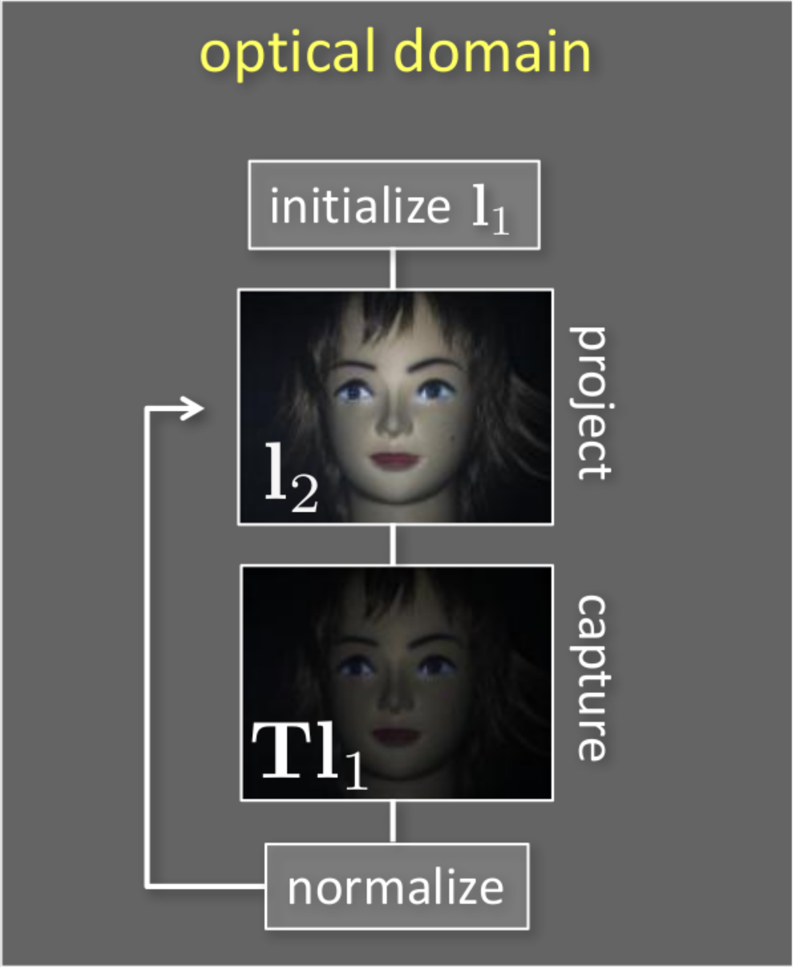
initialize $\mathbf{l}_1$

normalize

optical power iteration

goal: find principal eigenvector of $\mathbf{T}$

observation: it is a fixed point of the sequence $\mathbf{l}, \mathbf{T}\mathbf{l}, \mathbf{T}^2\mathbf{l}, \mathbf{T}^3\mathbf{l}, \ldots$



optical domain

initialize $\mathbf{l}_1$

$\mathbf{l}_2$

project

$\mathbf{T}\mathbf{l}_1$

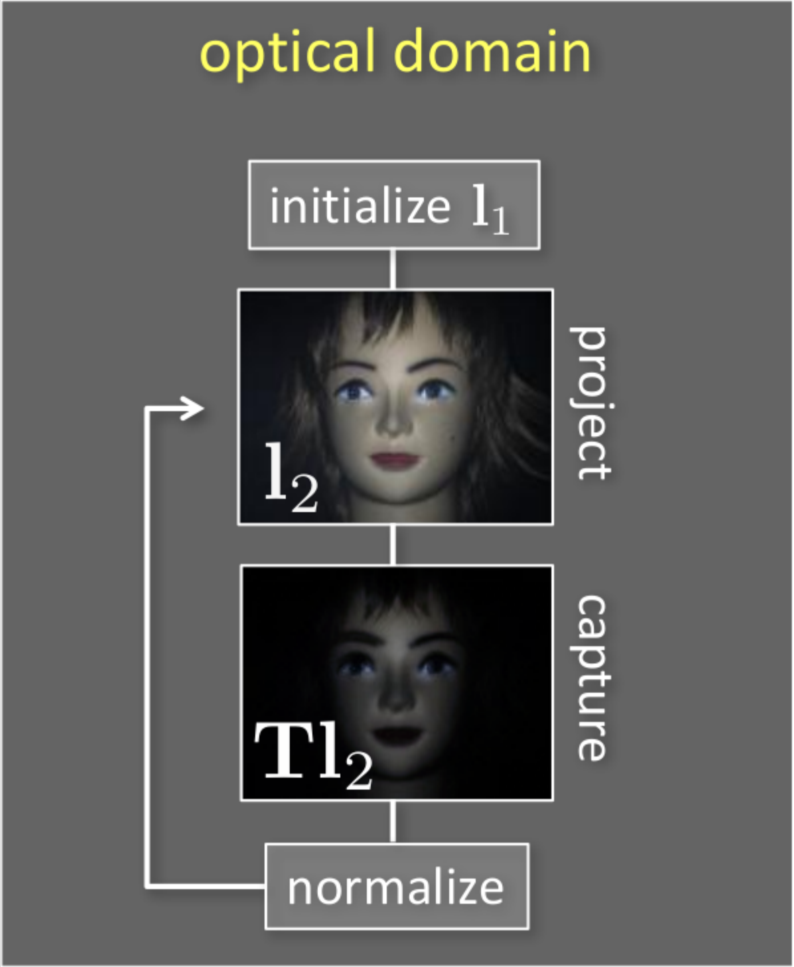
capture

normalize

optical power iteration

goal: find principal eigenvector of $\mathbf{T}$

observation: it is a fixed point of the sequence $\mathbf{l}, \mathbf{T}\mathbf{l}, \mathbf{T}^2\mathbf{l}, \mathbf{T}^3\mathbf{l}, \ldots$



optical domain

initialize $\mathbf{l}_1$

project

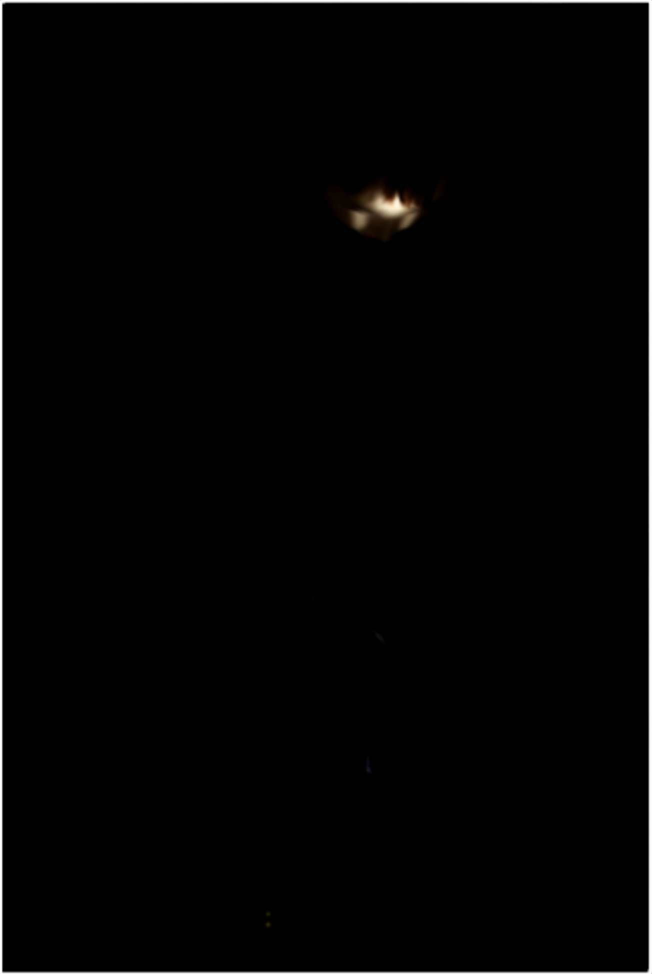
capture

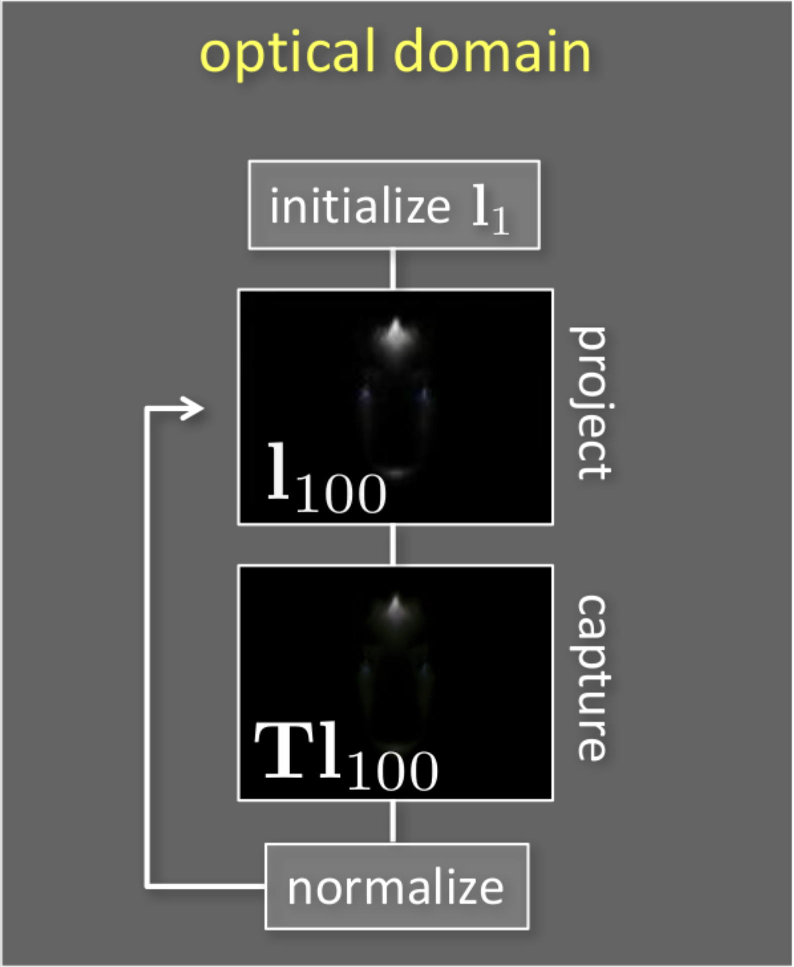
normalize

optical power iteration

goal: find principal eigenvector of $\mathbf{T}$

observation: it is a fixed point of the sequence $\mathbf{l}, \mathbf{T}\mathbf{l}, \mathbf{T}^2\mathbf{l}, \mathbf{T}^3\mathbf{l}, \ldots$

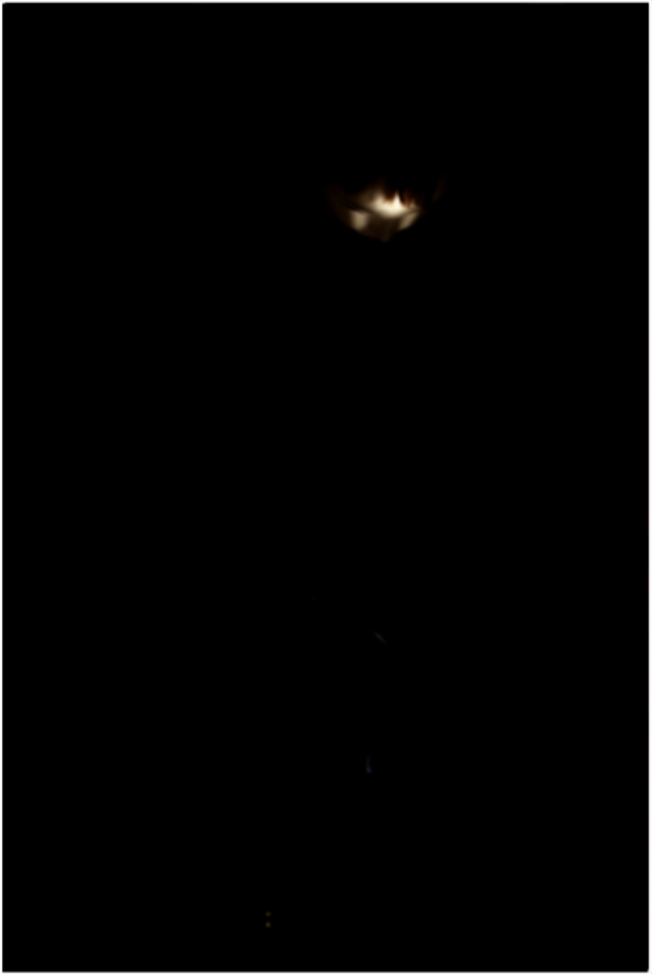




optical power iteration

goal: find principal eigenvector of $\mathbf{T}$

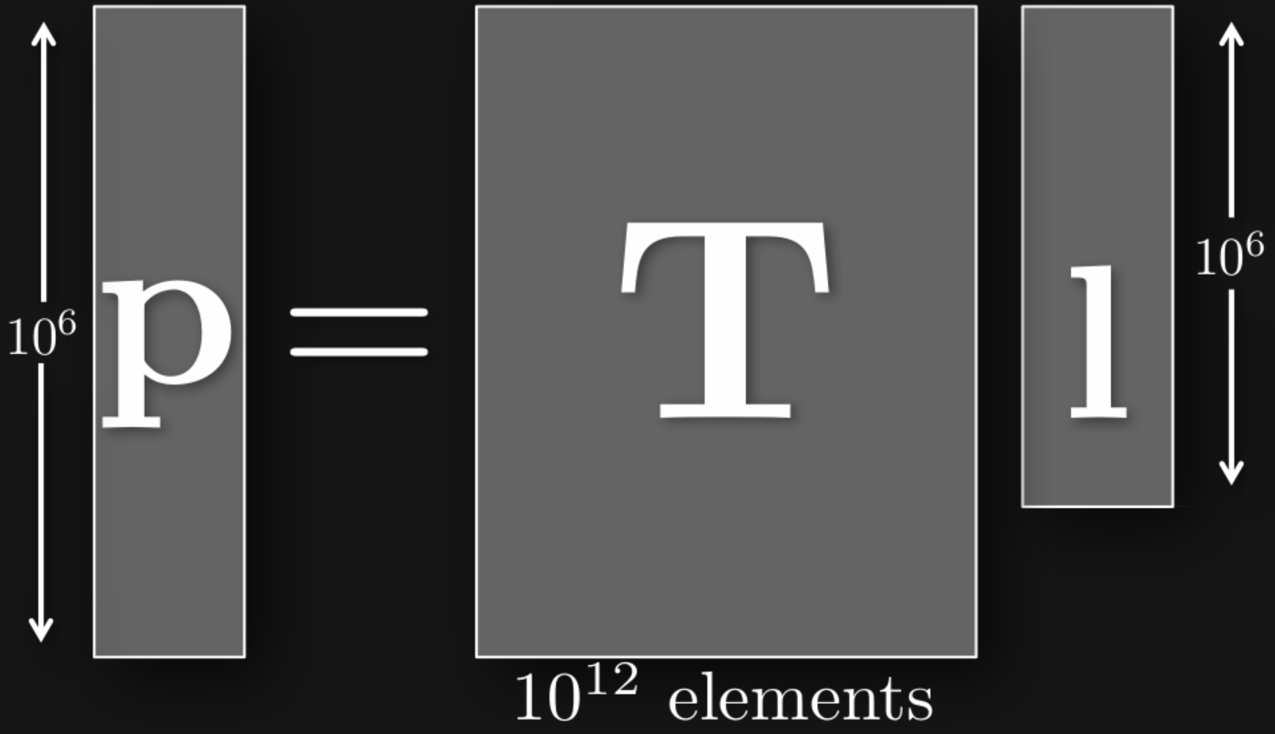
observation: it is a fixed point of the sequence $\mathbf{l}, \mathbf{T}\mathbf{l}, \mathbf{T}^2\mathbf{l}, \mathbf{T}^3\mathbf{l}, \ldots$



optical domain

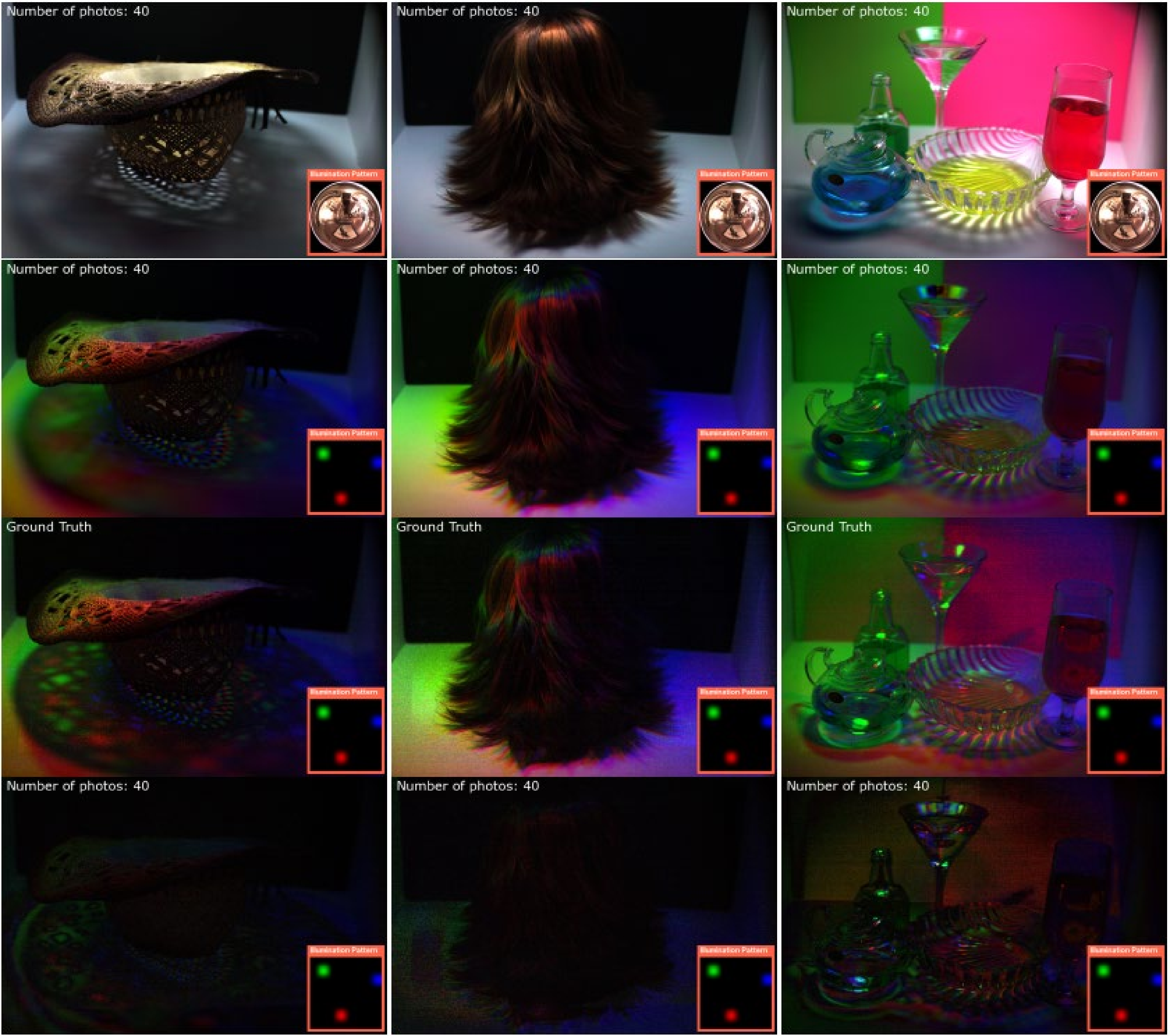
(approximate)
principal eigenvector

How would you measure the light transport matrix T?

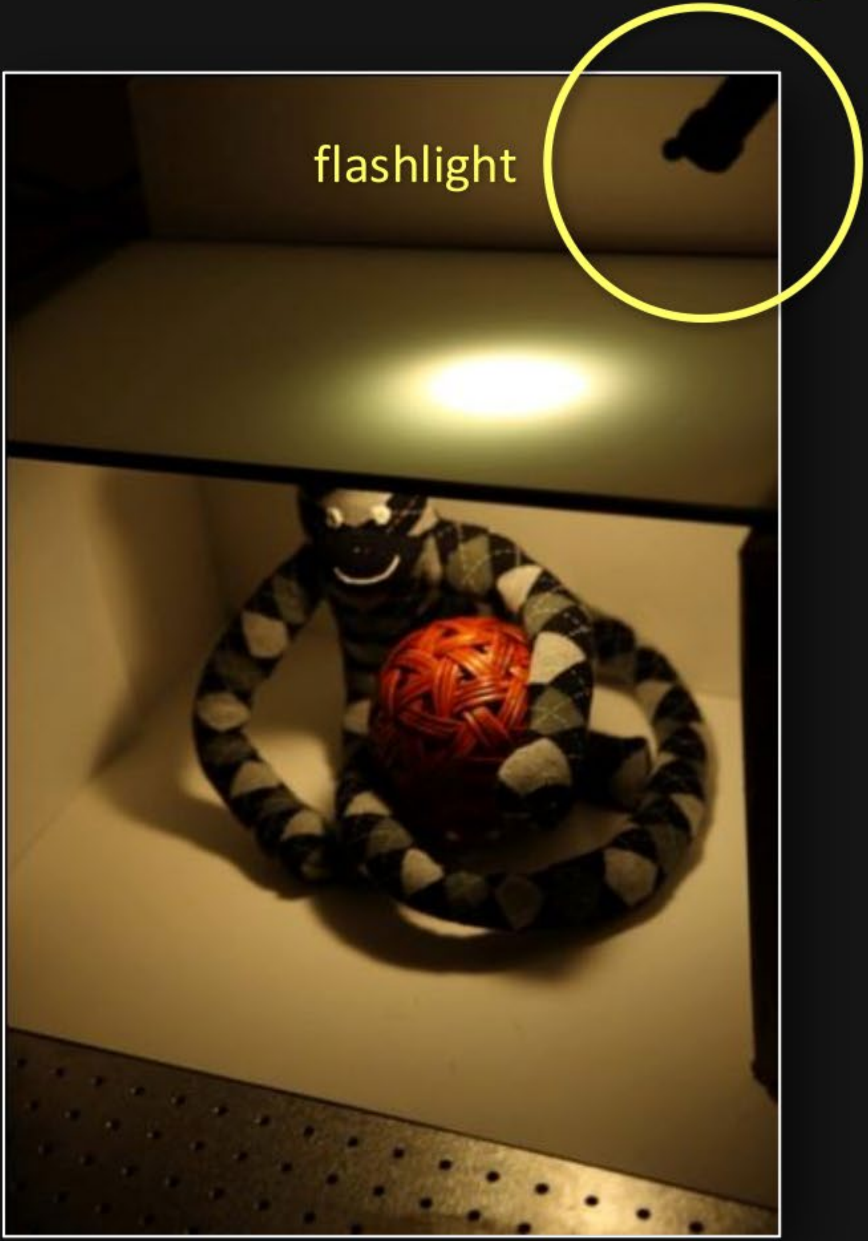


Alternative approach: use optical eigendecomposition to form a low-rank approximation to the light transport matrix.

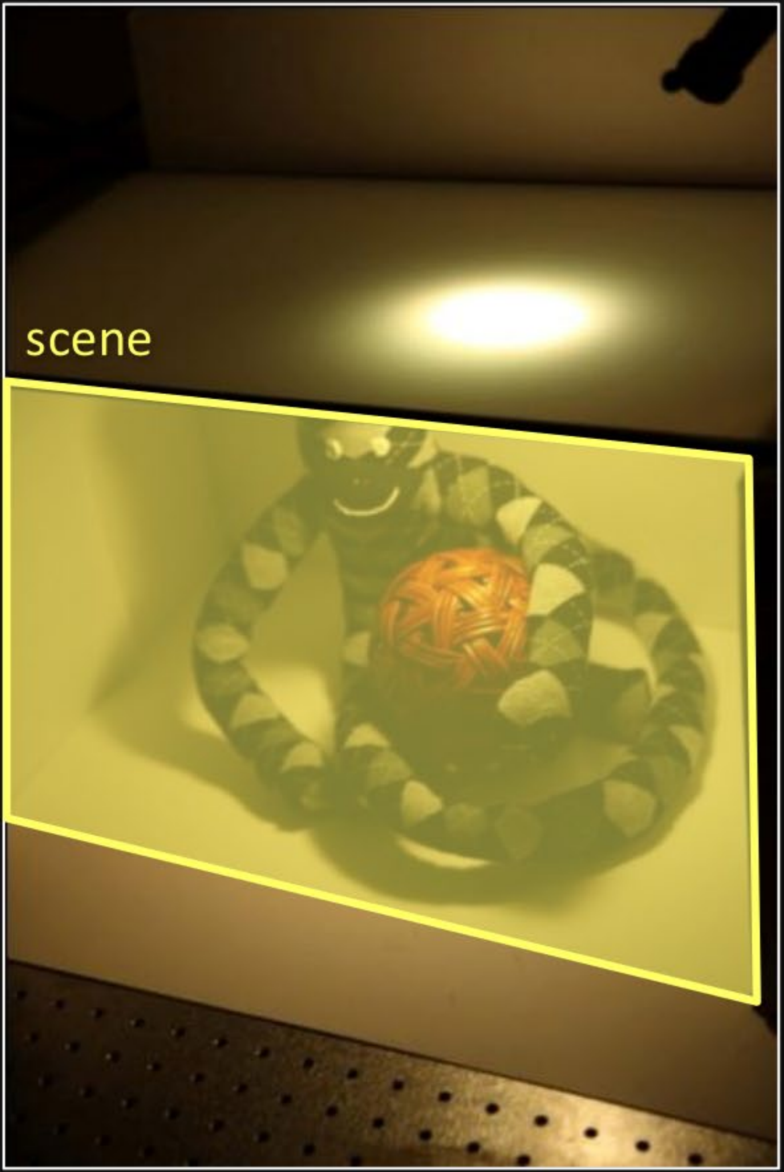
- How many photos do we need to capture?


Number of photos: 40
Illumination Pattern
Number of photos: 40
Illumination Pattern
Number of photos: 40
Illumination Pattern
Number of photos: 40
Illumination Pattern
Number of photos: 40
Illumination Pattern
Number of photos: 40
Illumination Pattern
Ground Truth
Illumination Pattern
Ground Truth
Illumination Pattern
Ground Truth
Illumination Pattern
Number of photos: 40
Illumination Pattern
Number of photos: 40
Illumination Pattern
Number of photos: 40
Illumination Pattern

Inverse transport

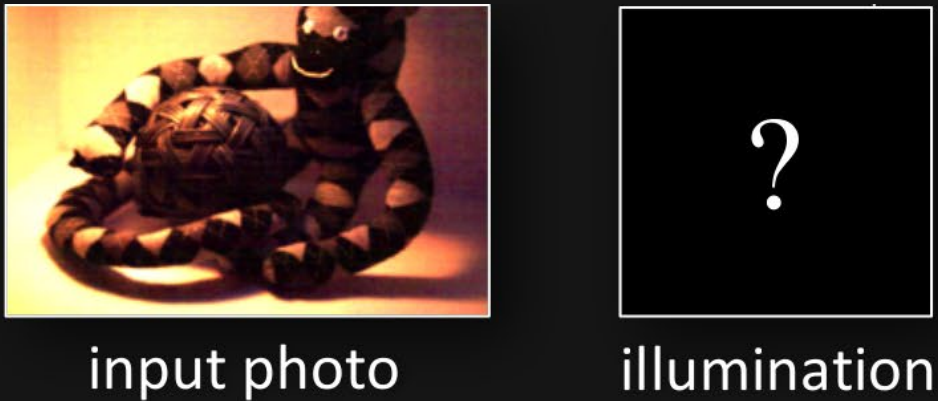

flashlight

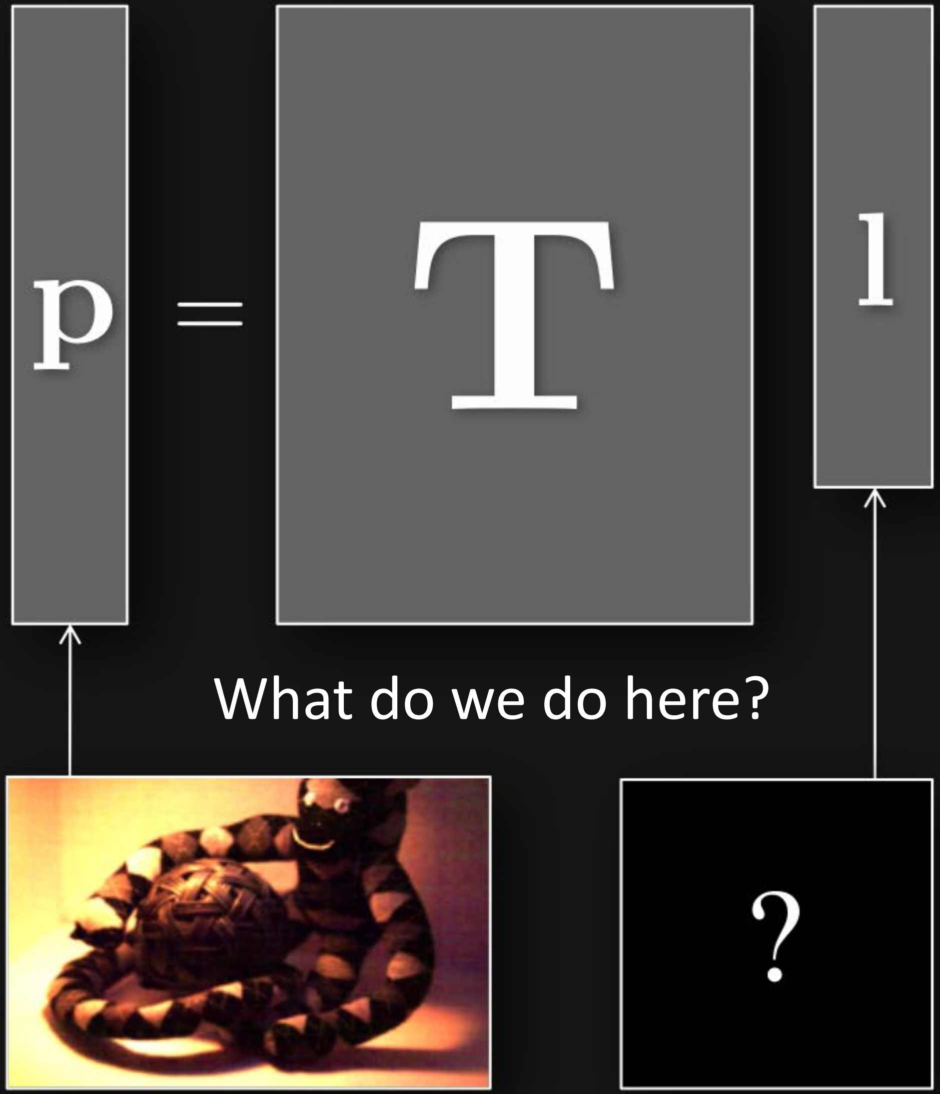
diffuser


scene

input photo

How do you solve this problem if you know the light transport matrix T?



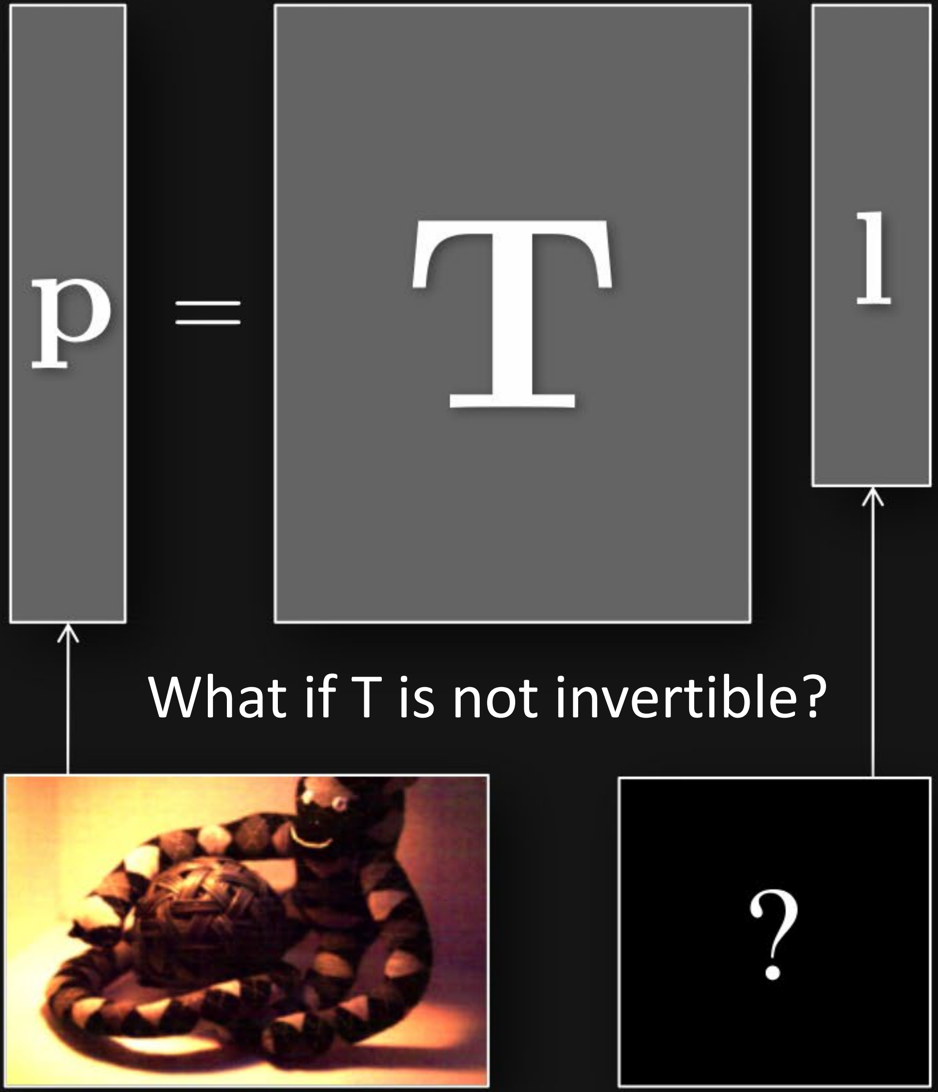


$$\mathbf{p} = \mathbf{T}\,\mathbf{l}$$

What do we do here?

input photo

illumination



$$\mathbf{p} = \mathbf{T}\,\mathbf{l}$$

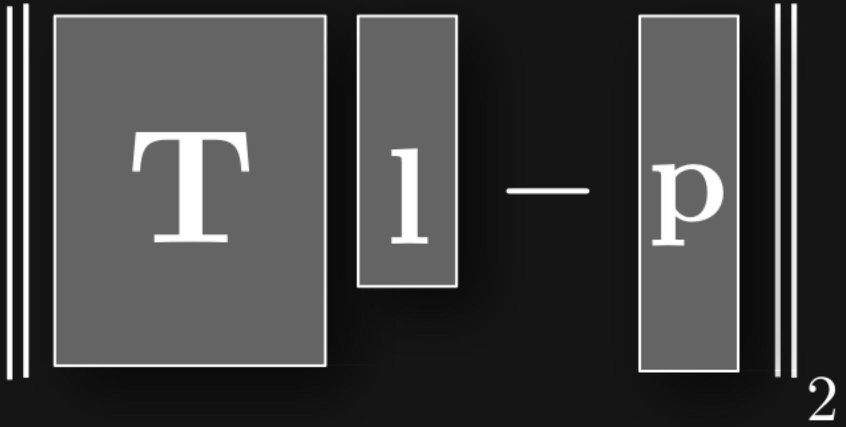
What if T is not invertible?

input photo

illumination

numerical goal

given photo $\mathbf{p}$, find illumination $\mathbf{l}$ that minimizes



$$\left\| \mathbf{T} \, \mathbf{l} - \mathbf{p} \right\|_2$$

How do you usually solve for l when T is large?

input photo

illumination

Reminder: gradient descent

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

Minimize by iteratively computing:

$$f^{i+1} = f^i - \eta^i r^i, \quad r^i = v - Af^i, \;\; \eta^i = \frac{\left(r^i\right)^T r^i}{(r^i)^T A r^i} \qquad \text{for } \mathrm{i} = 0, 1, \ldots, \mathrm{N}$$

Is this cheaper than the pseudo-inverse approach?

- We never need to compute A, only its products with vectors f, r.
- Vectors f, r are images.
- Because A is the *Laplacian matrix*, these matrix-vector products can be efficiently computed using *convolutions* with the *Laplacian kernel*.

Gradient descent in this case

Given the loss function:

$$E(f) = \|Gf - v\|^2$$

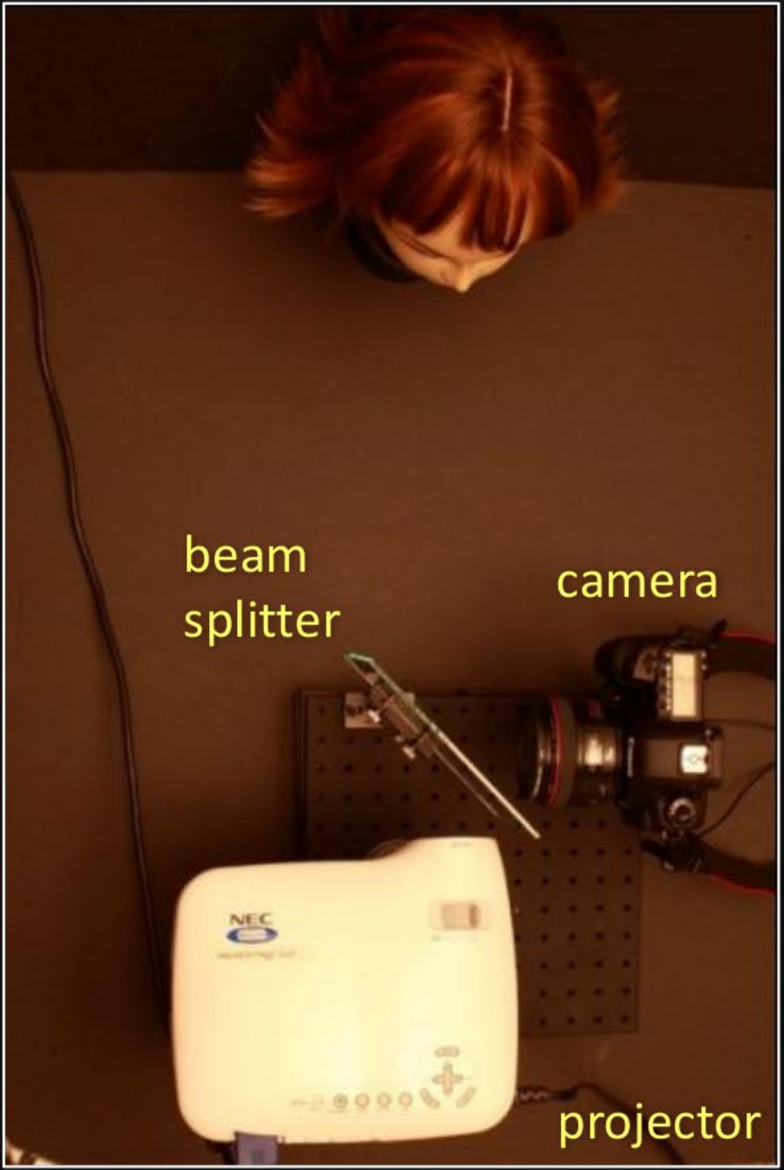
Minimize by iteratively computing:

$$f^{i+1} = f^i - \eta^i r^i, \quad r^i = v - Af^i, \;\; \eta^i = \frac{\left(r^i\right)^T r^i}{(r^i)^T A r^i} \qquad \text{for } i = 0, 1, \ldots, N$$

Is this cheaper than the pseudo-inverse approach?

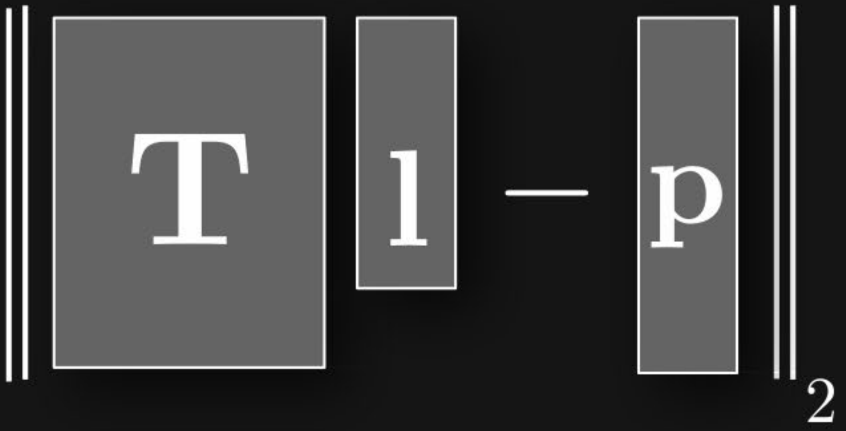
- We never need to compute A, only its products with vectors f, r.
- ~~Vectors f, r are images.~~ What are f, r in this case?
- ~~Because A is the *Laplacian matrix*, these matrix-vector products can be efficiently computed using *convolutions* with the *Laplacian kernel*.~~ How do we compute matrix-vector products efficiently in this case?

inverting light transport



numerical goal

given photo $\mathbf{p}$, find illumination $\mathbf{l}$ that minimizes



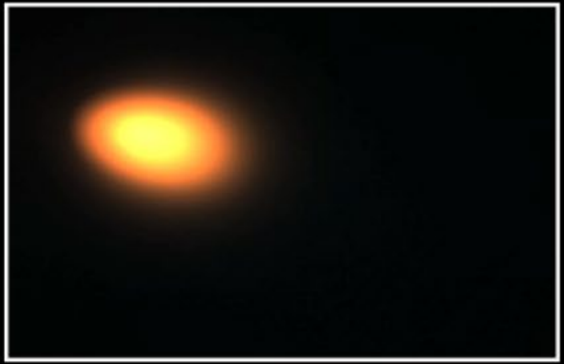
$$\left\| \mathbf{T}\,\mathbf{l} - \mathbf{p} \right\|_2$$

remarks

- $\mathbf{T}$ low-rank or high-rank
- $\mathbf{T}$ unknown & not acquired
- illumination sequence will be specific to input photo

inverting light transport

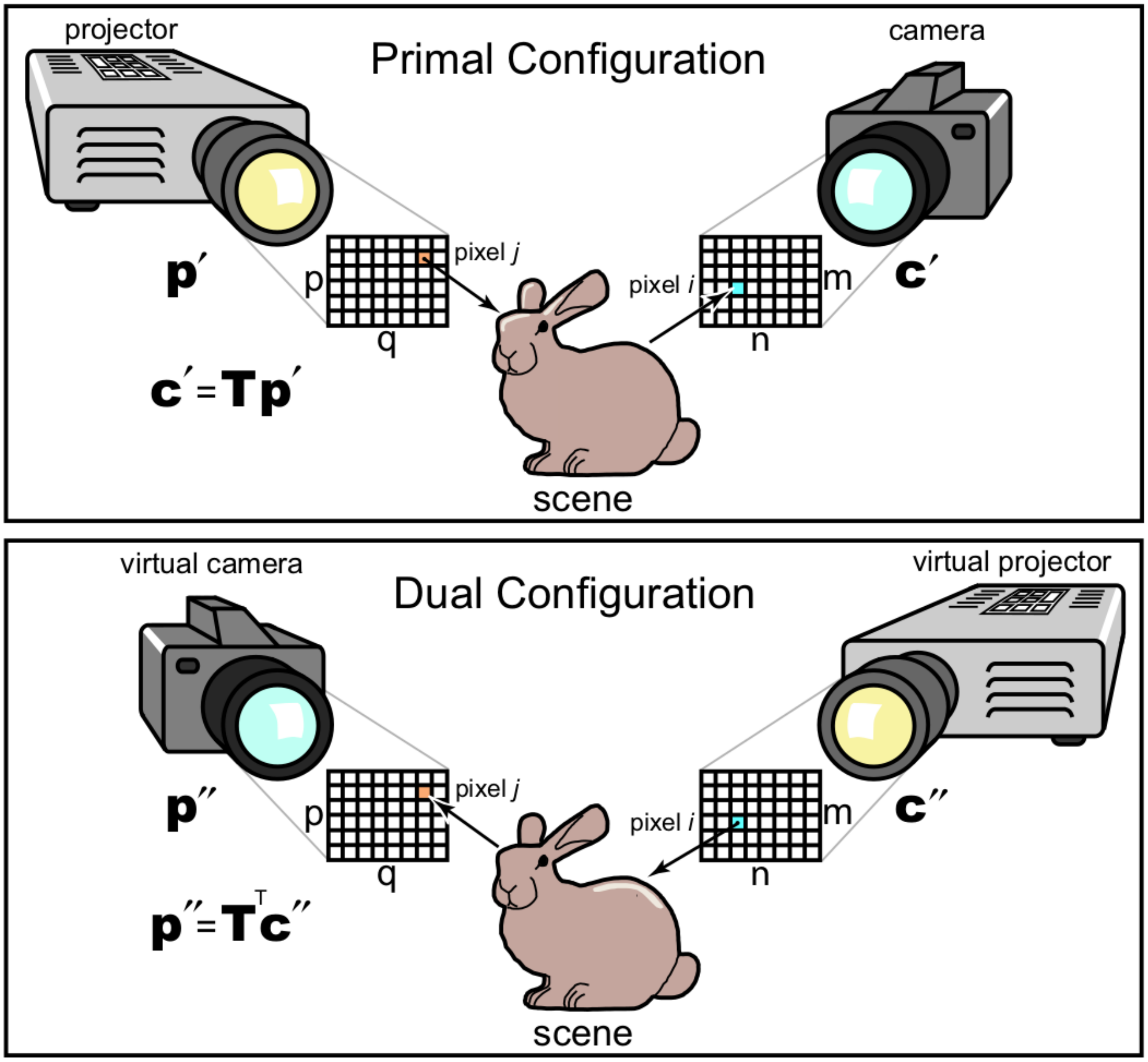
input photo

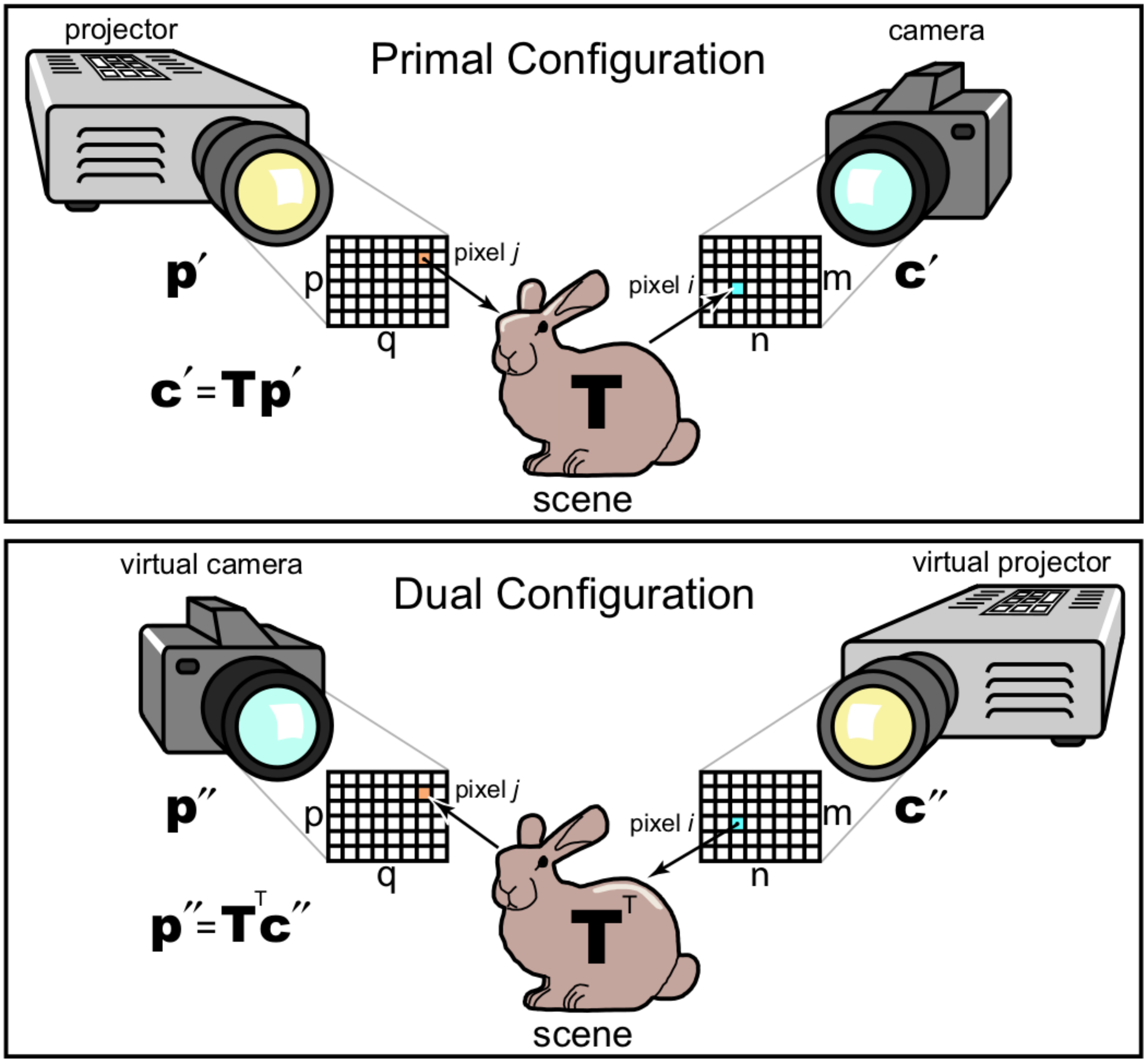


actual illumination

Dual photography

How do the light transport matrices for these two scenes relate to each other?





Helmholtz reciprocity: The two matrices are the transpose of each other.

Great demonstration: https://www.youtube.com/watch?v=eV58Ko3iFuI

References

Basic reading:

- Sloan et al., “Precomputed radiance transfer for real-time rendering in dynamic, low-frequency lighting environments,” SIGGRAPH 2002.
- Ng et al., “All-frequency shadows using non-linear wavelet lighting approximation,” SIGGRAPH 2003.
- Seitz et al., “A theory of inverse light transport,” ICCV 2005.

These three papers all discuss the concept of light transport matrix in detail.

- Debevec et al., “Acquiring the reflectance field of a human face,” SIGGRAPH 2000.

The paper on image-based relighting.

- O’Toole and Kutulakos, “Optical computing for fast light transport analysis,” SIGGRAPH Asia 2010.

The paper on eigenanalysis and optical computing using light transport matrices.

- Sen et al., “Dual photography,” SIGGRAPH 2005.

The dual photography paper.

Additional reading:

- Peers et al., “Compressive light transport sensing,” TOG 2009.
- Wang et al., “Kernel Nyström method for light transport,” SIGGRAPH 2009.

These two papers discuss alternative ways for efficient acquisition of the light transport matrix, using assumptions on its algebraic structure.

- Durand et al., “A frequency analysis of light transport,” SIGGRAPH 2005.
- Mahajan et al., “A theory of locally low dimensional light transport,” SIGGRAPH 2007.
- Reddy et al., “Frequency-space decomposition and acquisition of light transport under spatially varying illumination,” ECCV 2012.

These papers more formally discuss the notion of light transport frequency, how it relates to light transport matrix rank, and the frequency/rank characteristics of different light transport effects (specular versus diffuse reflections, hard versus smooth shadows).