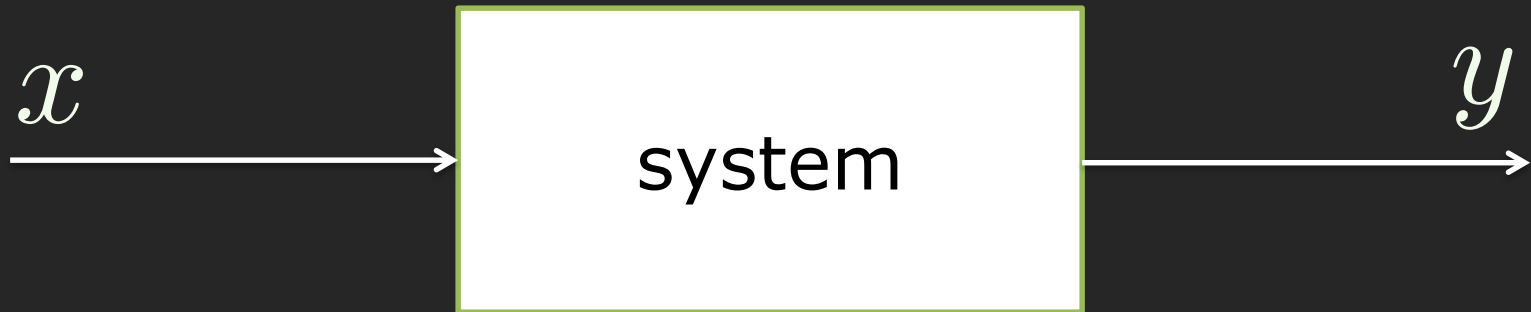


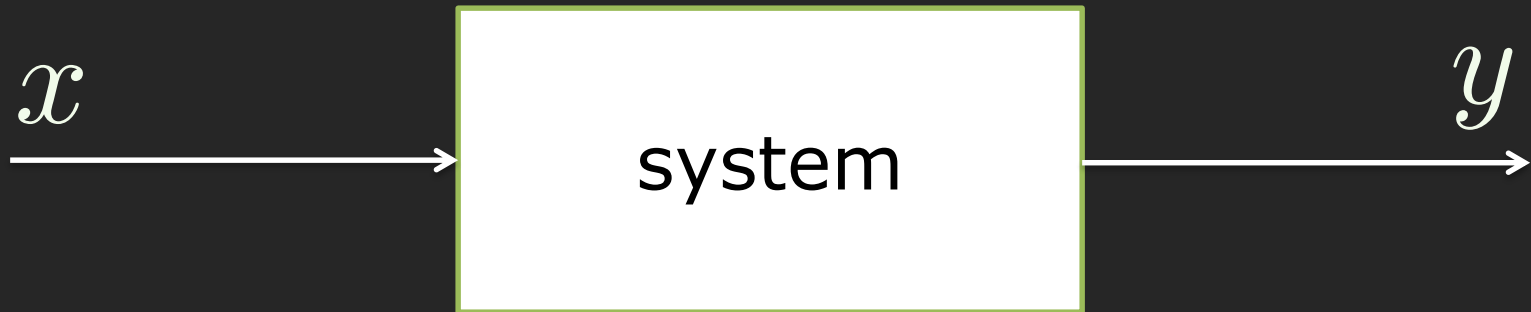
Introduction to Compressive Sensing

Aswin Sankaranarayanan



$$y = \Phi x$$

Is this system linear ?

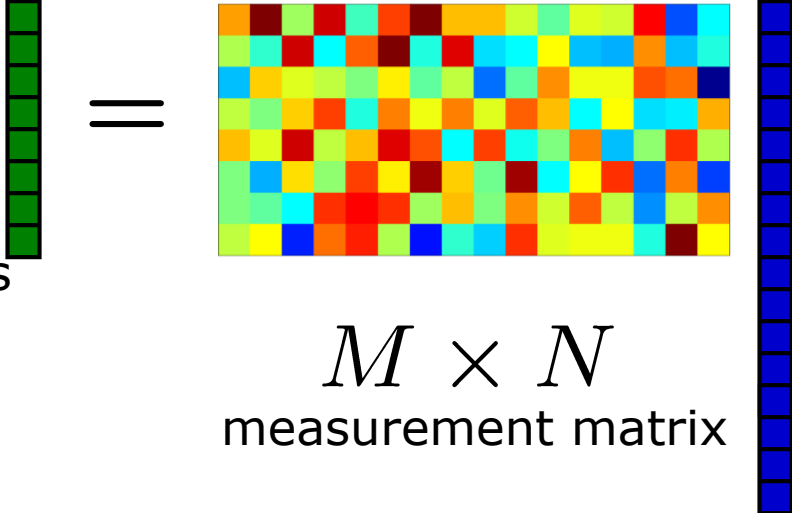


$$y = \Phi x$$

Is this system linear ?

Given y , can we recovery x ?

Under-determined problems

$$\begin{array}{c} y \\ M \times 1 \\ \text{measurements} \end{array} = \begin{array}{c} \Phi \\ M \times N \\ \text{measurement matrix} \end{array} \begin{array}{c} x \\ N \times 1 \\ \text{signal} \end{array}$$
The diagram illustrates the equation $y = \Phi x$. On the left, a vertical green bar represents the vector y , with the label $M \times 1$ and "measurements" below it. In the center, a colorful grid represents the measurement matrix Φ , with the label $M \times N$ and "measurement matrix" below it. On the right, a vertical blue bar represents the vector x , with the label $N \times 1$ and "signal" below it. An equals sign is placed between the vector y and the matrix Φ .

If $M < N$, then the system is **information lossy**



Image credit
Graeme Pope

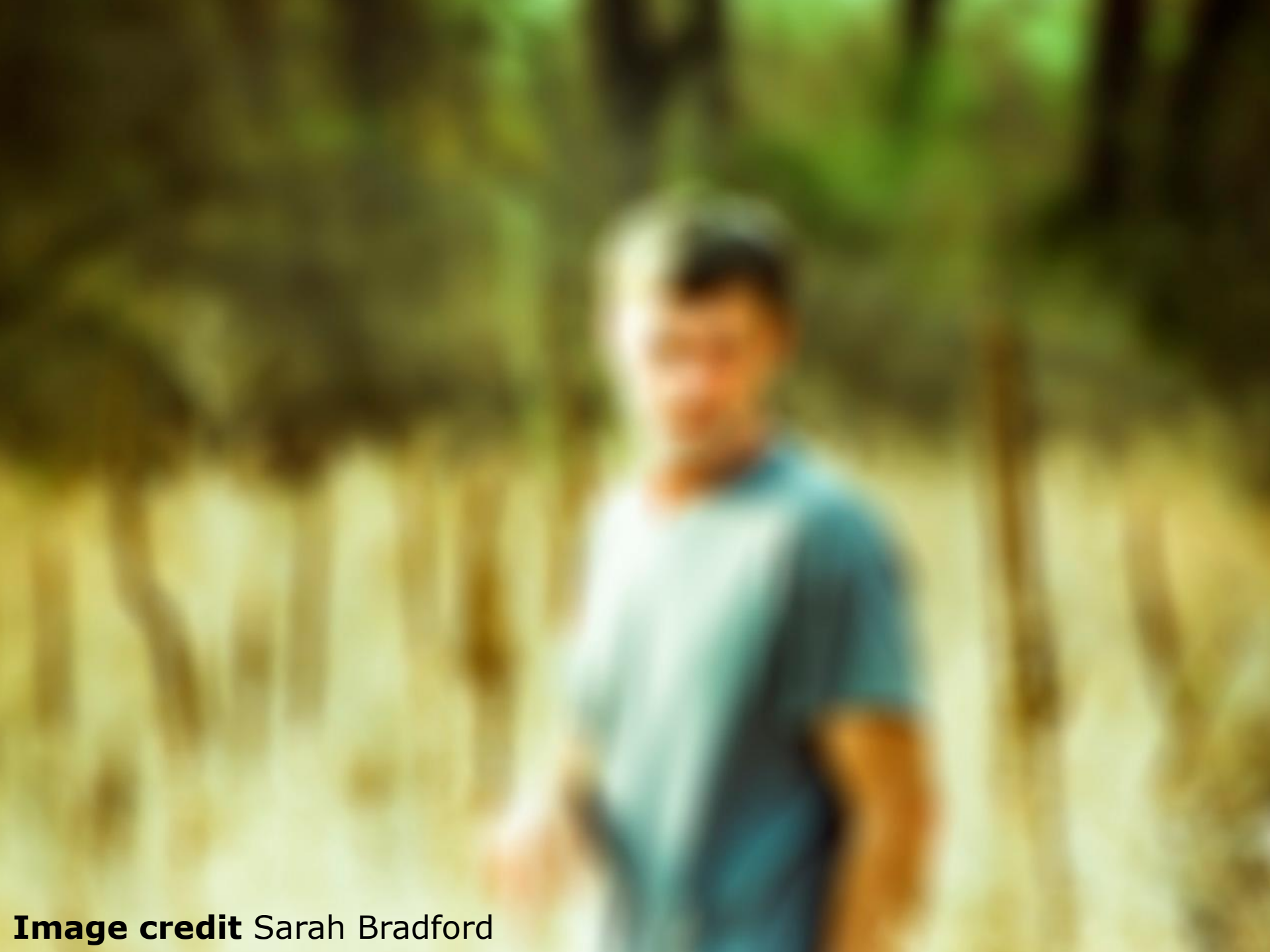
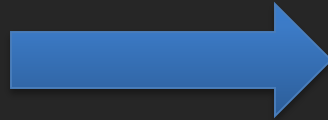


Image credit Sarah Bradford

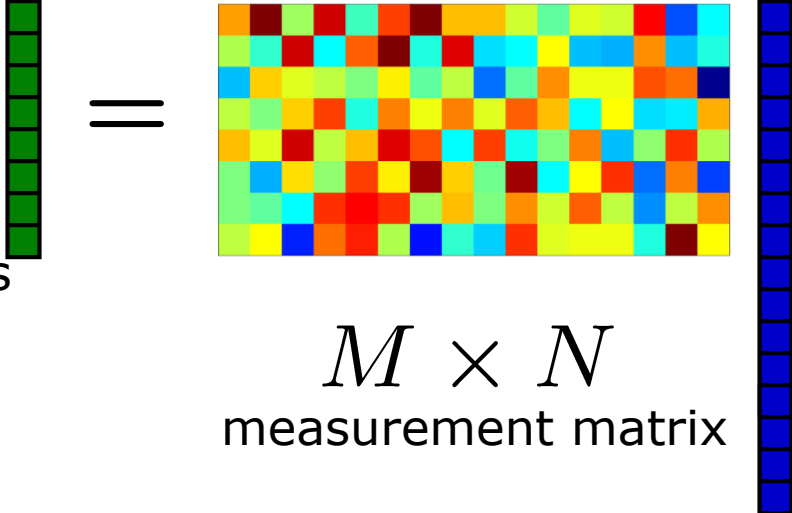
Super-resolution



Can we increase the resolution of this image ?

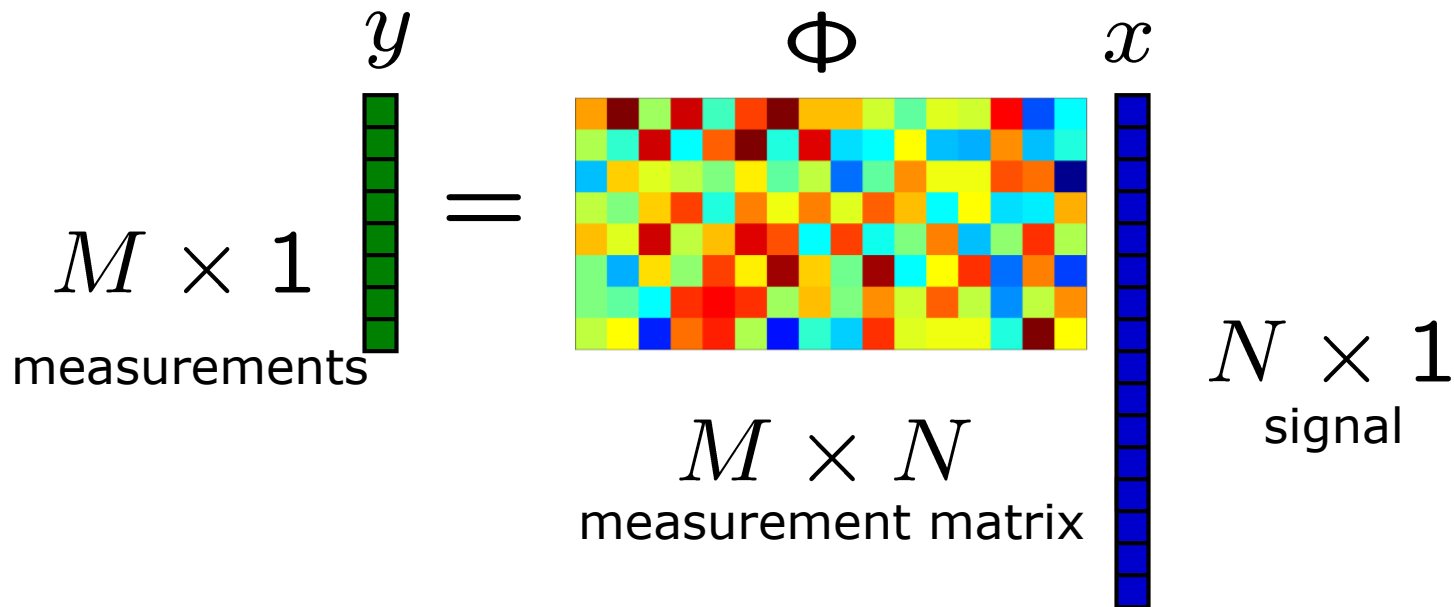
[\(Link: Depixelizing pixel art\)](#)

Under-determined problems

$$\begin{array}{c} y \\ M \times 1 \\ \text{measurements} \end{array} = \begin{array}{c} \Phi \\ M \times N \\ \text{measurement matrix} \end{array} \begin{array}{c} x \\ N \times 1 \\ \text{signal} \end{array}$$
The diagram illustrates the equation $y = \Phi x$. On the left, a green vertical vector y is labeled $M \times 1$ measurements. In the center, a colorful grid representing the measurement matrix Φ is labeled $M \times N$ measurement matrix. On the right, a blue vertical vector x is labeled $N \times 1$ signal. An equals sign is placed between the vector y and the matrix Φ .

Fewer knowns than unknowns!

Under-determined problems



The diagram illustrates an under-determined system of linear equations. On the left, a vertical column of 10 green squares represents the vector y , with the label $M \times 1$ measurements to its left. In the center is an equals sign. To the right of the equals sign is a 10x10 grid of colored squares (red, yellow, cyan, blue) representing the measurement matrix Φ , with the label $M \times N$ measurement matrix below it. To the right of the matrix is a vertical column of 10 blue squares representing the vector x , with the label $N \times 1$ signal to its right. The labels y , Φ , and x are positioned above their respective visual representations.

$$\begin{matrix} y \\ M \times 1 \\ \text{measurements} \end{matrix} = \begin{matrix} \Phi \\ M \times N \\ \text{measurement matrix} \end{matrix} \begin{matrix} x \\ N \times 1 \\ \text{signal} \end{matrix}$$

Fewer knowns than unknowns!

An infinite number of solutions to such problems



Credit: Rob Fergus and Antonio Torralba



Credit: Rob Fergus and Antonio Torralba



Is there anything we can do about this ?

Complete the sentences

I cnt blv I m bl t rd ths sntnc.

Wntr s cmng, n .. Wntr s hr

Hy, I m slvng n ndr-dtrmnd lnr system.

how: ?

Complete the matrix

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

how: ?

Complete the image



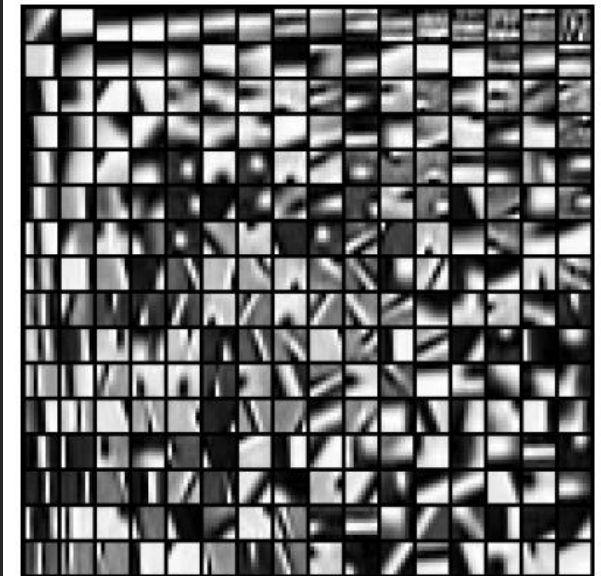
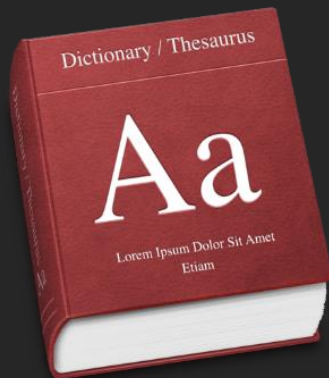
Model ?

Dictionary of visual words

I cnt blv I m bl t rd ths sntnc.

Shrlck s th vc f th drgn

Hy, I m slvng n ndr-dtrmnd
lnr systm.



Dictionary of visual words

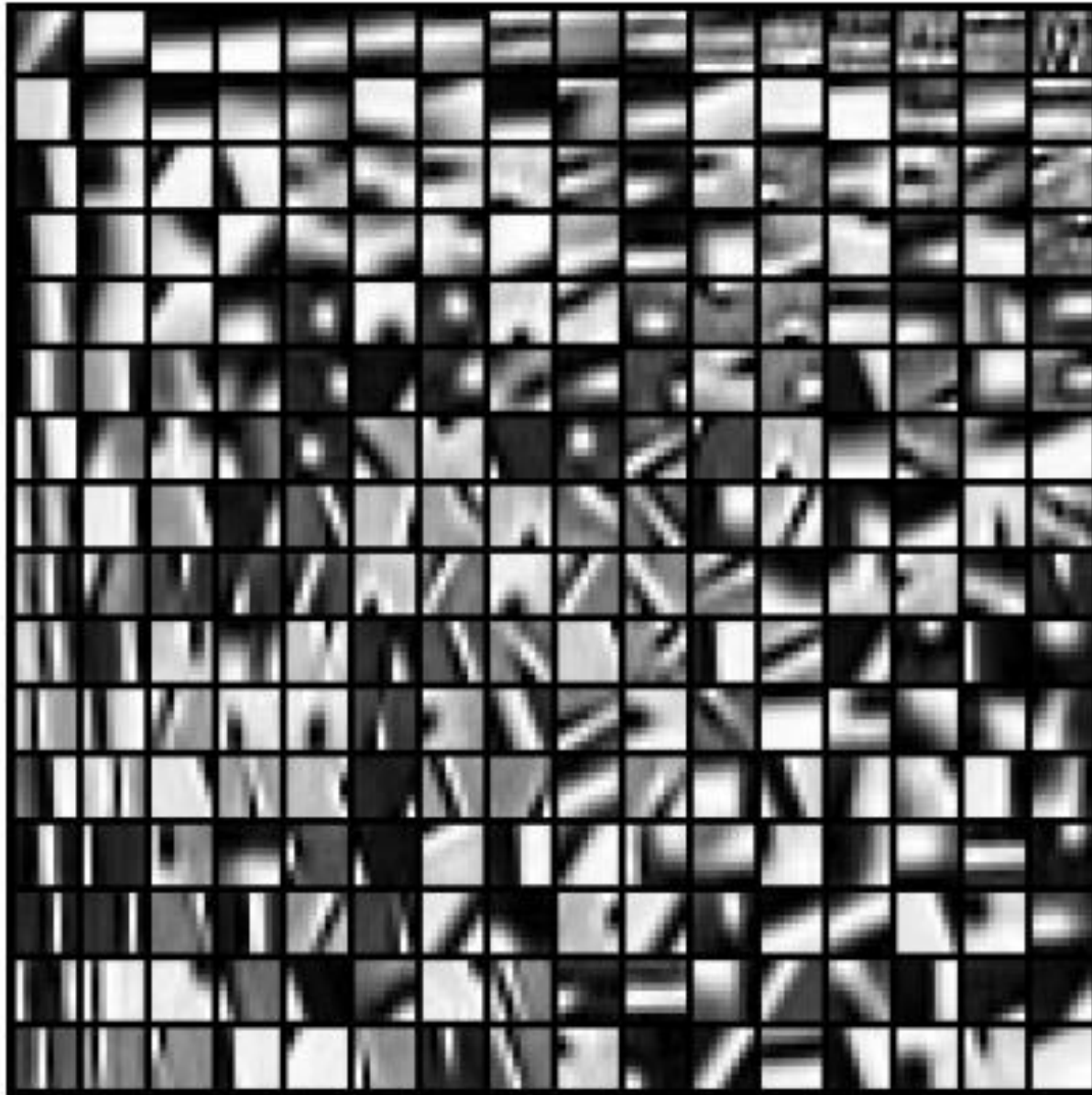




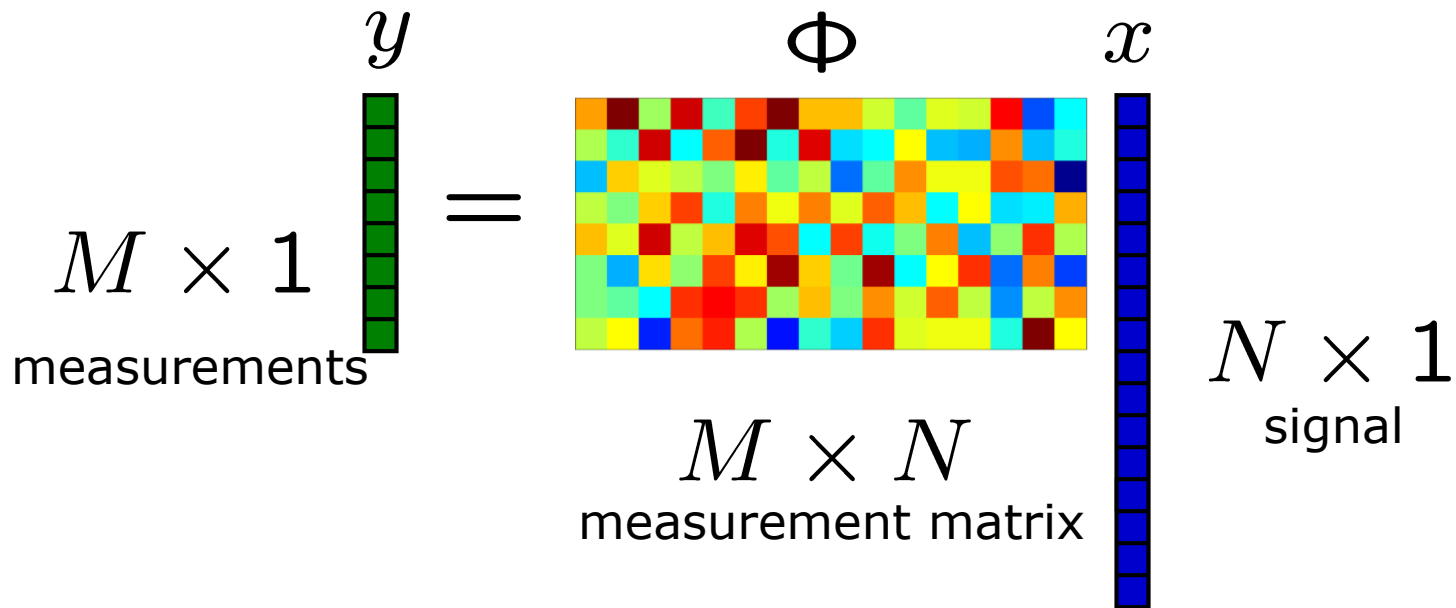
Image credit
Graeme Pope



Image credit
Graeme Pope

Result
Studer, Baraniuk, ACHA 2012

Compressive Sensing



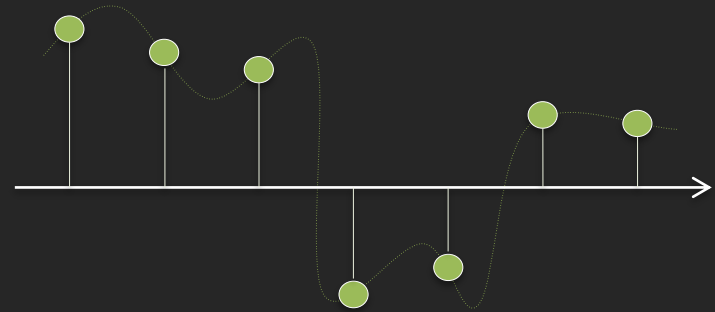
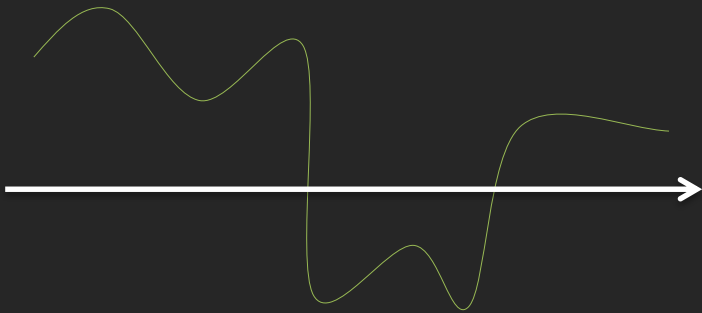
The diagram illustrates the compressive sensing equation $y = \Phi x$. On the left, a vertical column of 10 green squares represents the measurement vector y , with the label $M \times 1$ measurements to its left and the symbol y above it. In the center is an equals sign. To the right of the equals sign is a 10x10 grid of colored squares (red, green, blue, yellow) representing the measurement matrix Φ , with the label $M \times N$ measurement matrix below it. To the right of the grid is a vertical column of 10 blue squares representing the signal vector x , with the label $N \times 1$ signal to its right and the symbol x above it.

$$\begin{matrix} M \times 1 \\ \text{measurements} \end{matrix} \begin{matrix} y \\ \text{column of green squares} \end{matrix} = \begin{matrix} \Phi \\ \text{grid of colored squares} \\ M \times N \\ \text{measurement matrix} \end{matrix} \begin{matrix} x \\ \text{column of blue squares} \\ N \times 1 \\ \text{signal} \end{matrix}$$

A toolset to solve **under-determined systems** by exploiting additional structure/models on the signal we are trying to recover.

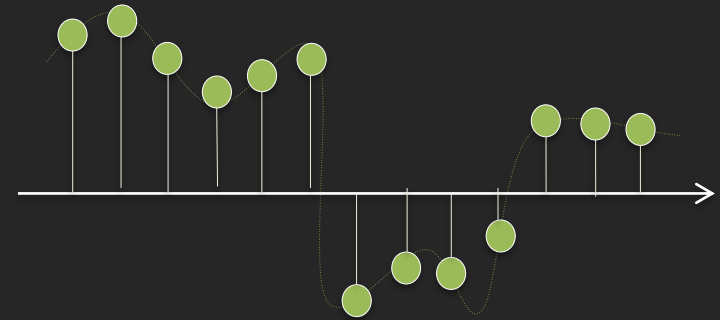
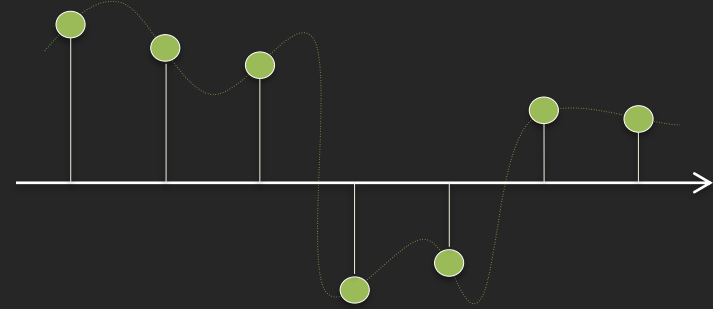
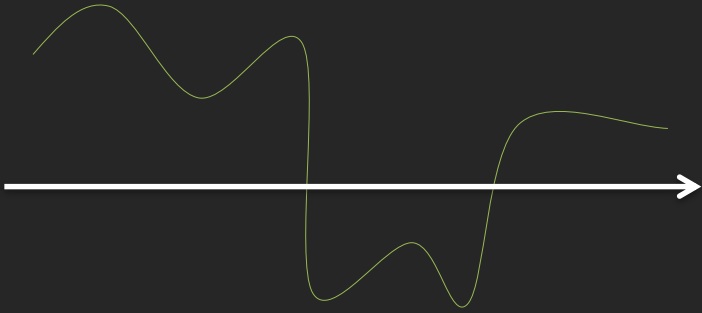
modern sensors are linear
systems!!!

Sampling



Can we recover the analog signal from its discrete time samples ?

Nyquist Theorem



An analog signal can be reconstructed perfectly from discrete samples *provided you sample it densely*.

The Nyquist Recipe

sample faster

sample denser

the more you sample,
the more detail is preserved

The Nyquist Recipe

sample faster

sample denser

the more you sample,
the more detail is preserved

But what happens if you do not follow the Nyquist recipe ?



Credit: Rob Fergus and Antonio Torralba



Image credit: Boston.com

The Nyquist Recipe

sample faster

sample denser

the more you sample,
the more detail is preserved

But what happens if you do not follow the Nyquist recipe ?



What you must learn is that these rules are no different than the rules of a computer system. Some of them can be bent. Others can be broken.

Breaking resolution barriers

- Observing a 40 fps spinning tool with a 25 fps camera

Normal Video:
25fps



Compressively
obtained video:
25fps



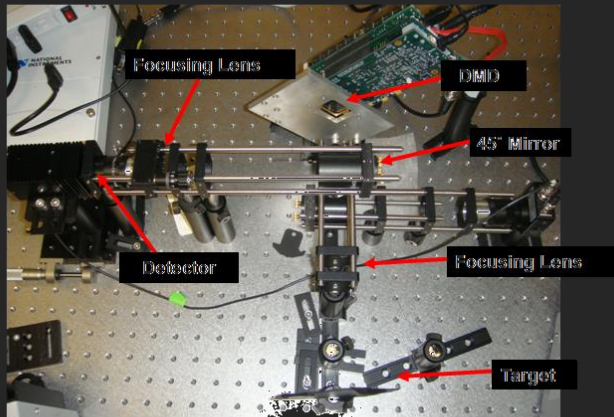
Recovered Video:
2000fps



Compressive Sensing

Use of **motion flow-models** in the context of compressive video recovery

128x128 images sensed at 61x comp.



single pixel camera

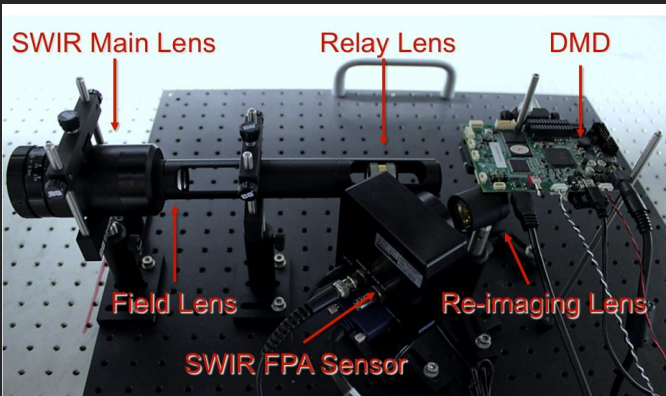


Naïve frame-to-frame recovery



CS-MUVI at 61x compression

Compressive Imaging Architectures

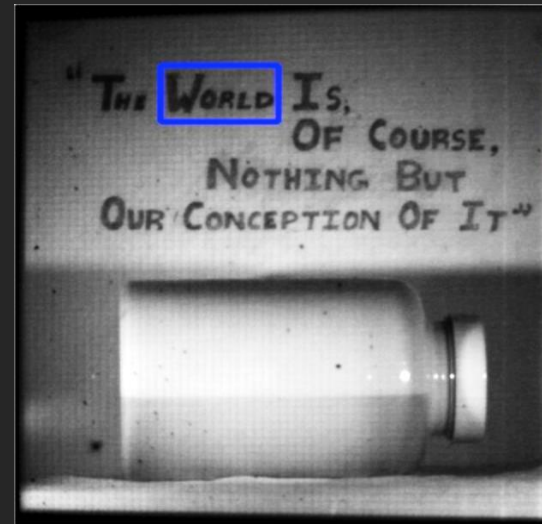


Scalable imaging architectures that deliver videos at **mega-pixel resolutions** in infrared

visible image



SWIR image



A mega-pixel image
obtained from a 64x64
pixel array sensor

Advances in Compressive Imaging

Carnegie Mellon University

Linear Inverse Problems

- Many classic problems in computer can be posed as linear inverse problems

- Notation

- **Signal** of interest

$$x \in \mathbb{R}^N$$

- **Observations**

$$y \in \mathbb{R}^M$$

- Measurement model

$$y = \Phi x + e$$

measurement
matrix



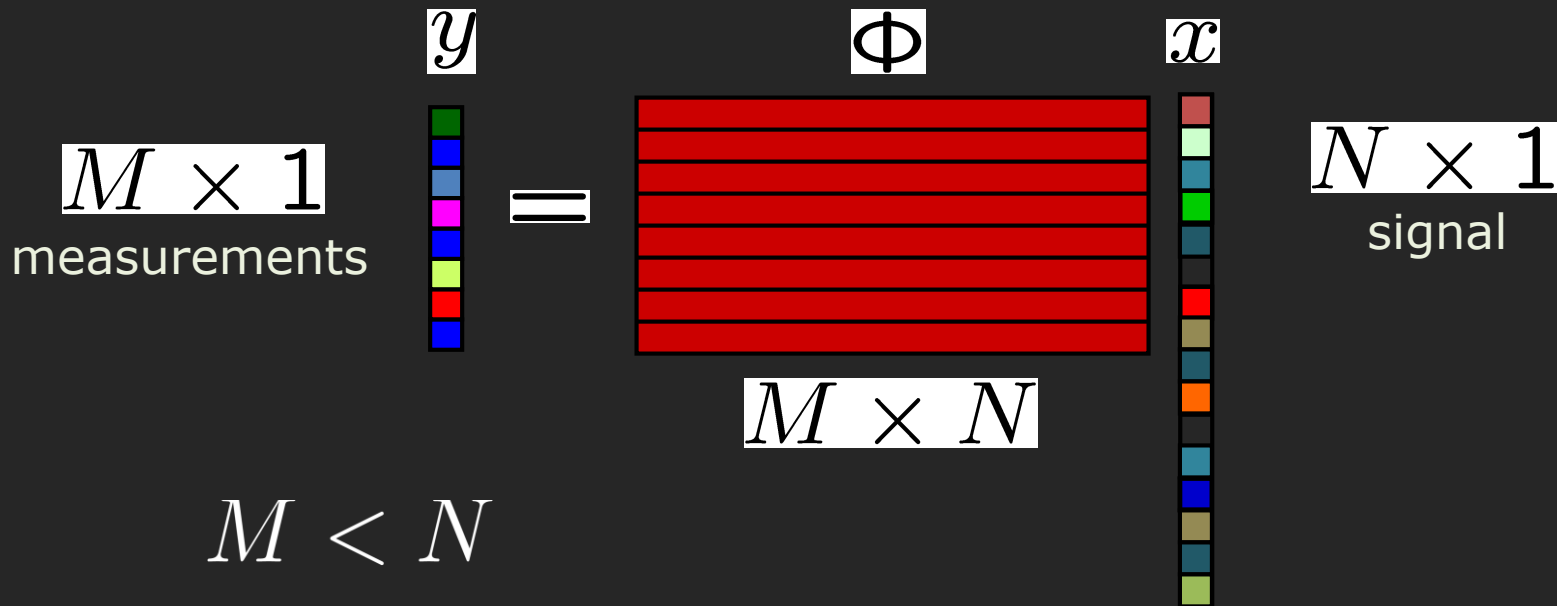
measurement
noise



- Problem definition: given y , recover x

Linear Inverse Problems

$$y = \Phi x$$

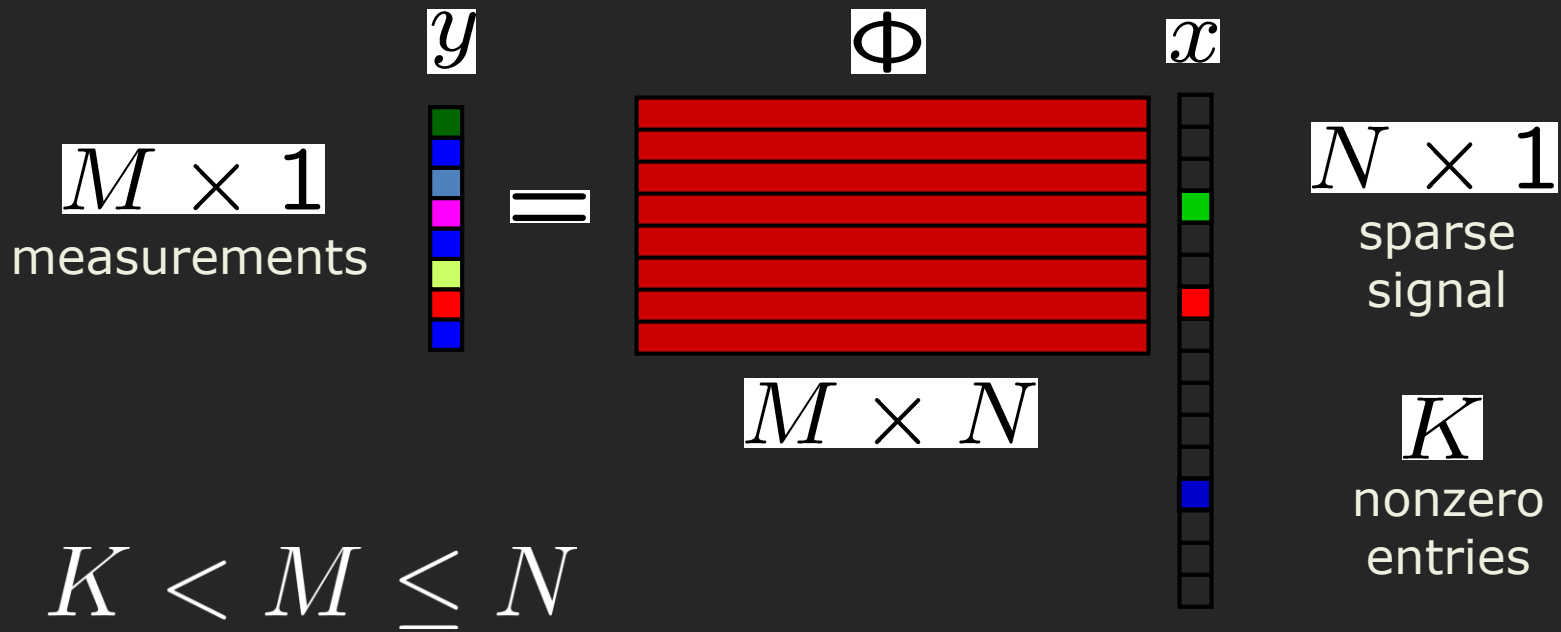


Measurement matrix has a $(N-M)$ dimensional **null-space**

Solution is no longer **unique**

Sparse Signals

$$y = \Phi x$$

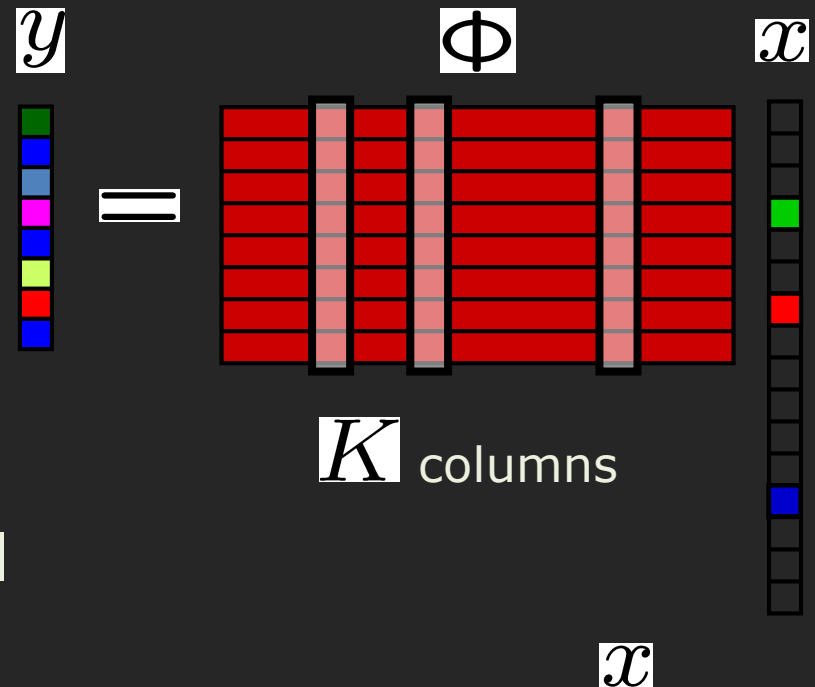


How Can It Work?

- Matrix Φ
not full rank...

$$M < N$$

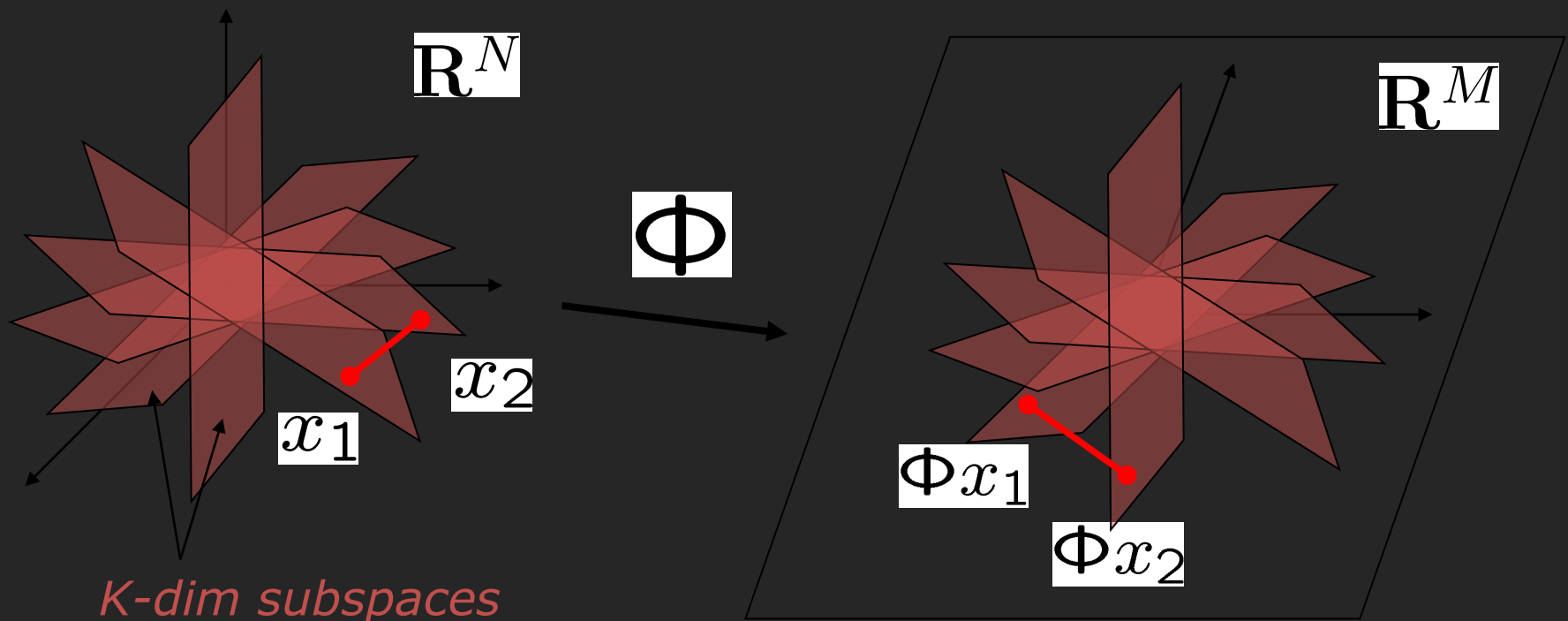
... and so
loses information in general



- But we are only interested in ***sparse*** vectors

Restricted Isometry Property (RIP)

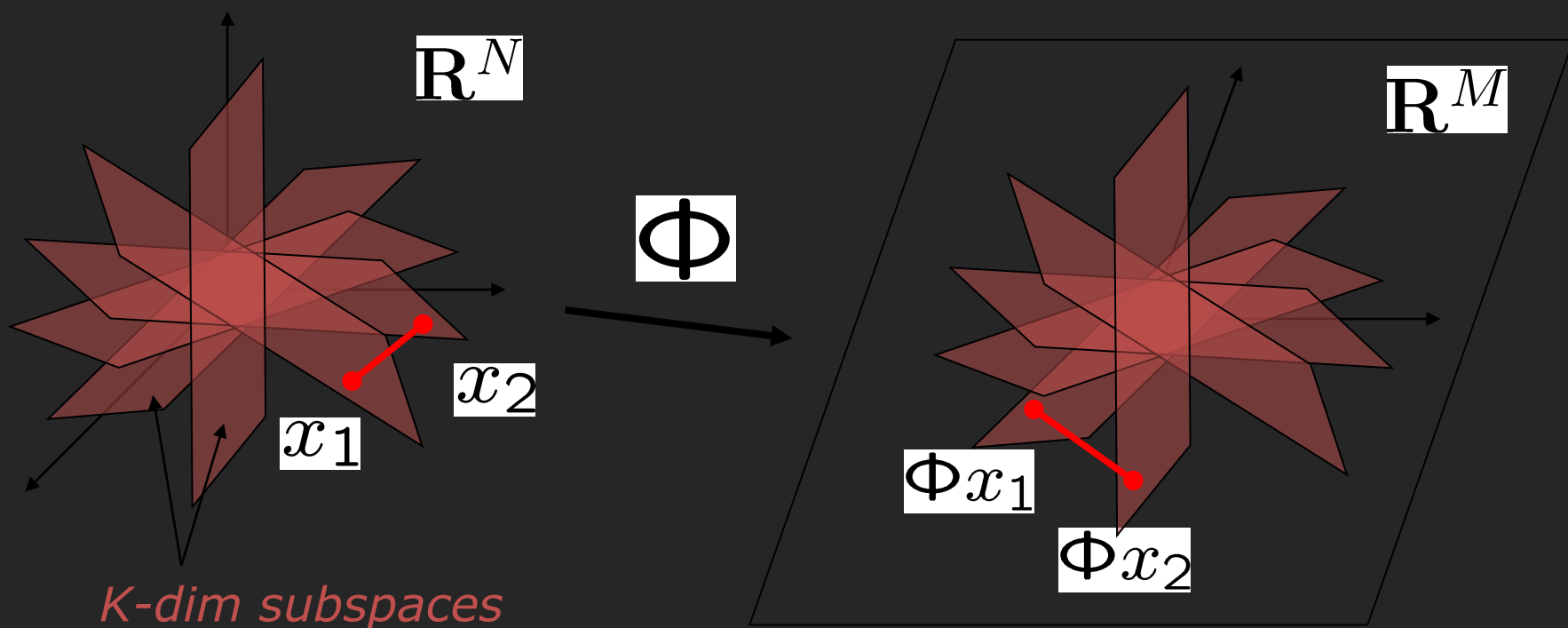
- Preserve the structure of sparse/compressible signals



Restricted Isometry Property (RIP)

- RIP of order $2K$ implies: for all K -sparse x_1 and x_2

$$(1 - \delta_{2K}) \leq \frac{\|\Phi x_1 - \Phi x_2\|_2^2}{\|x_1 - x_2\|_2^2} \leq (1 + \delta_{2K})$$



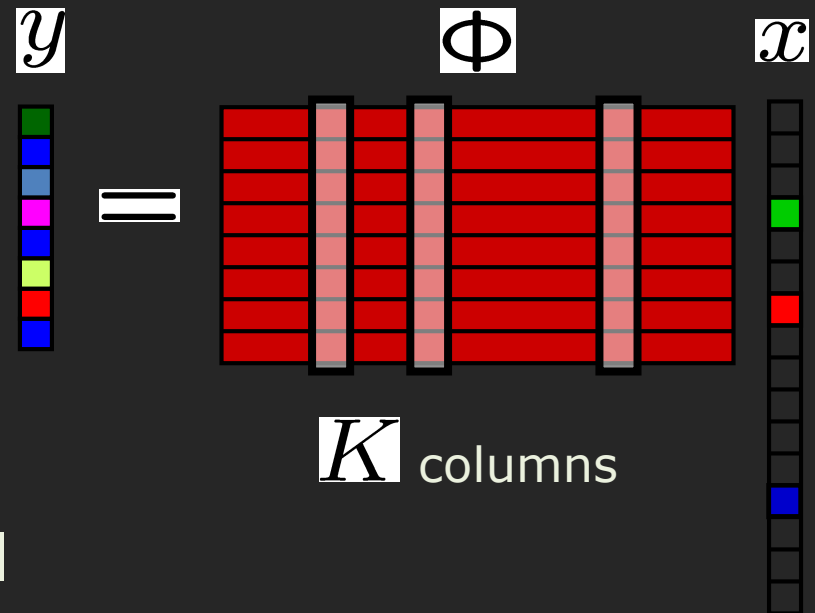
How Can It Work?

- Matrix Φ
not full rank...

$$M < N$$

... and so
loses information in general

- Design** Φ so that each of its $M \times 2K$ submatrices
are full rank (RIP)

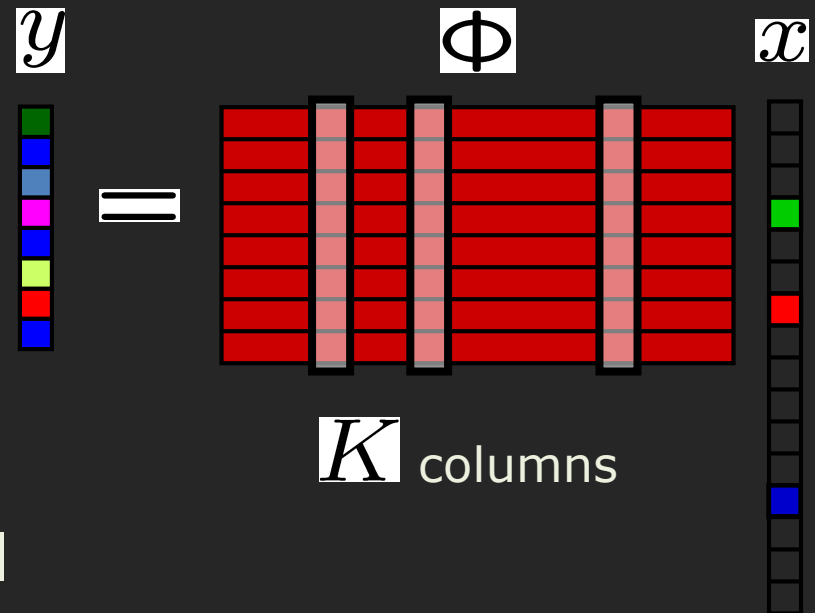


How Can It Work?

- Matrix Φ
not full rank...

$$M < N$$

... and so
loses information in general



- Design** Φ so that each of its $M \times 2K$ submatrices are full rank (RIP)
- Random measurements provide RIP with

$$M = O(K \log(N/K))$$

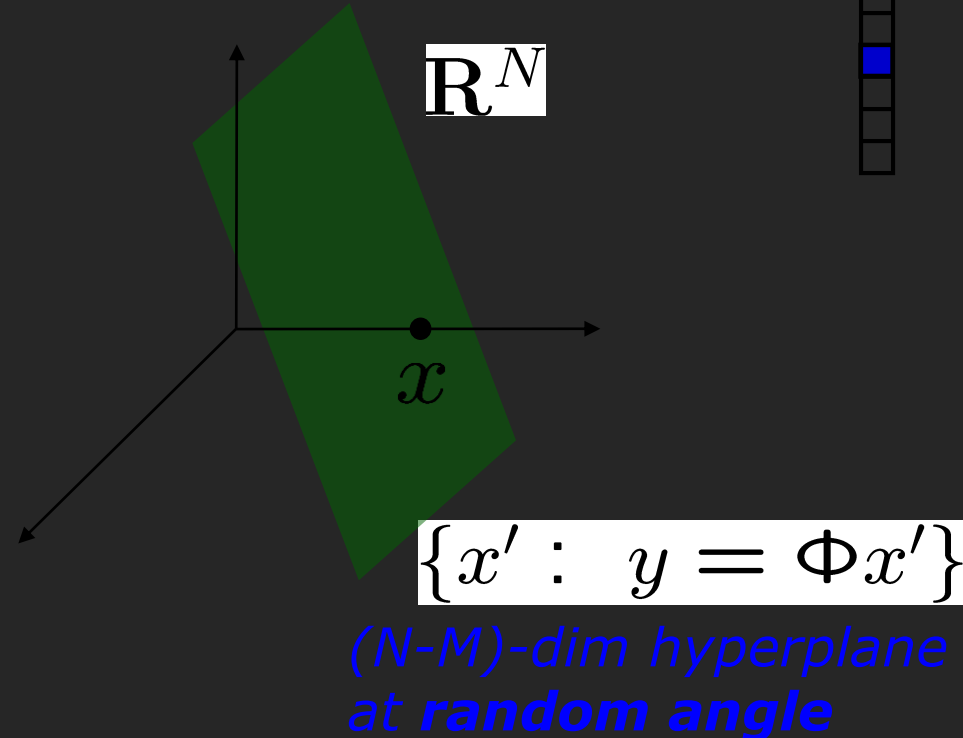
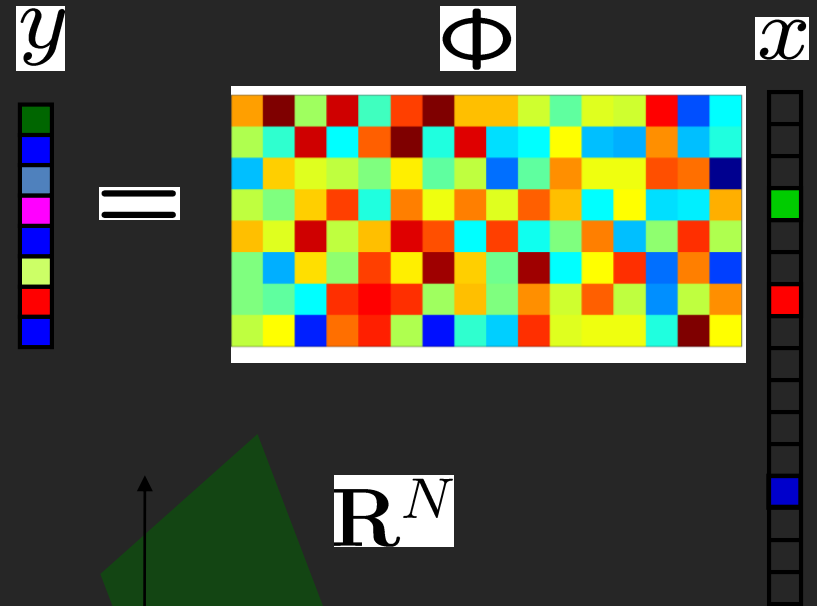
CS Signal Recovery

- Random projection Φ
not full rank

- Recovery problem:
given $y = \Phi x$
find x

- Null space**

- Search in null space
for the “sparsest” x



ℓ_1 Signal Recovery

- Recovery:
(ill-posed inverse problem) given $y = \Phi x$
find $\hat{x}$ (sparse)

- Optimization: $\hat{x} = \arg \min_{y=\Phi x} \|x\|_1$

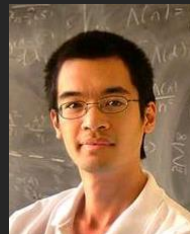
- Convexify** the ℓ_0 optimization



Candes



Romberg



Tao



Donoho

ℓ_1 Signal Recovery

- Recovery:
(ill-posed inverse problem)

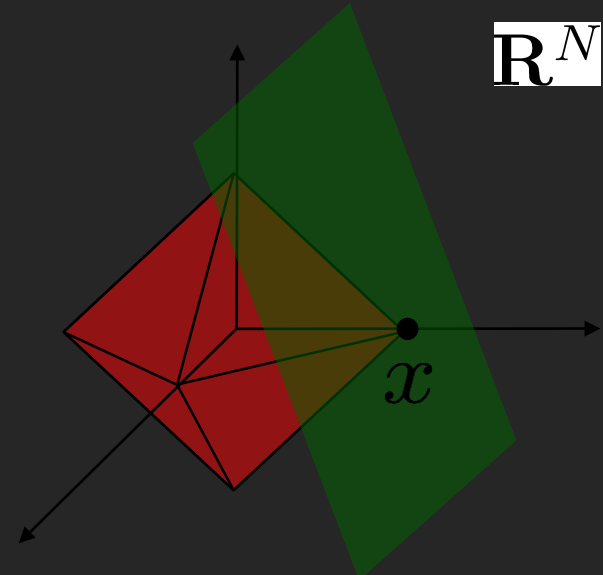
given $y = \Phi x$
find x (sparse)

- Optimization:

$$\hat{x} = \arg \min_{y=\Phi x} \|x\|_1$$

- **Convexify** the ℓ_0 optimization

- **Polynomial time** alg
(linear programming)



Compressive Sensing

Let. $y = \Phi x_0 + e$

$$\hat{x} = \arg \min_x \|x\|_1 \quad s.t. \quad \|y - \Phi x\|_2 \leq \|e\|$$

If Φ satisfies RIP with $\delta_{2K} \leq \sqrt{2} - 1$,

Then

$$\|\hat{x} - x_0\|_1 \leq C_1 \|e\|_2 + C_2 \|x_0 - x_{0,K}\|_2 / \sqrt{K}$$

 **Best K-sparse approximation**