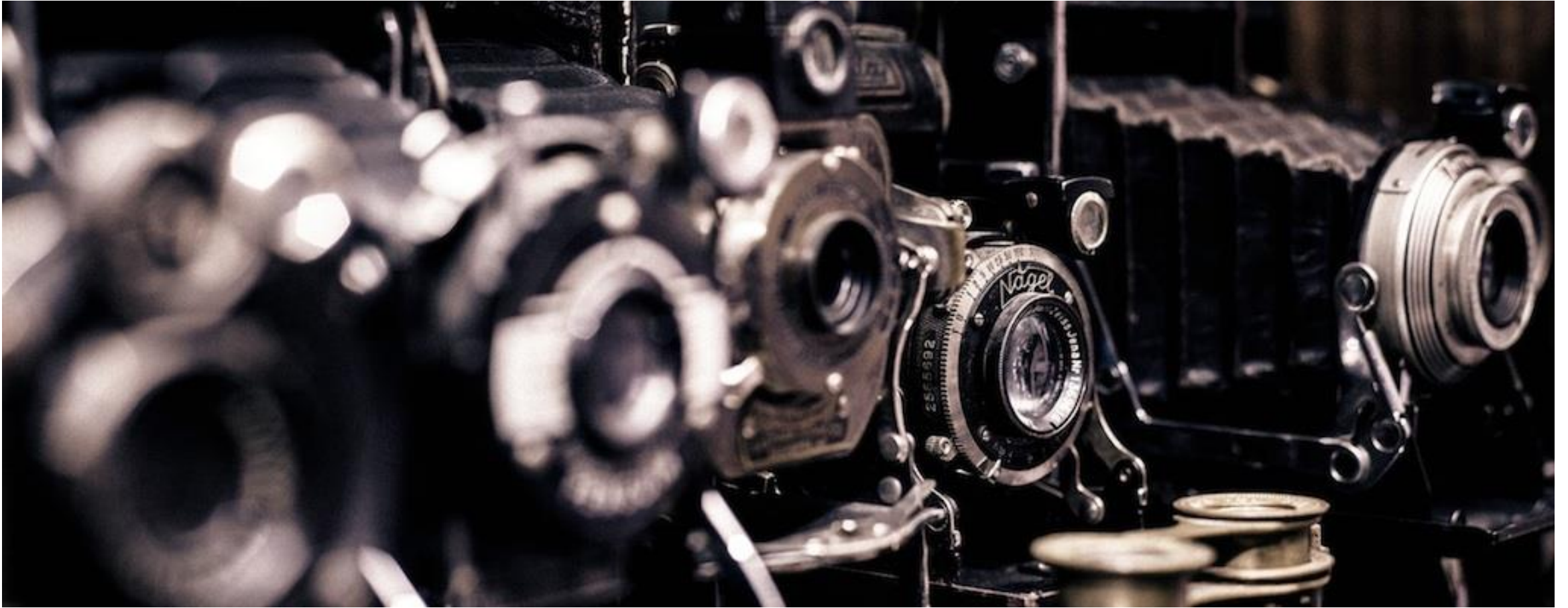


Geometric camera models and calibration



15-463, 15-663, 15-862
Computational Photography
Fall 2018, Lecture 13

Course announcements

- Homework 3 is out.
 - Due October 12th tonight.
 - Any questions?
- Homework 4 will be posted tonight and will be on lightfields.
- Due October 21st: Project ideas posted on Piazza.
- Next week I'll schedule extra office hours for those of you who cannot make it to the make-up lecture, and also so that you can discuss with me about the final project.
- Additional guest lecture next Monday: Anat Levin, "Coded photography."

Overview of today's lecture

- Leftover from lecture 12
- Reminder about pinhole and lens cameras
- Camera matrix.
- Perspective distortion.
- Other camera models.
- Geometric camera calibration.

Slide credits

Many of these slides were adapted from:

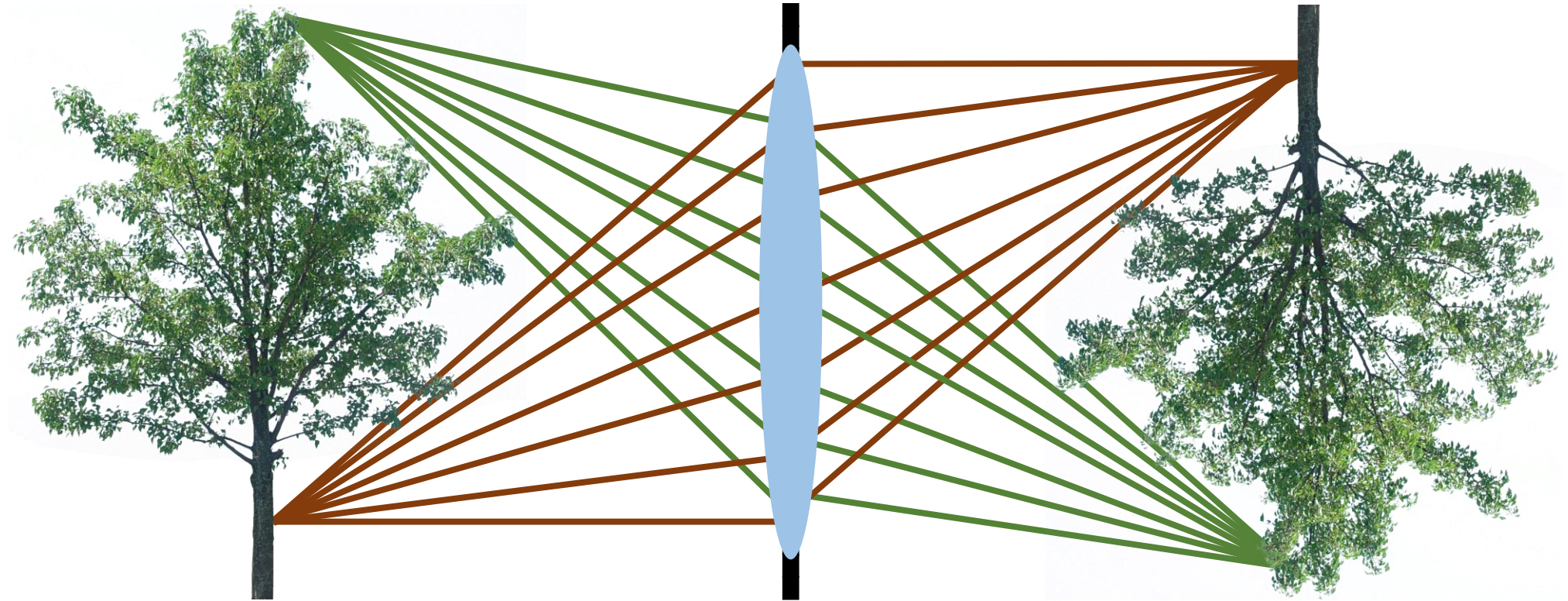
- Kris Kitani (15-463, Fall 2016).
- Srinivasa Narasimhan (16-720, Fall 2017).
- Noah Snavely (Cornell).

Some slides inspired from:

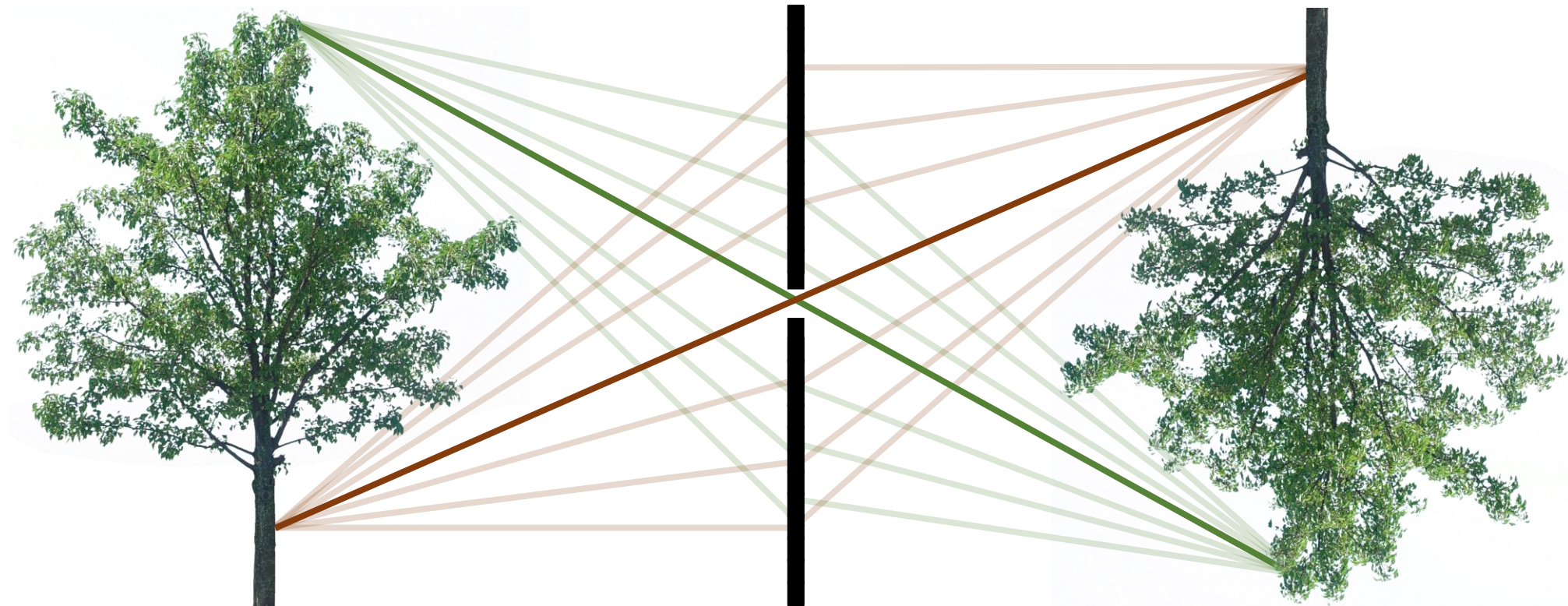
- Fredo Durand (MIT).

Pinhole and lens cameras

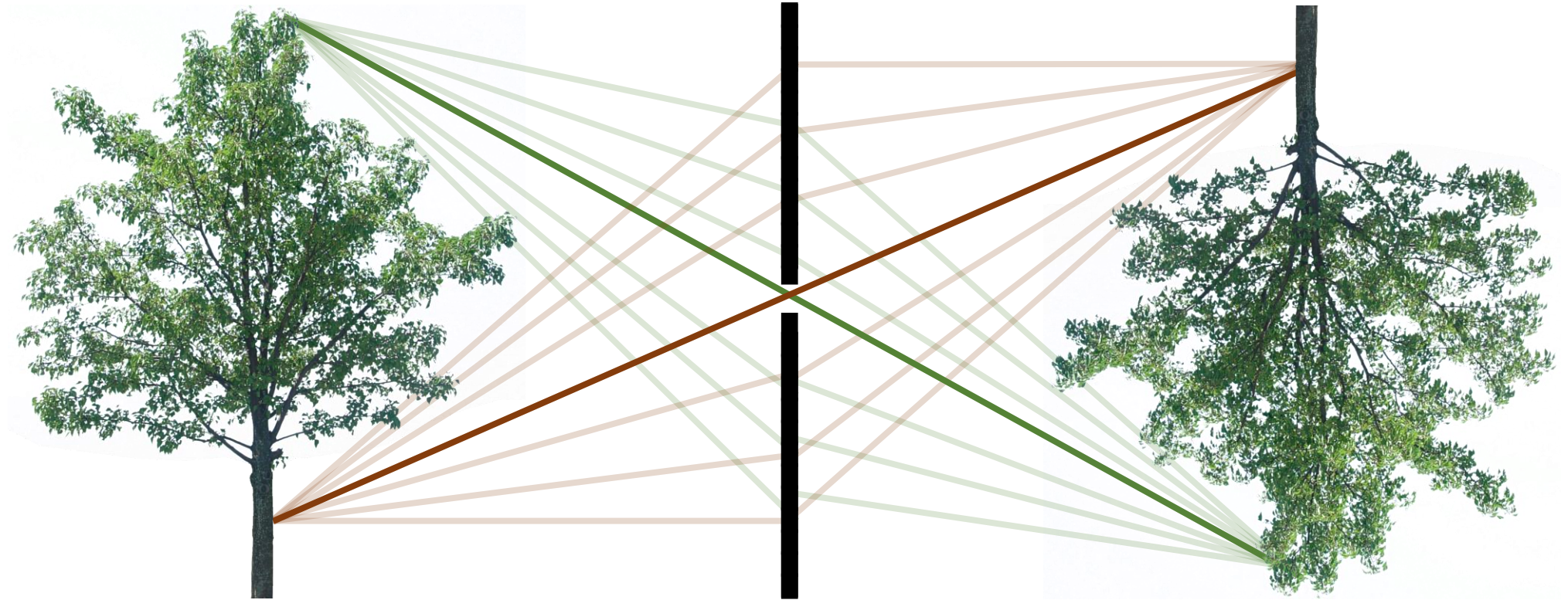
The lens camera



The pinhole camera

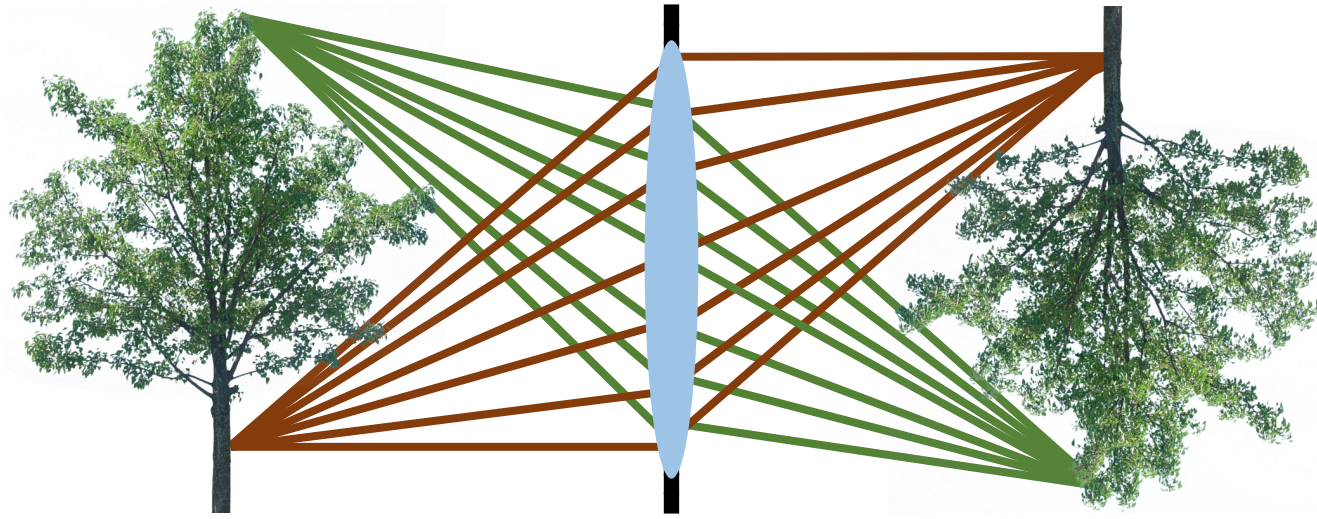


The pinhole camera



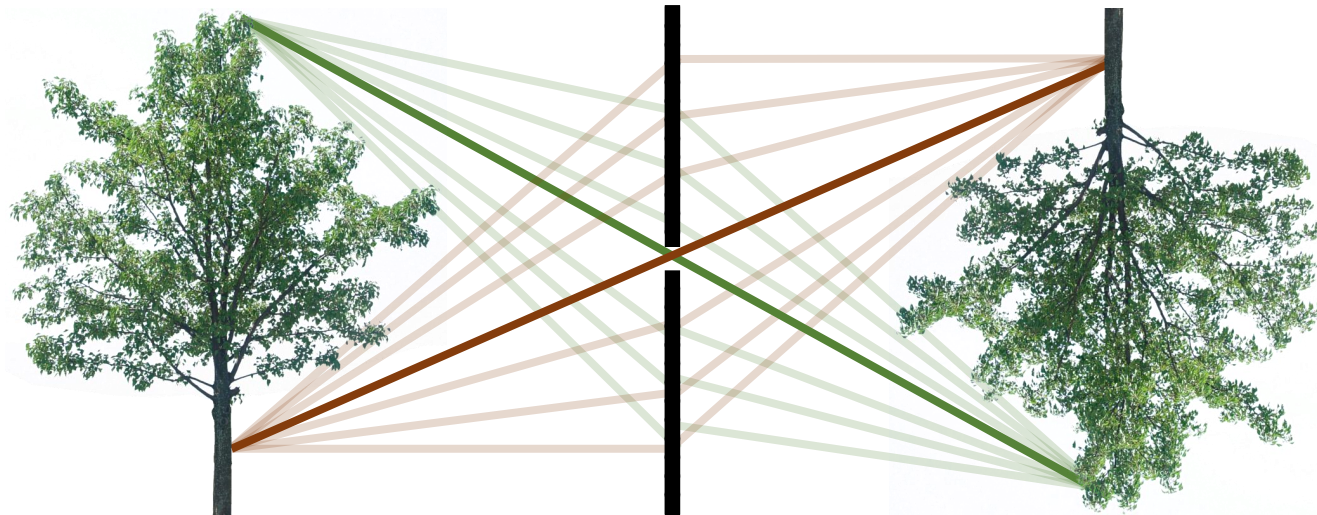
Central rays propagate in the same way for both models!

Describing both lens and pinhole cameras



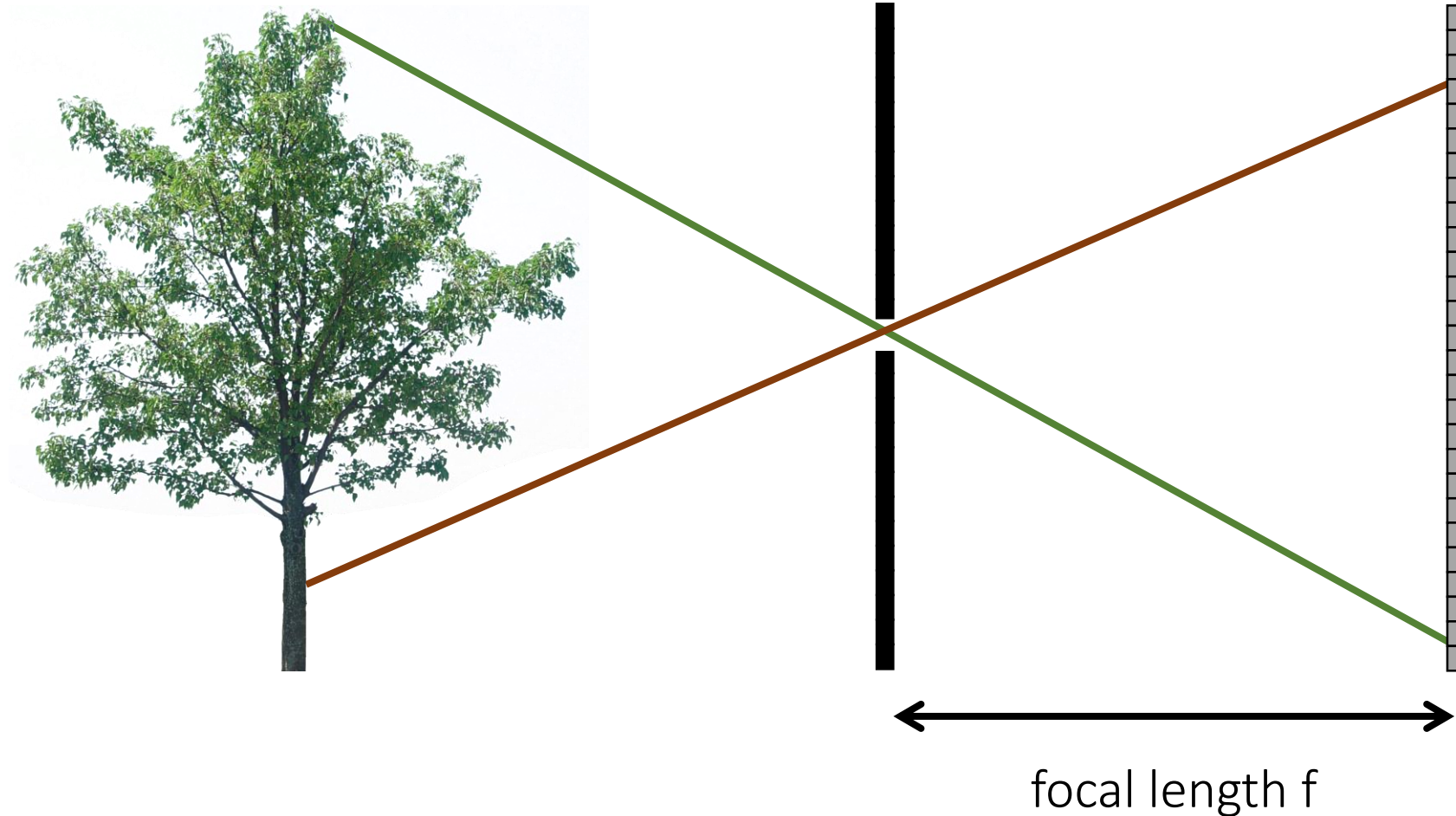
We can derive properties and descriptions that hold for both camera models if:

- We use only central rays.
- We assume the lens camera is in focus.



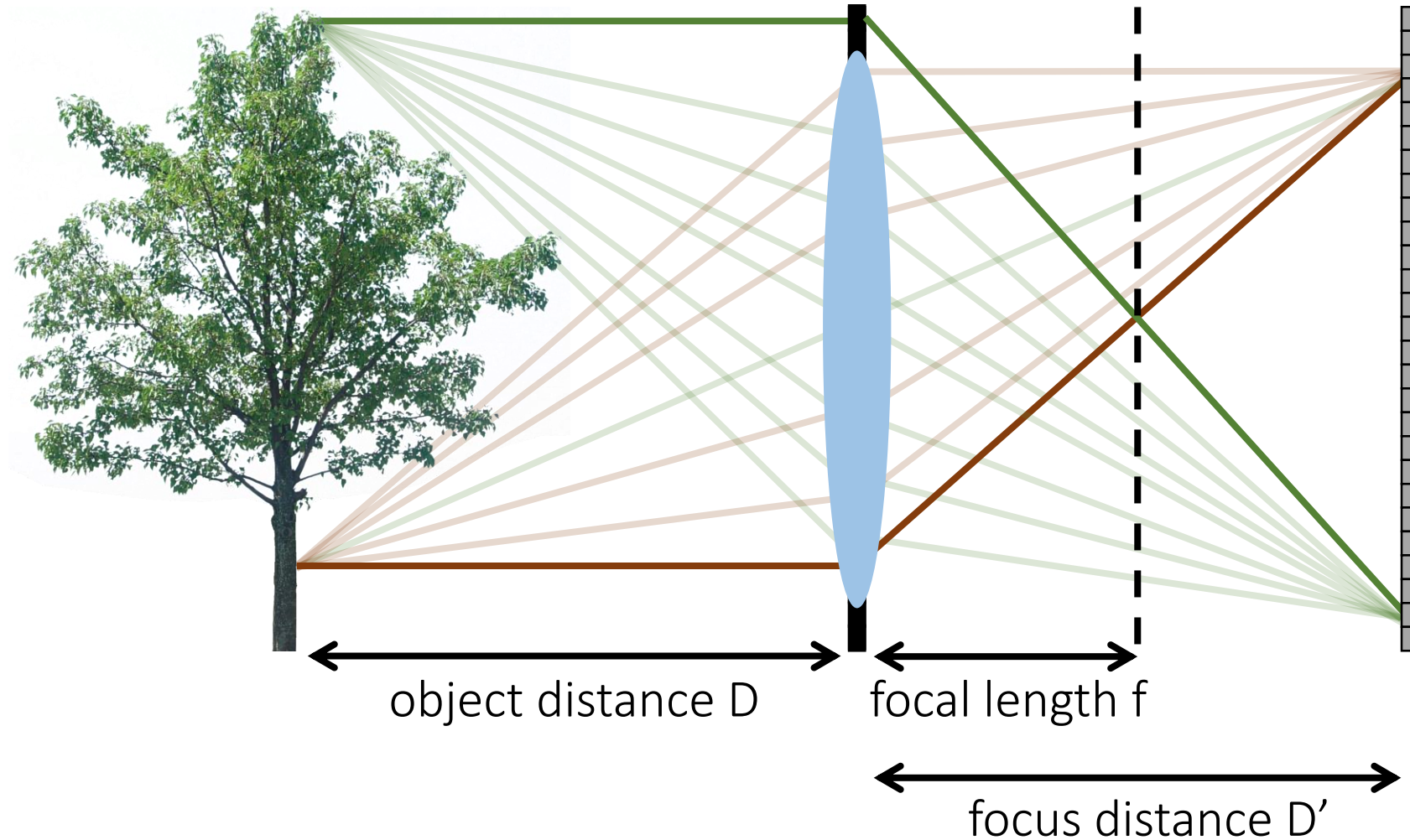
Important difference: focal length

In a pinhole camera, focal length is distance between aperture and sensor

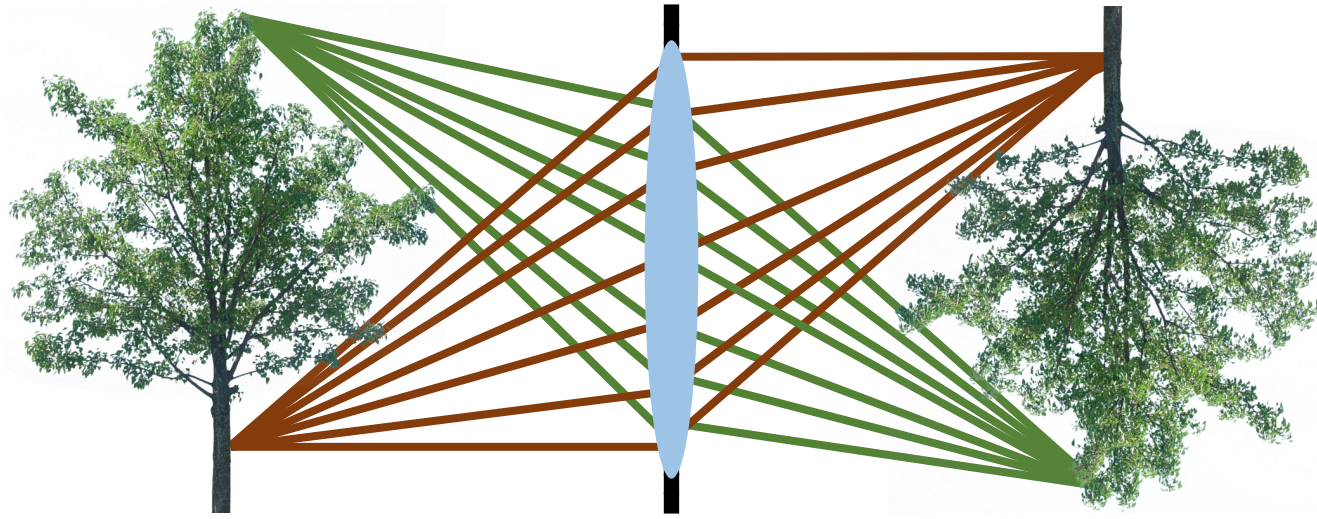


Important difference: focal length

In a lens camera, focal length is distance where parallel rays intersect

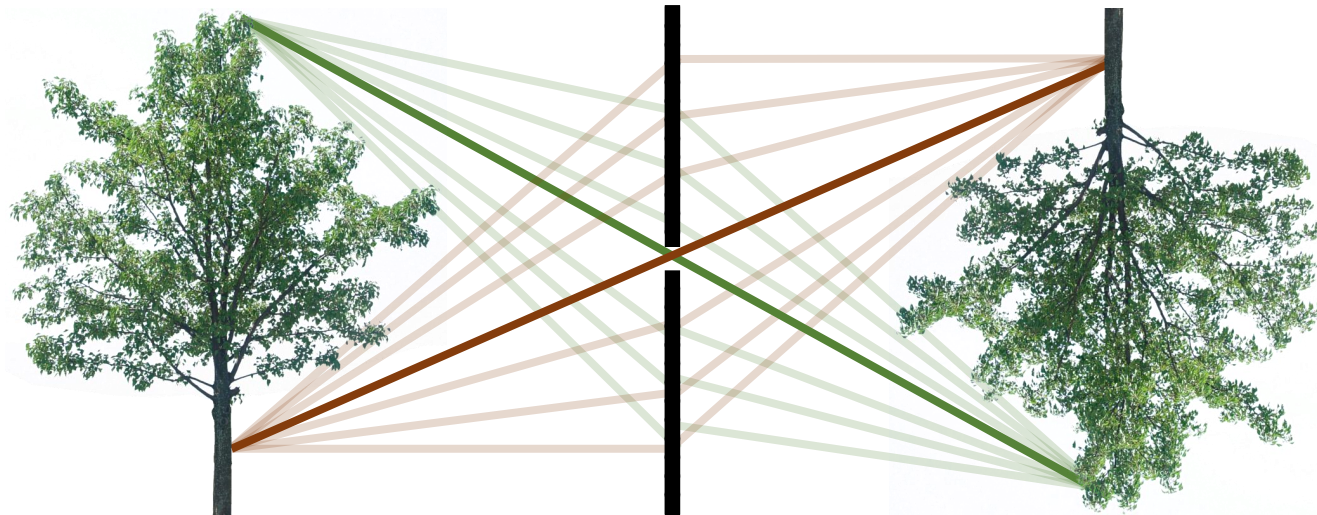


Describing both lens and pinhole cameras



We can derive properties and descriptions that hold for both camera models if:

- We use only central rays.
- We assume the lens camera is in focus.
- We assume that the focus distance of the lens camera is equal to the focal length of the pinhole camera.

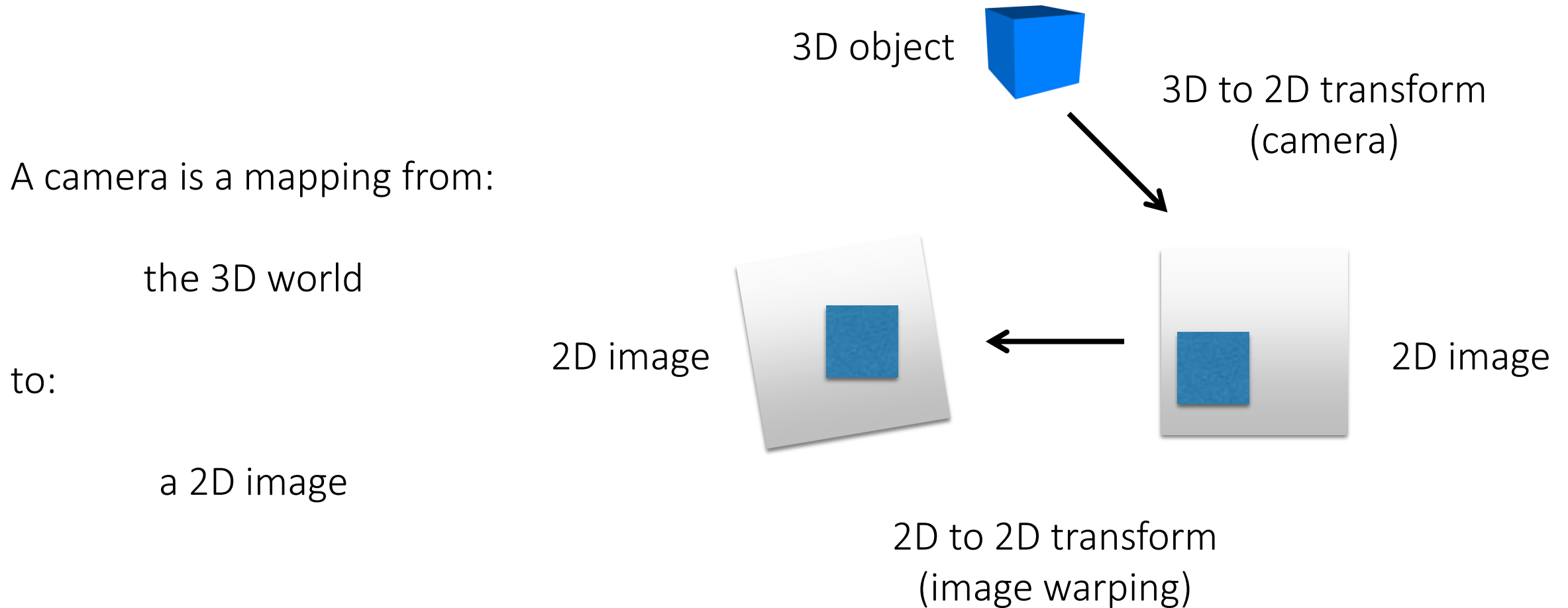


Remember: *focal length* f refers to different things for lens and pinhole cameras.

- In this lecture, we use it to refer to the aperture-sensor distance, as in the pinhole camera case.

Camera matrix

The camera as a coordinate transformation



The camera as a coordinate transformation

A camera is a mapping from:

the 3D world

to:

a 2D image

homogeneous coordinates

The diagram shows the equation $x = PX$. Above the equation, the text "homogeneous coordinates" has two arrows pointing down to the x and X terms. Below the equation, the terms are labeled: "2D image point" under x , "camera matrix" under P , and "3D world point" under X .

$$x = PX$$

2D image point camera matrix 3D world point

What are the dimensions of each variable?

The camera as a coordinate transformation

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

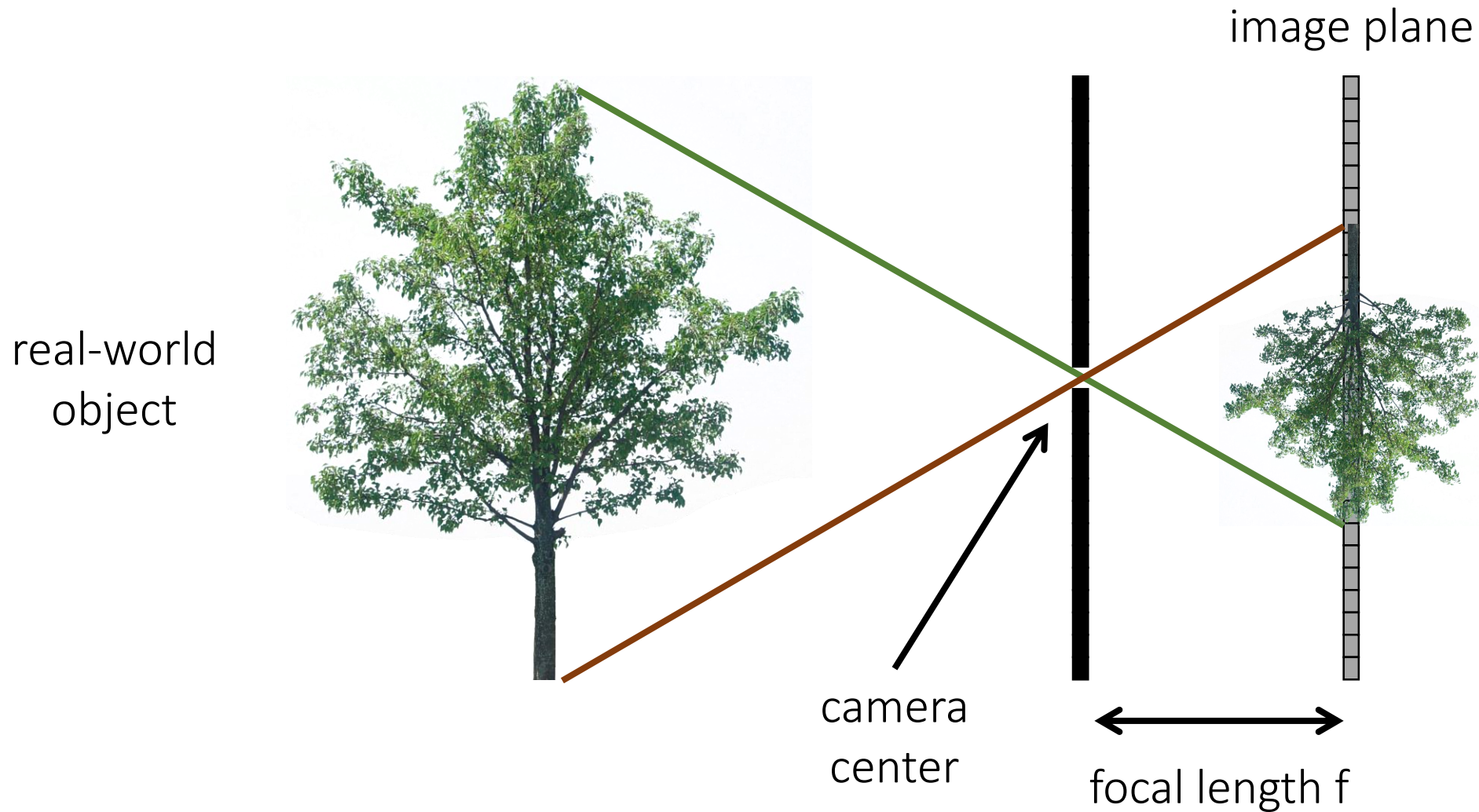
$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

homogeneous
image coordinates
3 x 1

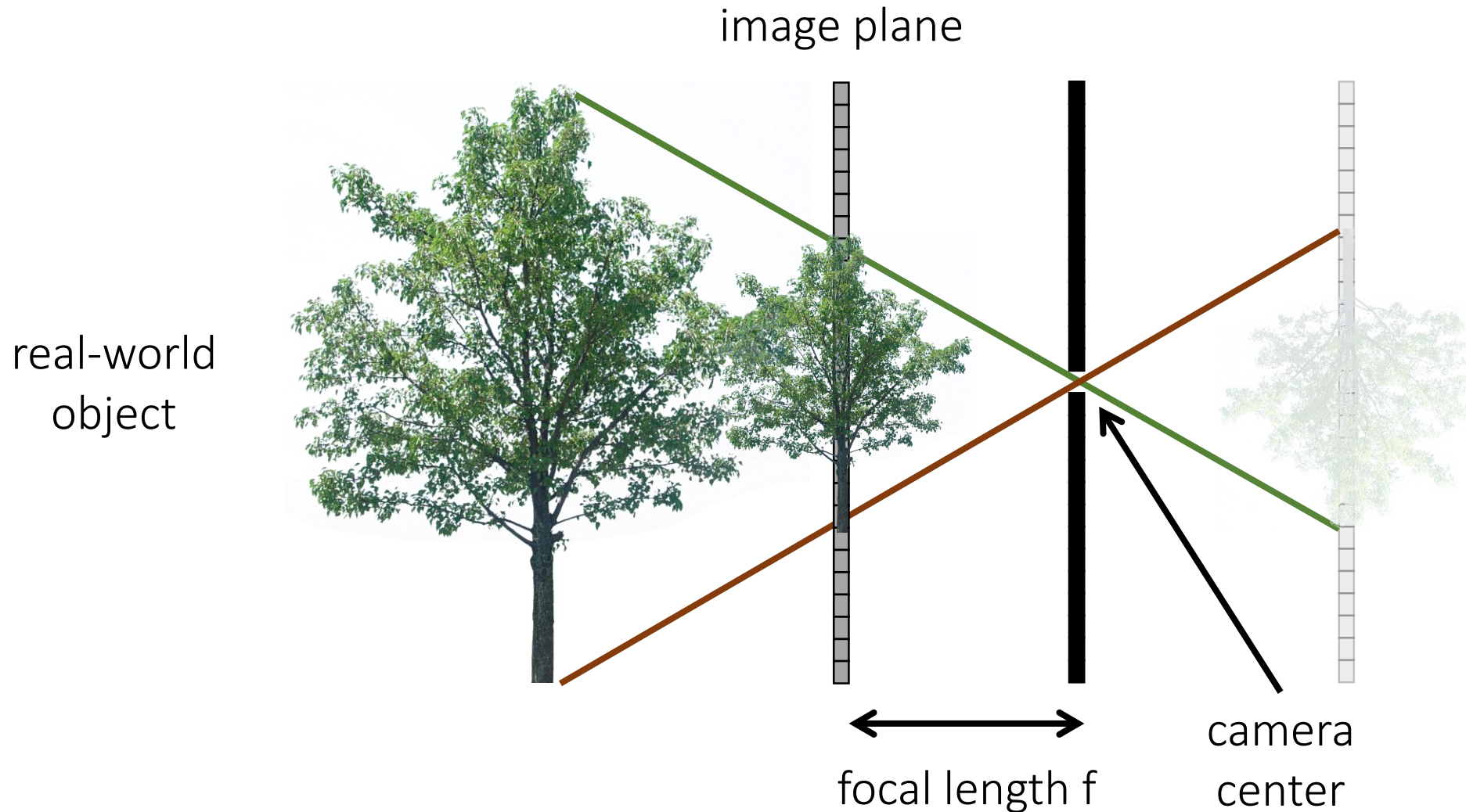
camera
matrix
3 x 4

homogeneous
world coordinates
4 x 1

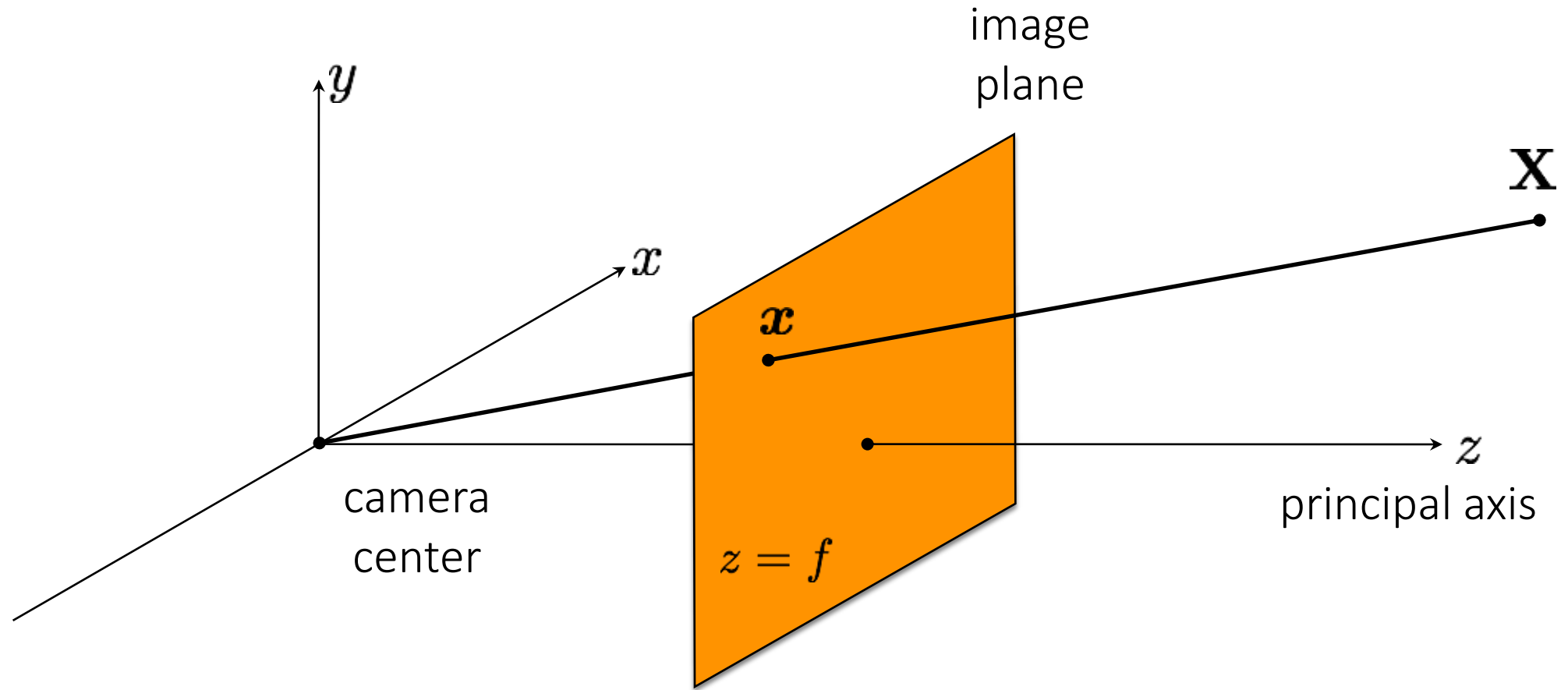
The pinhole camera



The (rearranged) pinhole camera

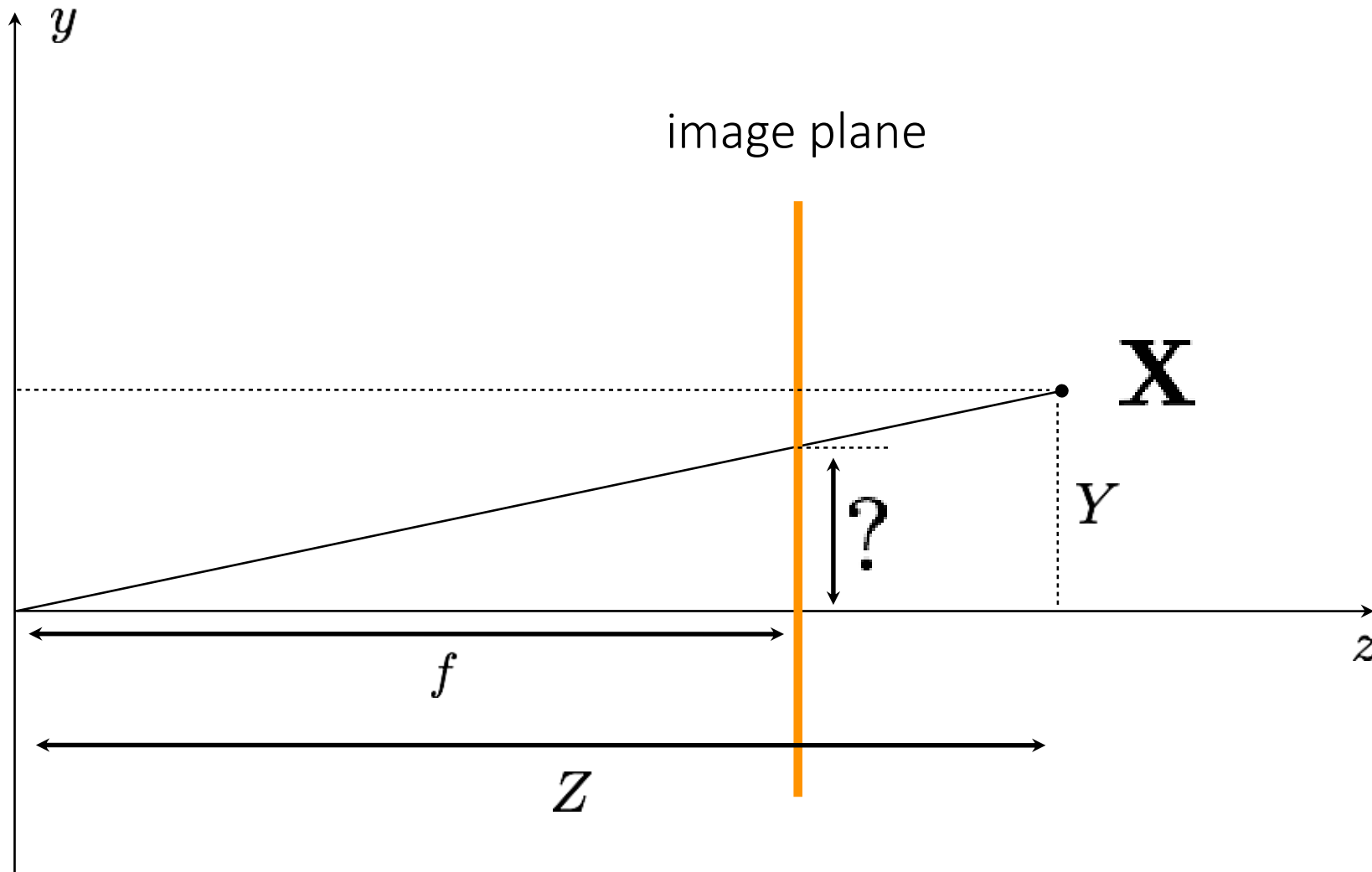


The (rearranged) pinhole camera



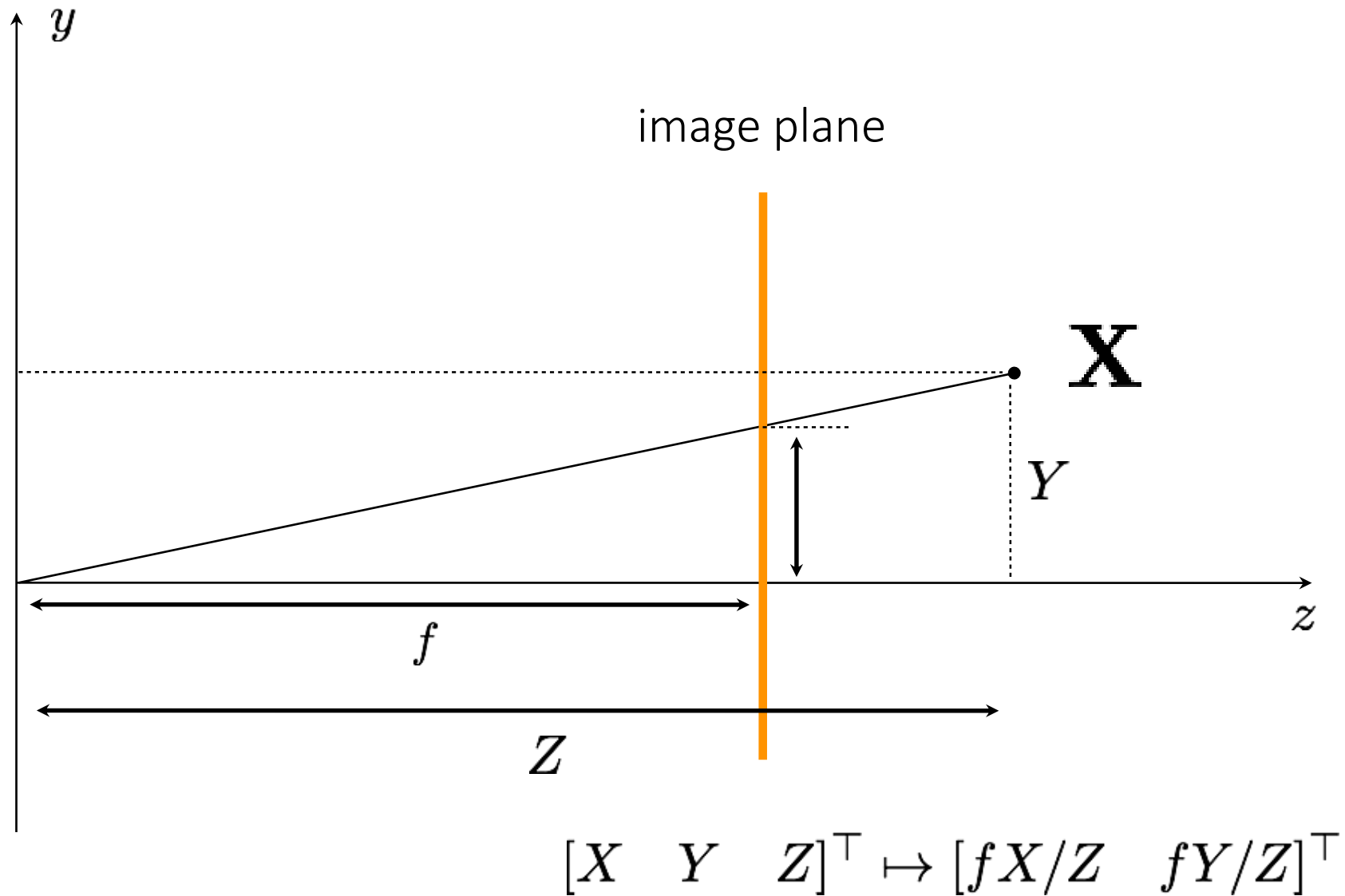
What is the equation for image coordinate $\mathbf{x}$ in terms of $\mathbf{X}$?

The 2D view of the (rearranged) pinhole camera

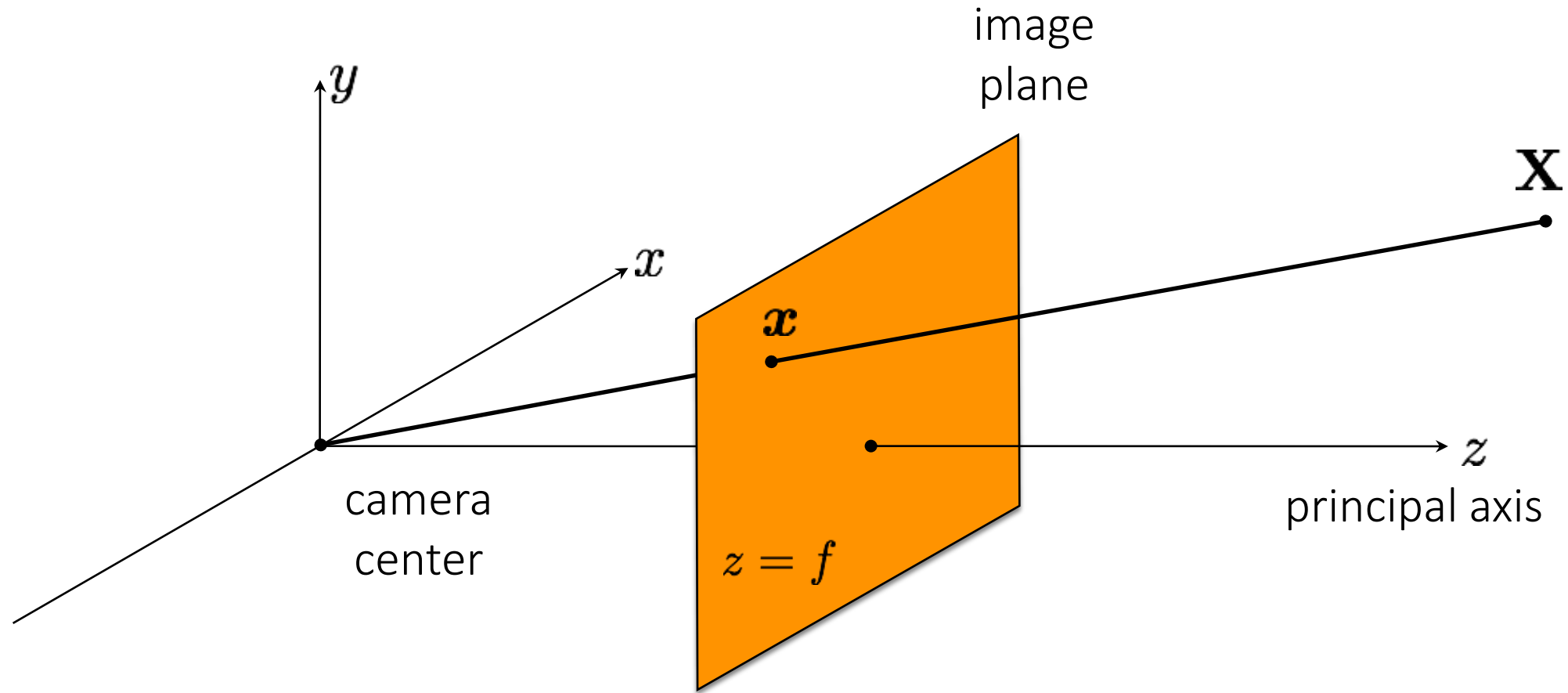


What is the equation for image coordinate $\mathbf{x}$ in terms of $\mathbf{X}$?

The 2D view of the (rearranged) pinhole camera



The (rearranged) pinhole camera



What is the camera matrix $\mathbf{P}$ for a pinhole camera?

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

The pinhole camera matrix

Relationship from similar triangles:

$$[X \quad Y \quad Z]^\top \mapsto [fX/Z \quad fY/Z]^\top$$

General camera model:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What does the pinhole camera projection look like?

$$\mathbf{P} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

The pinhole camera matrix

Relationship from similar triangles:

$$[X \quad Y \quad Z]^{\top} \mapsto [fX/Z \quad fY/Z]^{\top}$$

General camera model:

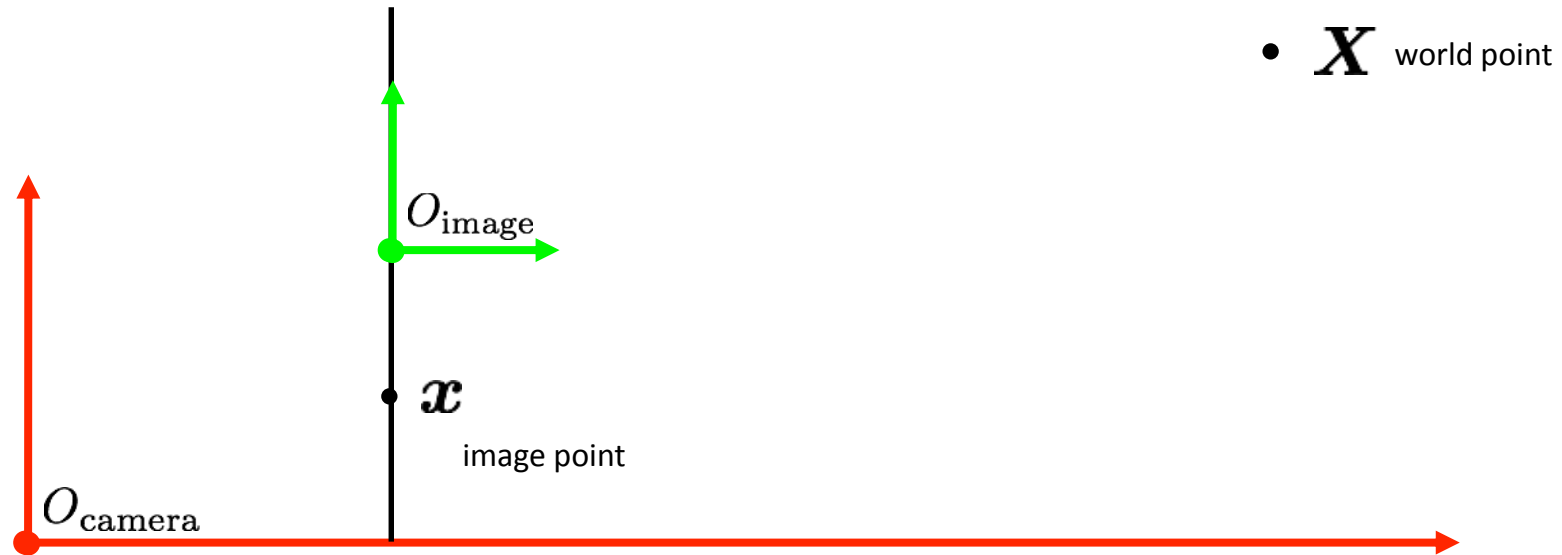
$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What does the pinhole camera projection look like?

$$\mathbf{P} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

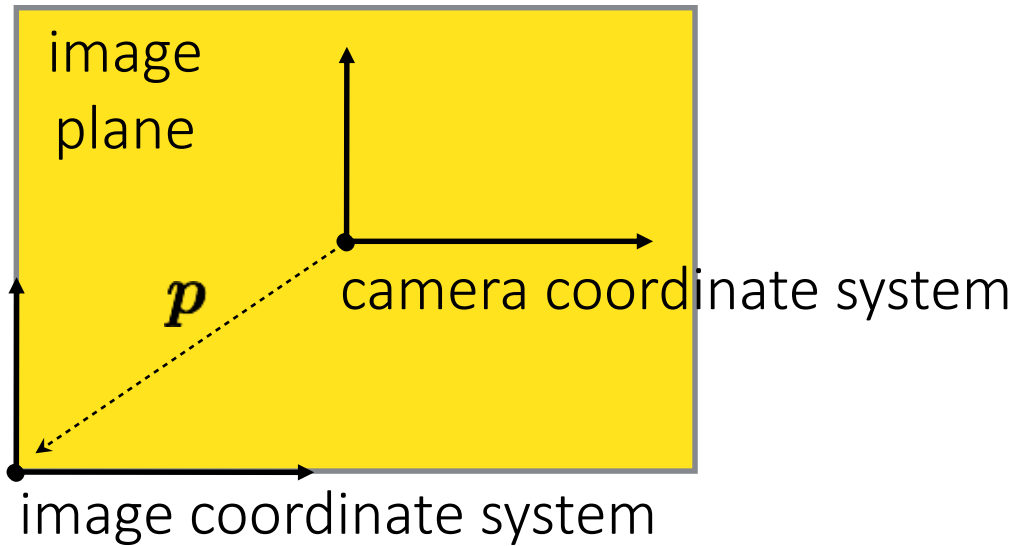
Generalizing the camera matrix

In general, the camera and image have *different* coordinate systems.



Generalizing the camera matrix

In particular, the camera origin and image origin may be different:

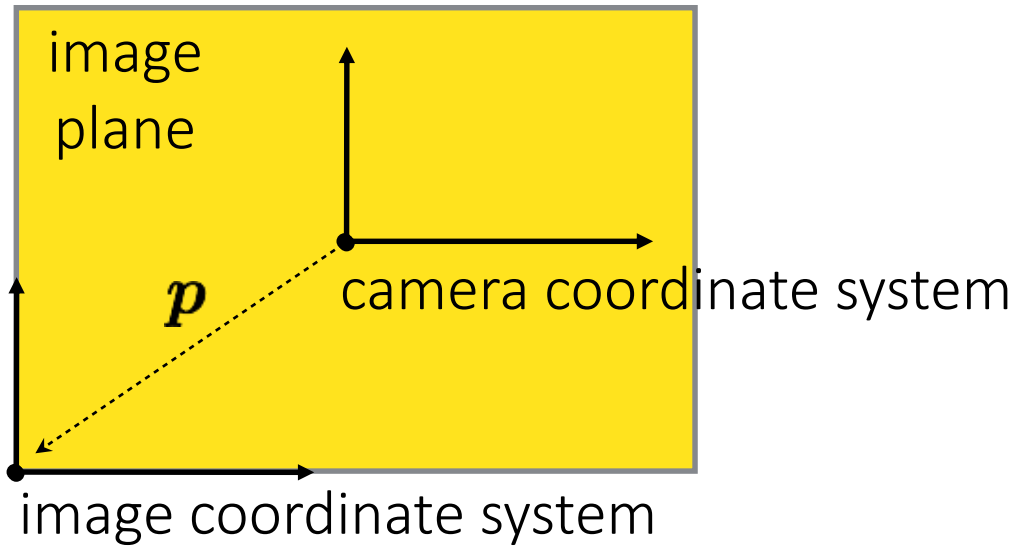


How does the camera matrix change?

$$\mathbf{P} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Generalizing the camera matrix

In particular, the camera origin and image origin may be different:



How does the camera matrix change?

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

shift vector
transforming
camera origin to
image origin

Camera matrix decomposition

We can decompose the camera matrix like this:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

What does each part of the matrix represent?

Camera matrix decomposition

We can decompose the camera matrix like this:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$



(homogeneous) transformation
from 2D to 2D, accounting for not
unit focal length and origin shift

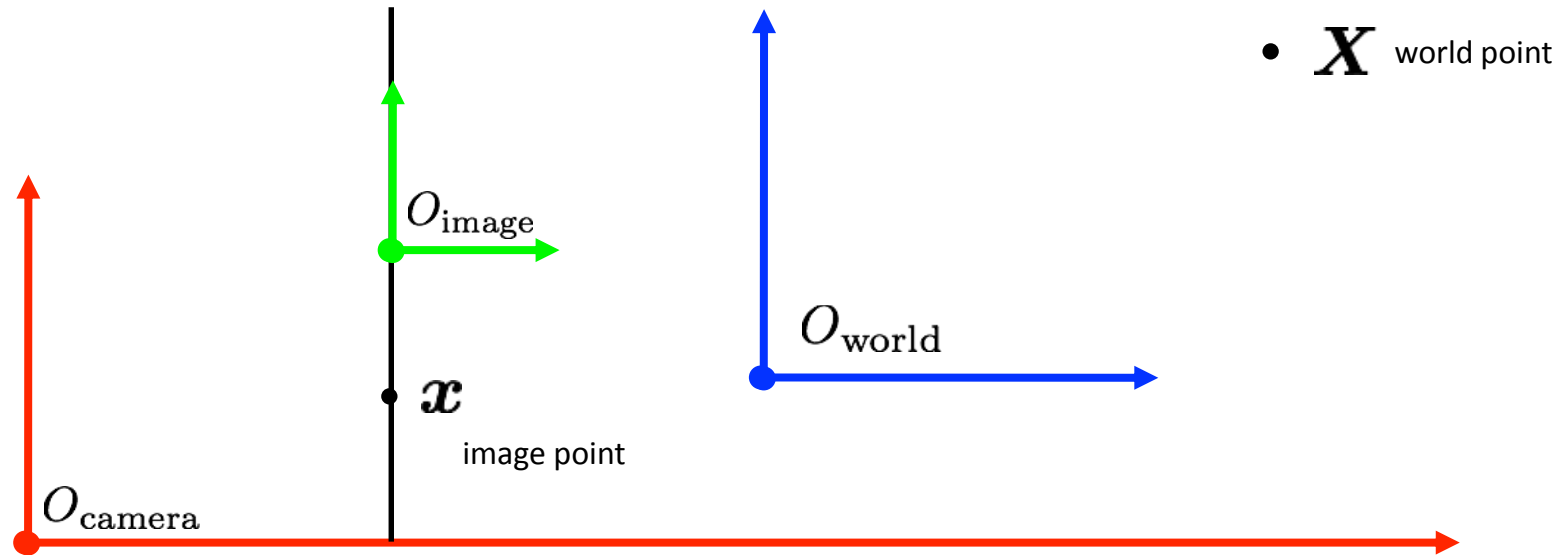


(homogeneous) projection from 3D
to 2D, assuming image plane at $z = 1$
and shared camera/image origin

Also written as: $\mathbf{P} = \mathbf{K}[\mathbf{I}|\mathbf{0}]$ where $\mathbf{K} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}$

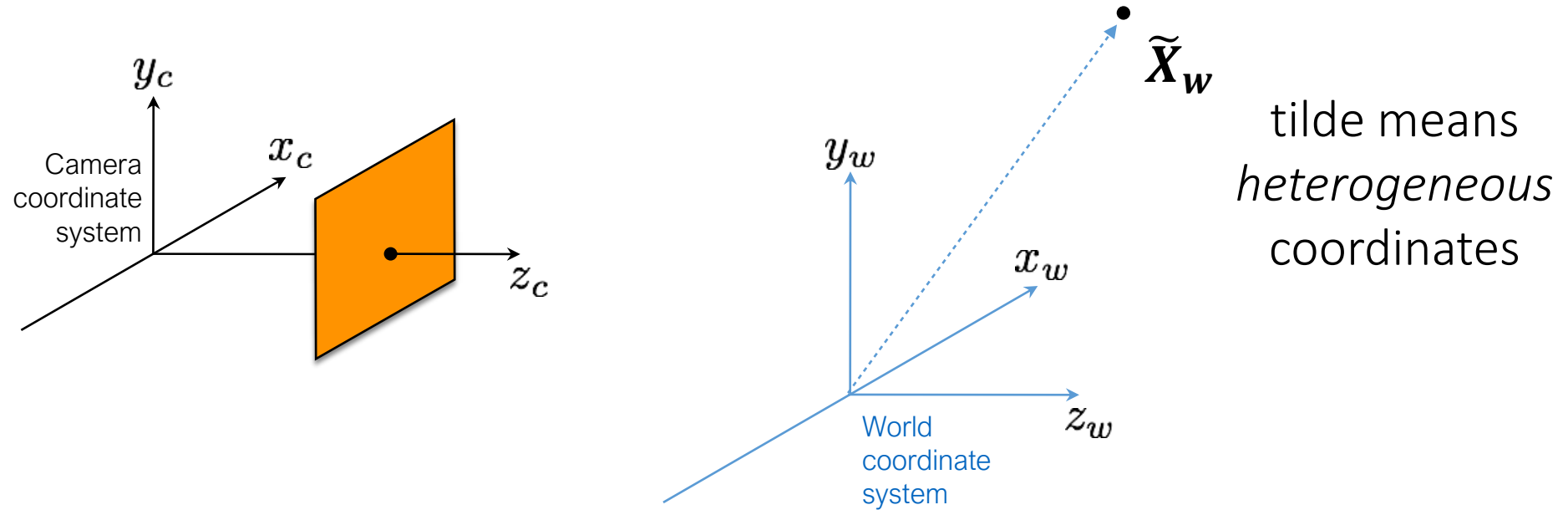
Generalizing the camera matrix

In general, there are *three*, generally different, coordinate systems.

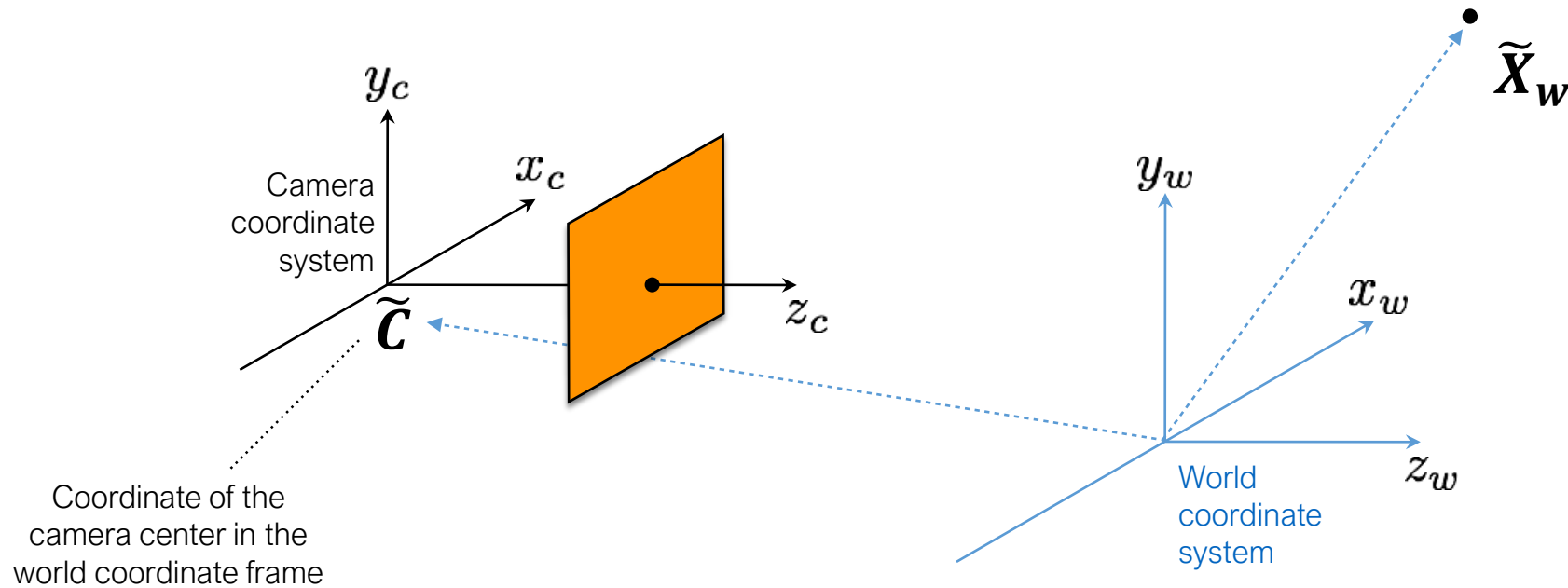


We need to know the transformations between them.

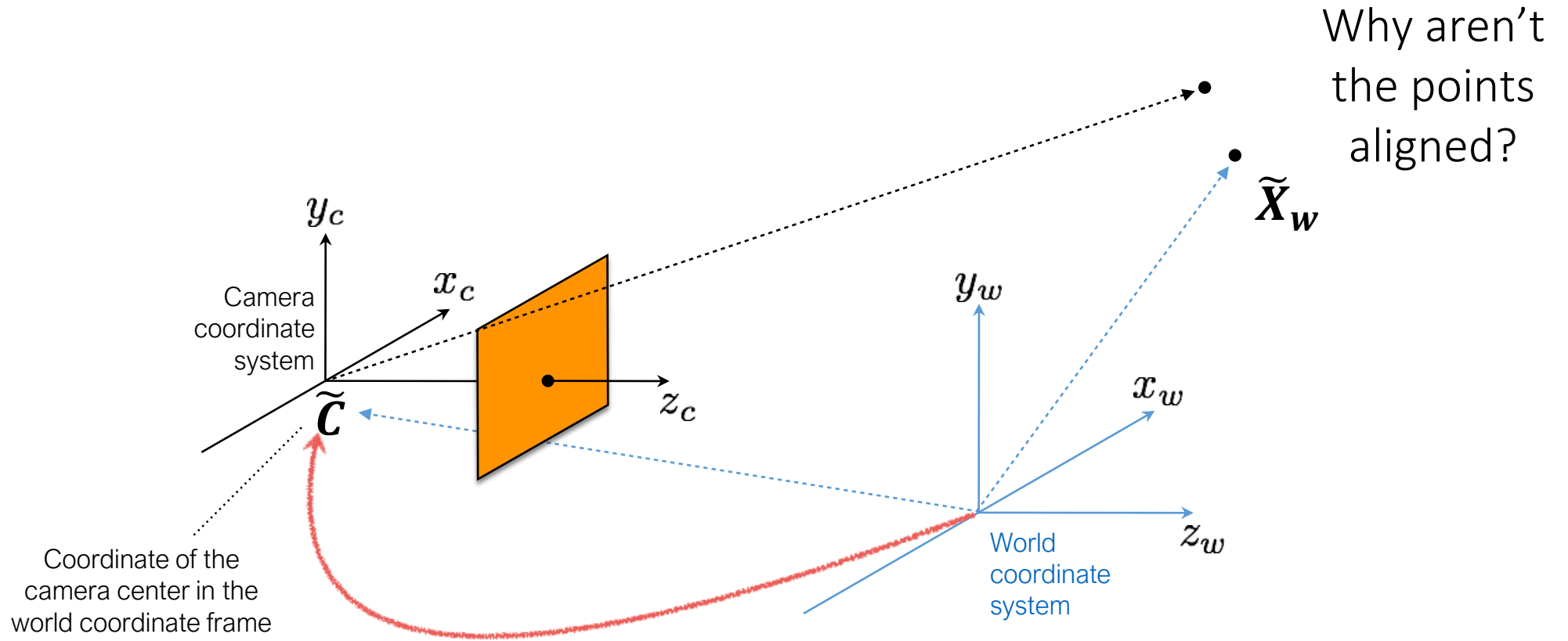
World-to-camera coordinate system transformation



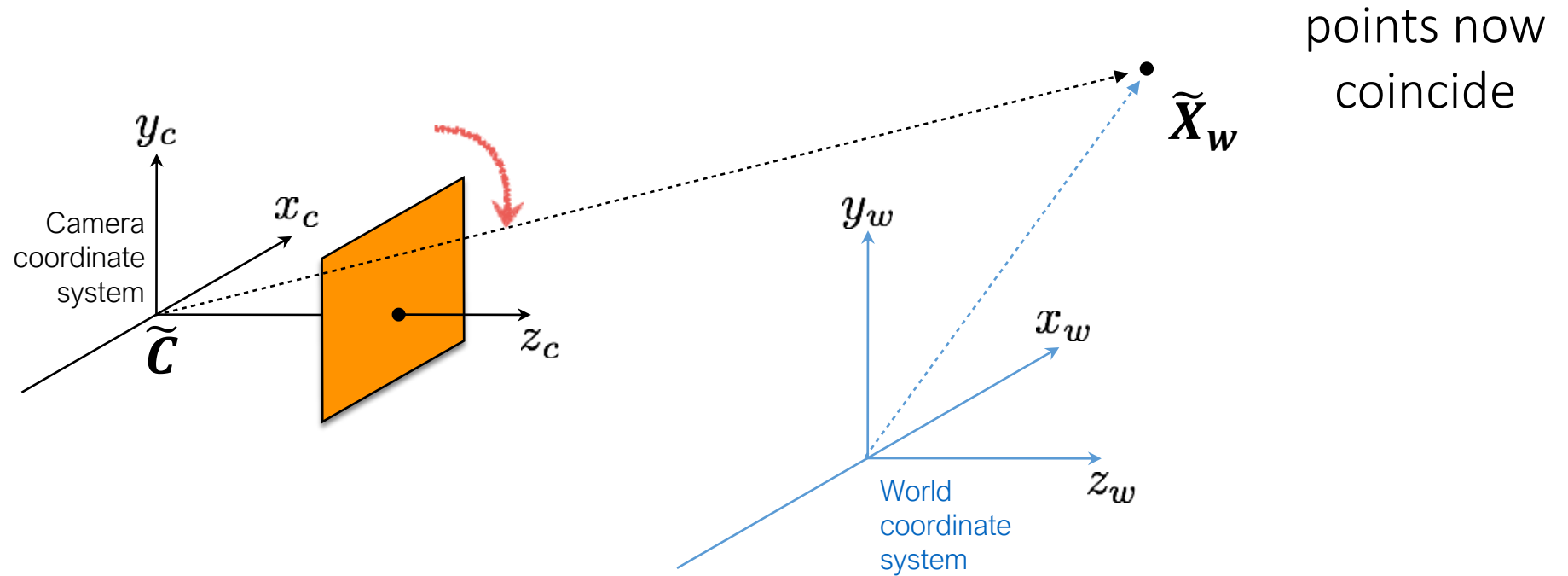
World-to-camera coordinate system transformation



World-to-camera coordinate system transformation



World-to-camera coordinate system transformation



$$\underset{\text{rotate}}{R} \cdot (\underset{\text{translate}}{\tilde{\mathbf{X}}_w - \tilde{\mathbf{C}}})$$

Modeling the coordinate system transformation

In heterogeneous coordinates, we have:

$$\tilde{\mathbf{X}}_{\mathbf{c}} = \mathbf{R} \cdot (\tilde{\mathbf{X}}_{\mathbf{w}} - \tilde{\mathbf{C}})$$

How do we write this transformation in homogeneous coordinates?

Modeling the coordinate system transformation

In heterogeneous coordinates, we have:

$$\tilde{\mathbf{X}}_c = \mathbf{R} \cdot (\tilde{\mathbf{X}}_w - \tilde{\mathbf{C}})$$

In homogeneous coordinates, we have:

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\mathbf{C} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} \quad \text{or} \quad \mathbf{X}_c = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{X}_w$$

Incorporating the transform in the camera matrix

The previous camera matrix is for homogeneous 3D coordinates in camera coordinate system:

$$\mathbf{x} = \mathbf{P}\mathbf{X}_c = \mathbf{K}[\mathbf{I}|\mathbf{0}]\mathbf{X}_c$$

We also just derived:

$$\mathbf{X}_c = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{X}_w$$

Putting it all together

We can write everything into a single projection:

$$\mathbf{x} = \mathbf{P}\mathbf{X}_w$$

The camera matrix now looks like:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix}$$

intrinsic parameters (3 x 3):
correspond to camera internals
(sensor not at $f = 1$ and origin shift)



extrinsic parameters (3 x 4):
correspond to camera externals
(world-to-camera transformation)



General pinhole camera matrix

We can decompose the camera matrix like this:

$$\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I} | -\mathbf{C}]$$

(translate first then rotate)

Another way to write the mapping:

$$\mathbf{P} = \mathbf{K}[\mathbf{R} | \mathbf{t}]$$

where $\mathbf{t} = -\mathbf{R}\mathbf{C}$

(rotate first then translate)

General pinhole camera matrix

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 & r_2 & r_3 & t_1 \\ r_4 & r_5 & r_6 & t_2 \\ r_7 & r_8 & r_9 & t_3 \end{bmatrix}$$

intrinsic
parameters

extrinsic
parameters

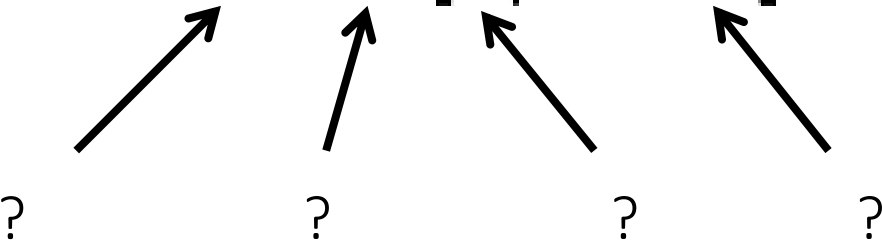
$$\mathbf{R} = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \quad \mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

3D rotation

3D translation

Recap

What is the size and meaning of each term in the camera matrix?

$$\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I} | -\mathbf{C}]$$


The diagram illustrates the components of the camera matrix equation $\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I} | -\mathbf{C}]$. Four arrows point from question marks below to the terms $\mathbf{K}$, $\mathbf{R}$, $[\mathbf{I} | -\mathbf{C}]$, and the overall expression $\mathbf{P}$, indicating a query about their size and meaning.

Recap

What is the size and meaning of each term in the camera matrix?

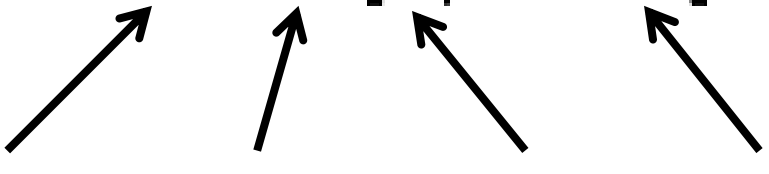
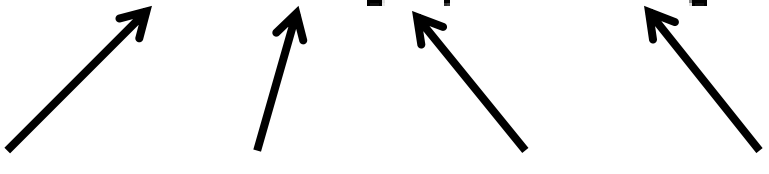
$$\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I} | -\mathbf{C}]$$


Diagram illustrating the components of the camera matrix equation $\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I} | -\mathbf{C}]$:

- $\mathbf{P}$: 3x3 intrinsics
- $\mathbf{K}$: ?
- $\mathbf{R}$: ?
- $[\mathbf{I} | -\mathbf{C}]$: ?

Recap

What is the size and meaning of each term in the camera matrix?

$$\mathbf{P} = \mathbf{K} \mathbf{R} [\mathbf{I} | \mathbf{C}]$$


The diagram shows four arrows pointing from labels below to terms in the equation $\mathbf{P} = \mathbf{K} \mathbf{R} [\mathbf{I} | \mathbf{C}]$. The first arrow points from '3x3 intrinsics' to $\mathbf{K}$. The second arrow points from '3x3 3D rotation' to $\mathbf{R}$. The third arrow points from a '?' to the $\mathbf{I}$ in the vector $[\mathbf{I} | \mathbf{C}]$. The fourth arrow points from a '?' to $\mathbf{C}$.

3x3
intrinsics

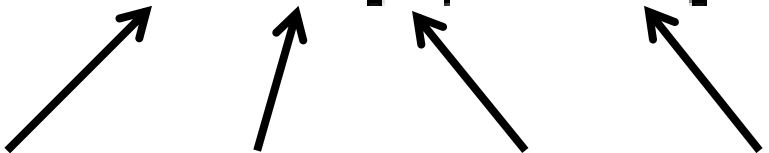
3x3
3D rotation

?

?

Recap

What is the size and meaning of each term in the camera matrix?

$$\mathbf{P} = \mathbf{K} \mathbf{R} [\mathbf{I} | \mathbf{C}]$$


The diagram shows four arrows pointing upwards from labels below to terms in the equation $\mathbf{P} = \mathbf{K} \mathbf{R} [\mathbf{I} | \mathbf{C}]$. The first arrow points from '3x3 intrinsics' to $\mathbf{K}$. The second arrow points from '3x3 3D rotation' to $\mathbf{R}$. The third arrow points from '3x3 identity' to $\mathbf{I}$. The fourth arrow points from '?' to $\mathbf{C}$.

3x3
intrinsics

3x3
3D rotation

3x3
identity

?

Recap

What is the size and meaning of each term in the camera matrix?

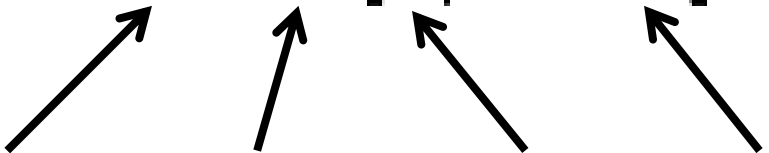
$$\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I} | -\mathbf{C}]$$


Diagram illustrating the components of the camera matrix $\mathbf{P}$:

- $\mathbf{K}$: 3x3 intrinsics
- $\mathbf{R}$: 3x3 3D rotation
- $[\mathbf{I}]$: 3x3 identity
- $-\mathbf{C}$: 3x1 3D translation

Quiz

The camera matrix relates what two quantities?

Quiz

The camera matrix relates what two quantities?

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

homogeneous 3D points to 2D image points

Quiz

The camera matrix relates what two quantities?

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

homogeneous 3D points to 2D image points

The camera matrix can be decomposed into?

Quiz

The camera matrix relates what two quantities?

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

homogeneous 3D points to 2D image points

The camera matrix can be decomposed into?

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

intrinsic and extrinsic parameters

More general camera matrices

The following is the standard camera matrix we saw.

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & -\mathbf{R}\mathbf{C} \\ \mathbf{0} & 1 \end{bmatrix}$$

More general camera matrices

CCD camera: pixels may not be square.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & 0 & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix}$$

How many degrees of freedom?

More general camera matrices

CCD camera: pixels may not be square.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & 0 & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & -\mathbf{R}\mathbf{C} \\ \mathbf{0} & 1 \end{bmatrix}$$

How many degrees of freedom?

10 DOF

More general camera matrices

Finite projective camera: sensor be skewed.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & -\mathbf{R}\mathbf{C} \\ \mathbf{0} & 1 \end{bmatrix}$$

How many degrees of freedom?

More general camera matrices

Finite projective camera: sensor be skewed.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix}$$

How many degrees of freedom?

11 DOF

Perspective distortion

Finite projective camera

$$\mathbf{P} = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & -\mathbf{R}\mathbf{C} \\ \mathbf{0} & 1 \end{bmatrix}$$



What does this matrix look like if
the camera and world have the
same coordinate system?

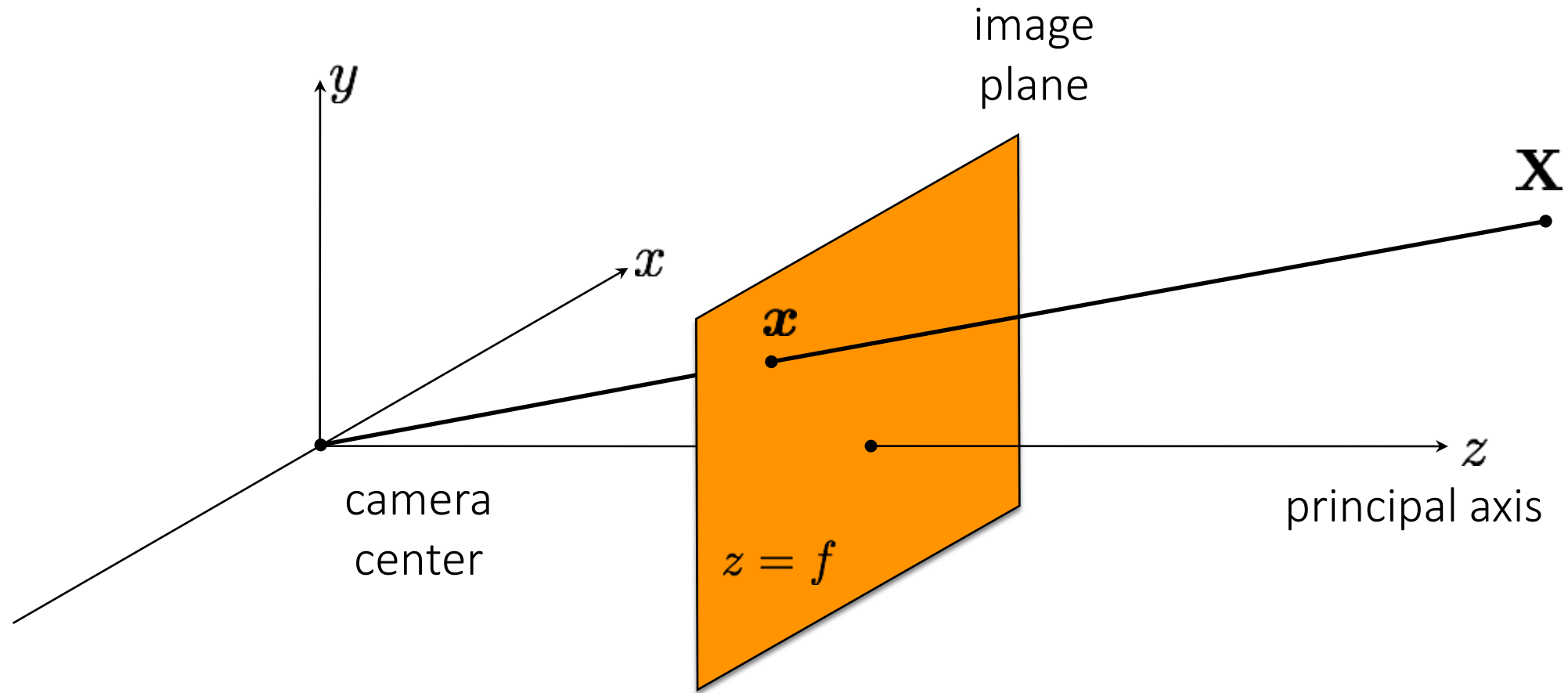
Finite projective camera

The pinhole camera and all of the more general cameras we have seen so far have “*perspective distortion*”.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$


Perspective projection from
(homogeneous) 3D to 2D coordinates

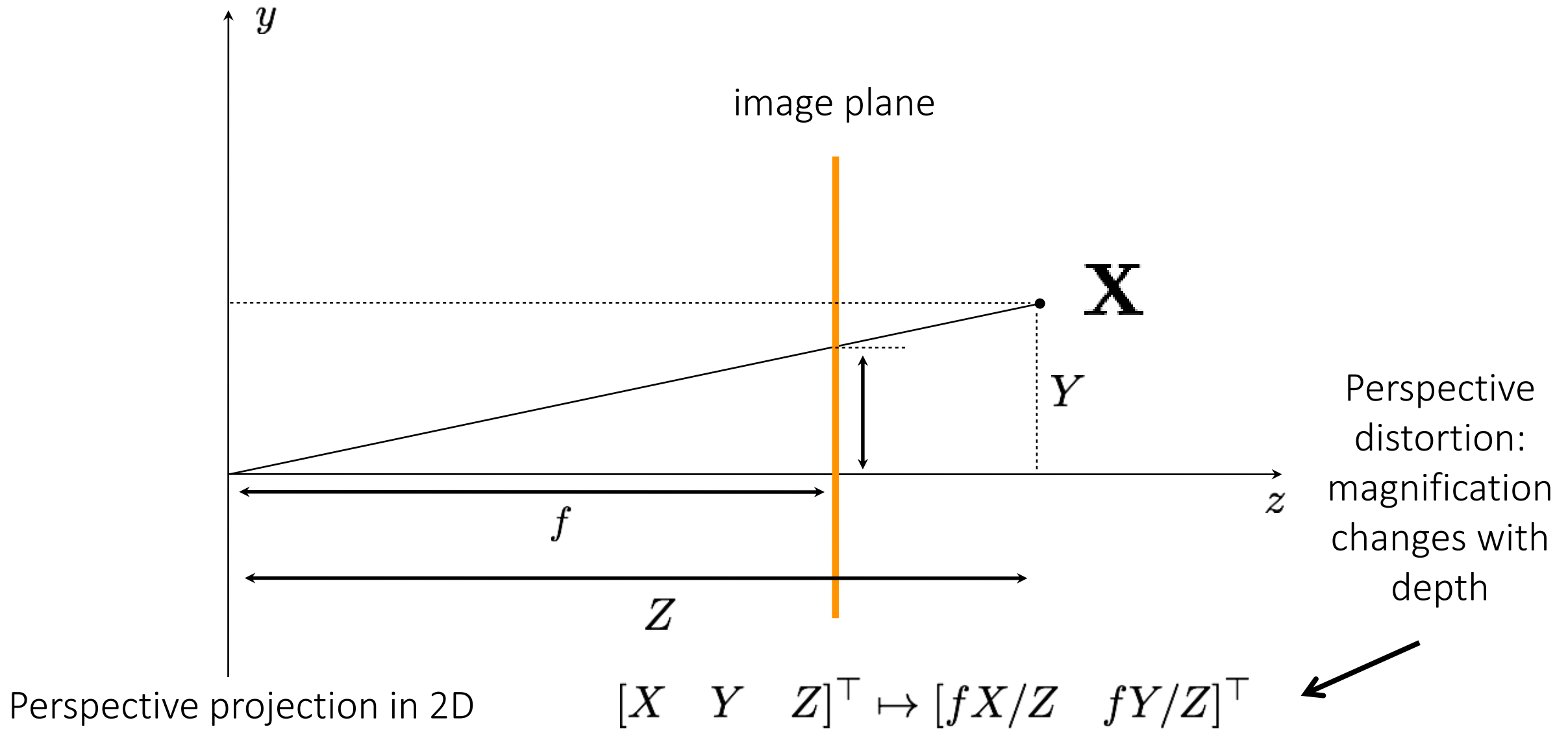
The (rearranged) pinhole camera



Perspective projection in 3D

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

The 2D view of the (rearranged) pinhole camera



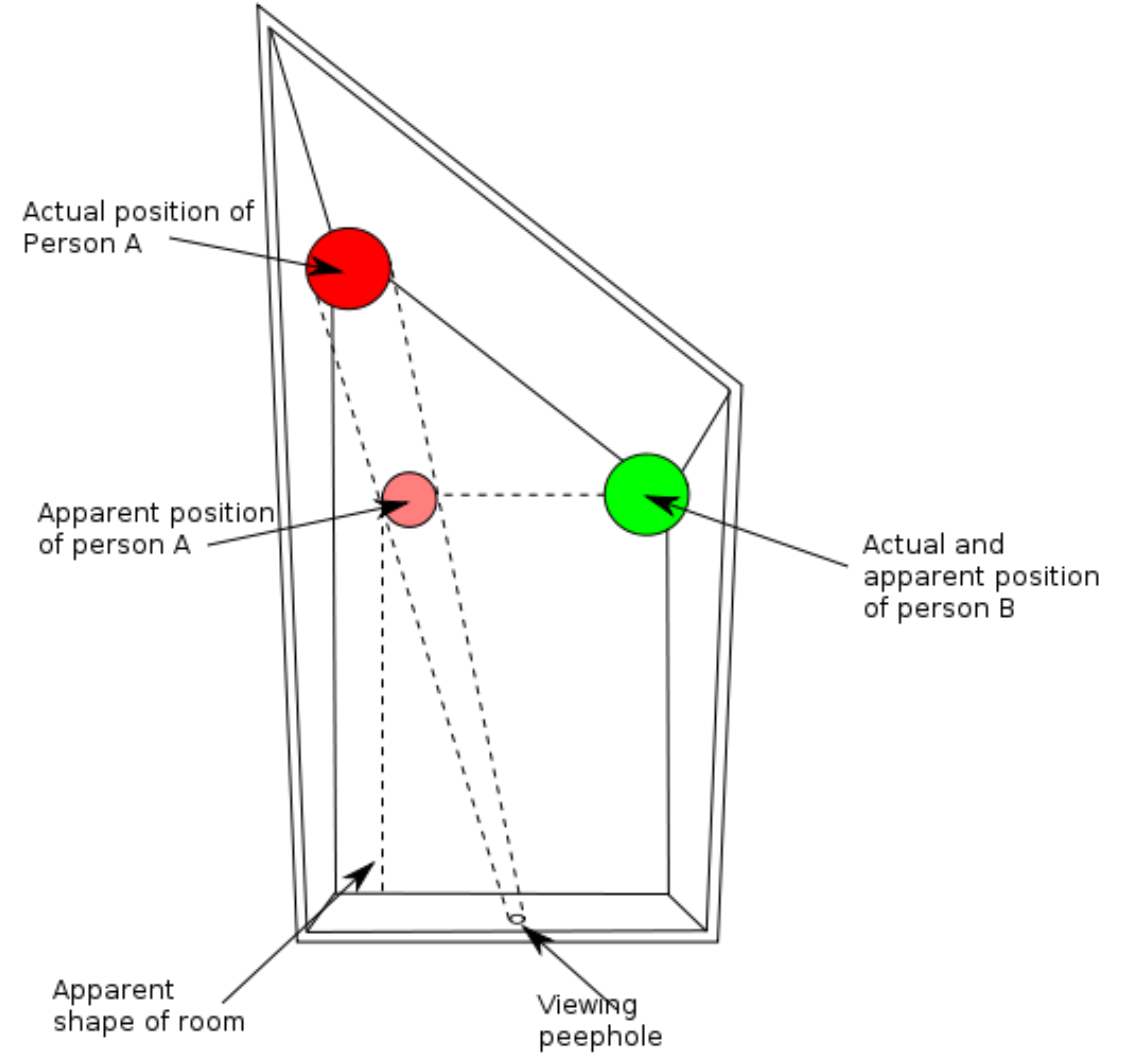
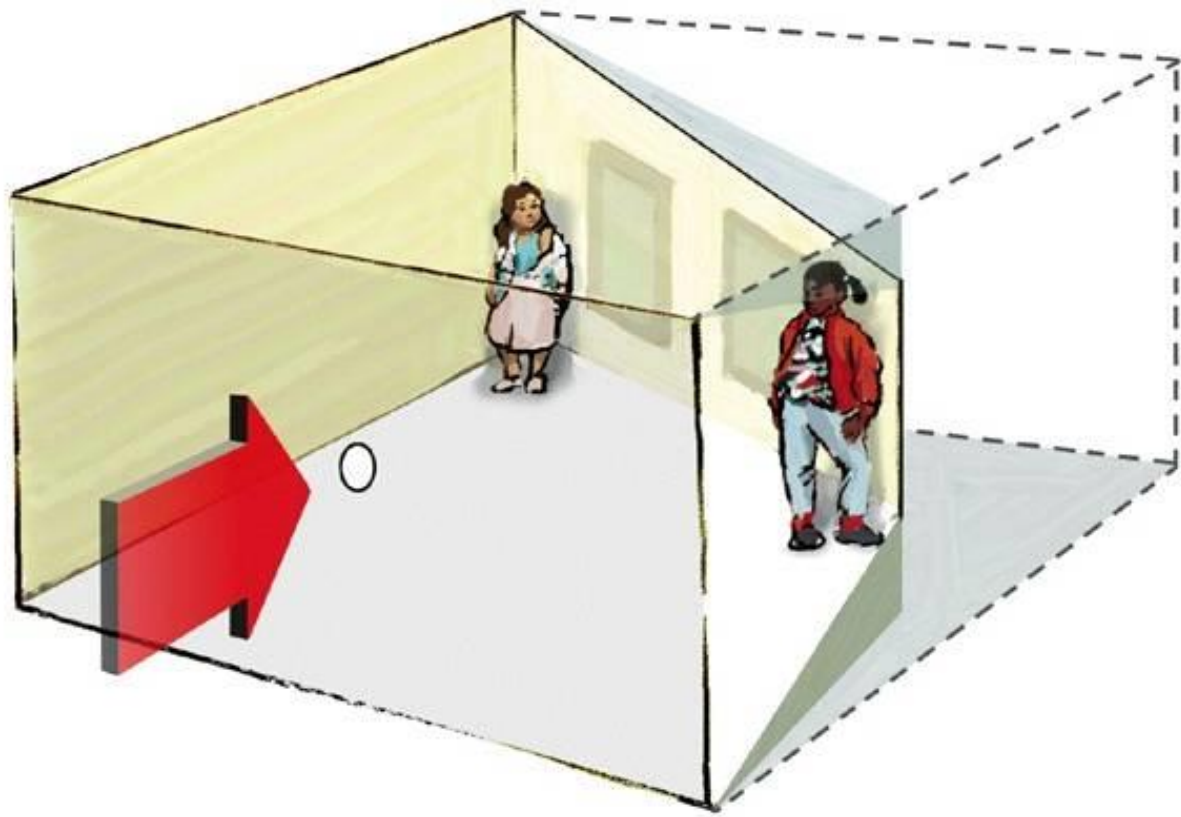
Forced perspective



The Ames room illusion



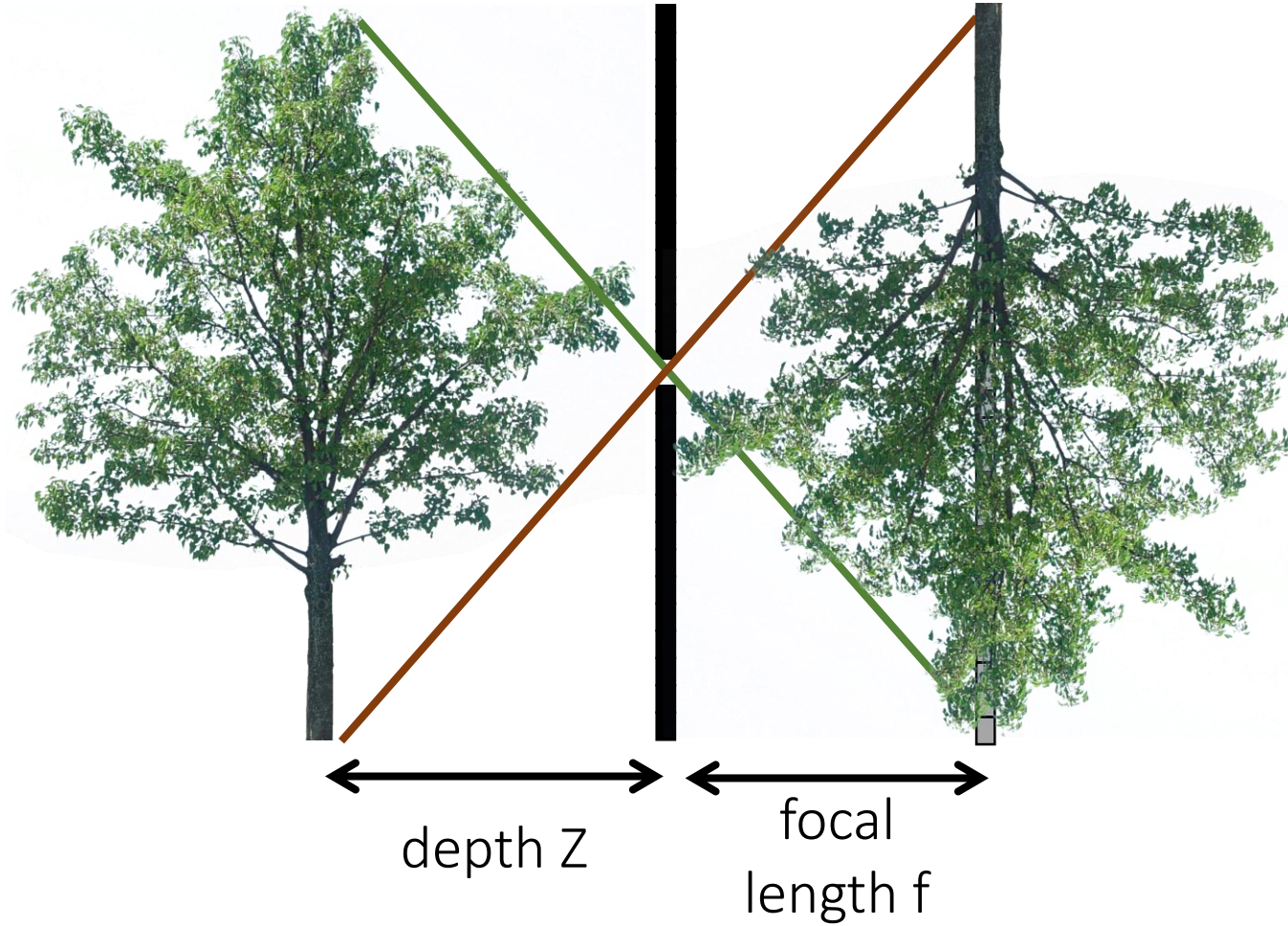
The Ames room illusion



Other camera models

What if...

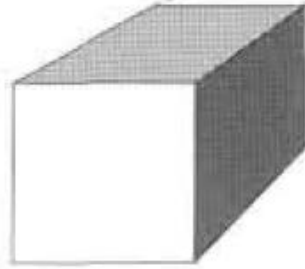
real-world
object



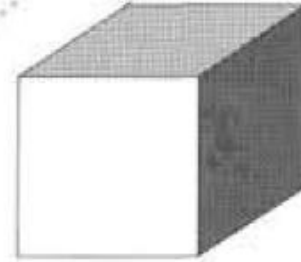
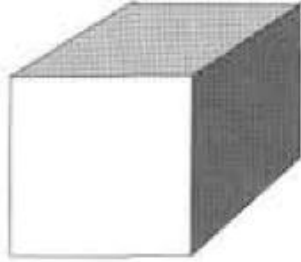
... we continue increasing Z
and f while maintaining
same magnification?

$$f \rightarrow \infty \text{ and } \frac{f}{Z} = \text{constant}$$

camera is *close*
to object and has
small focal length



perspective



camera is *far* from
object and has
large focal length

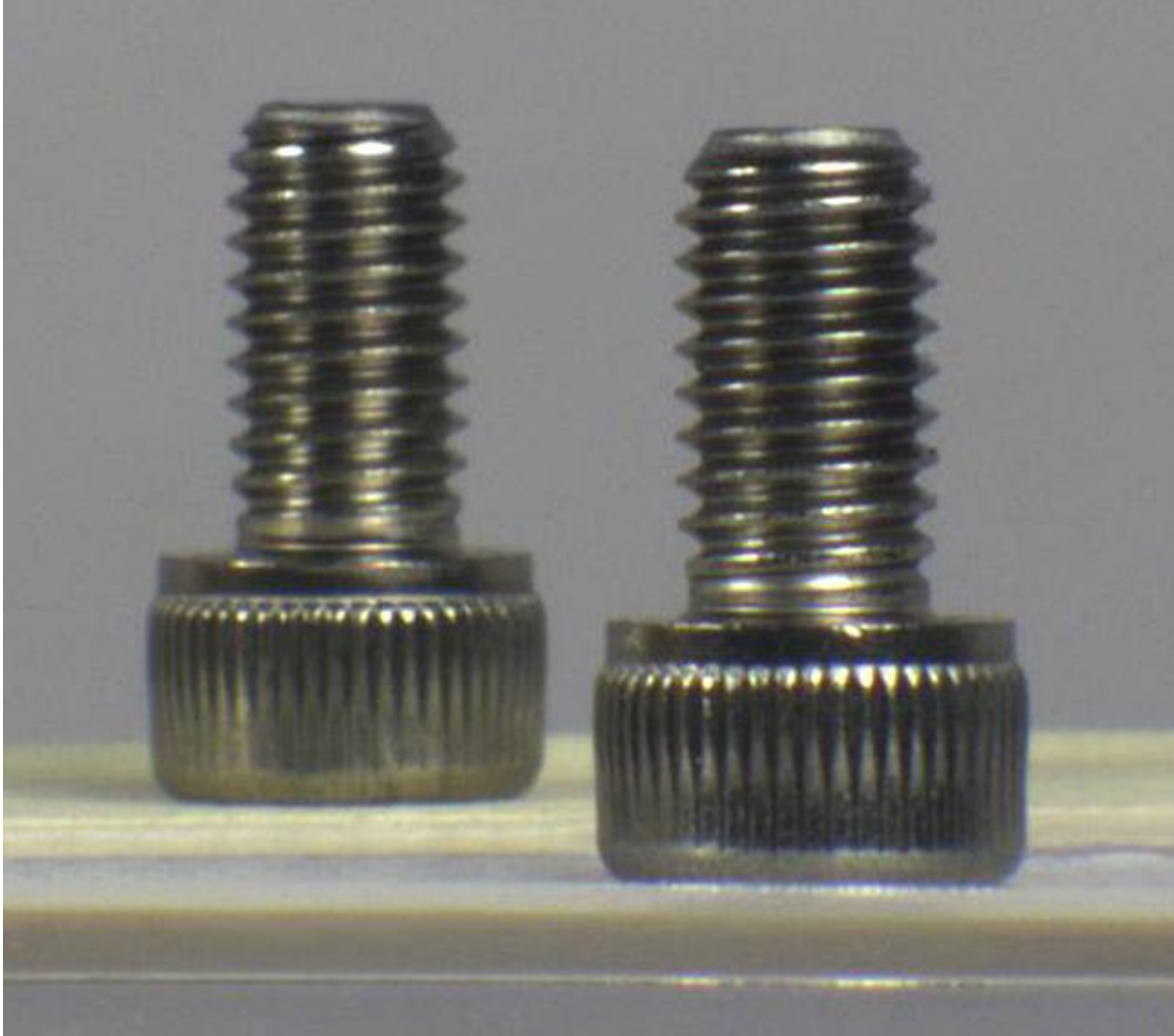
weak perspective

increasing focal length →

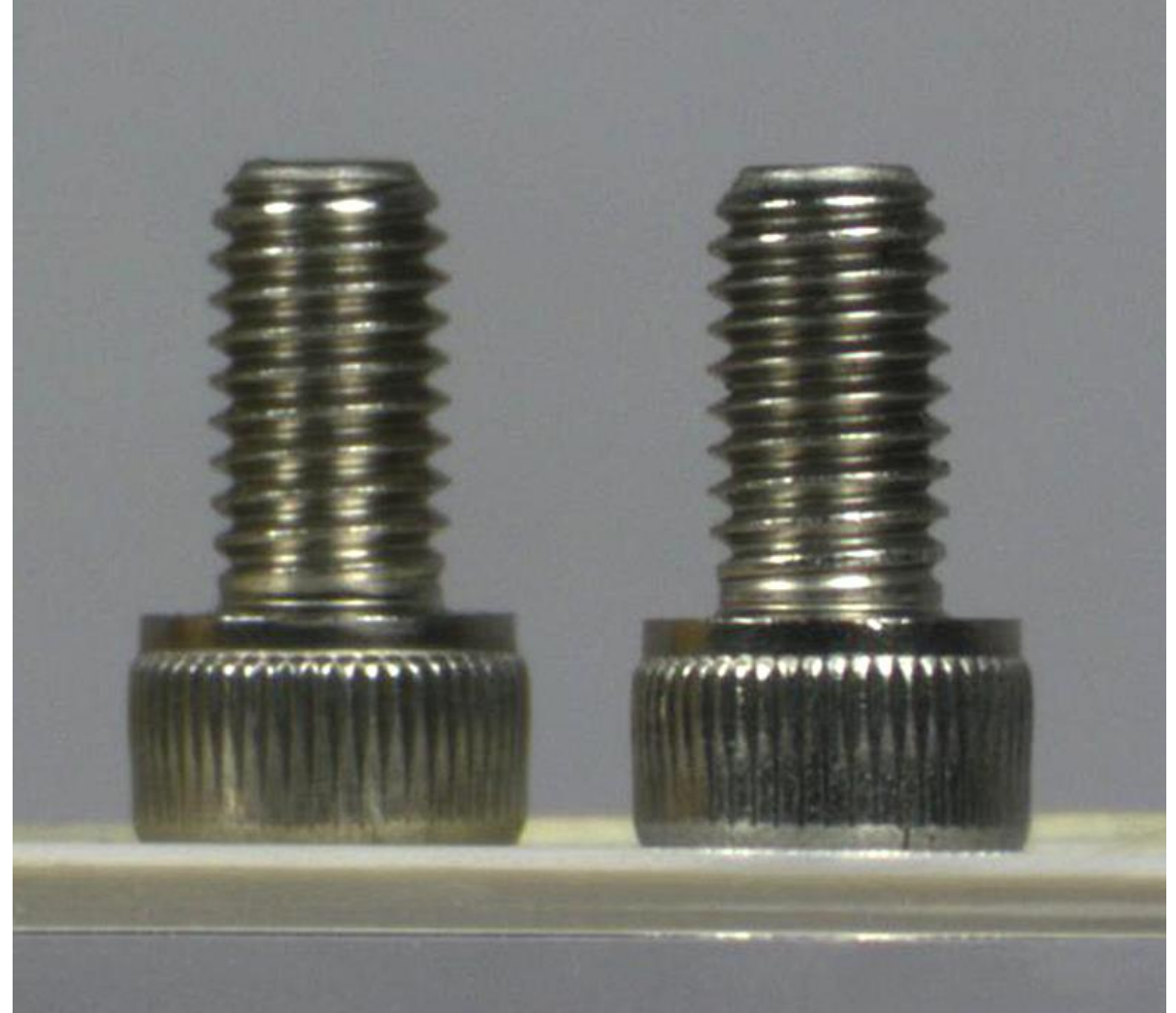
increasing distance from camera →



Different cameras

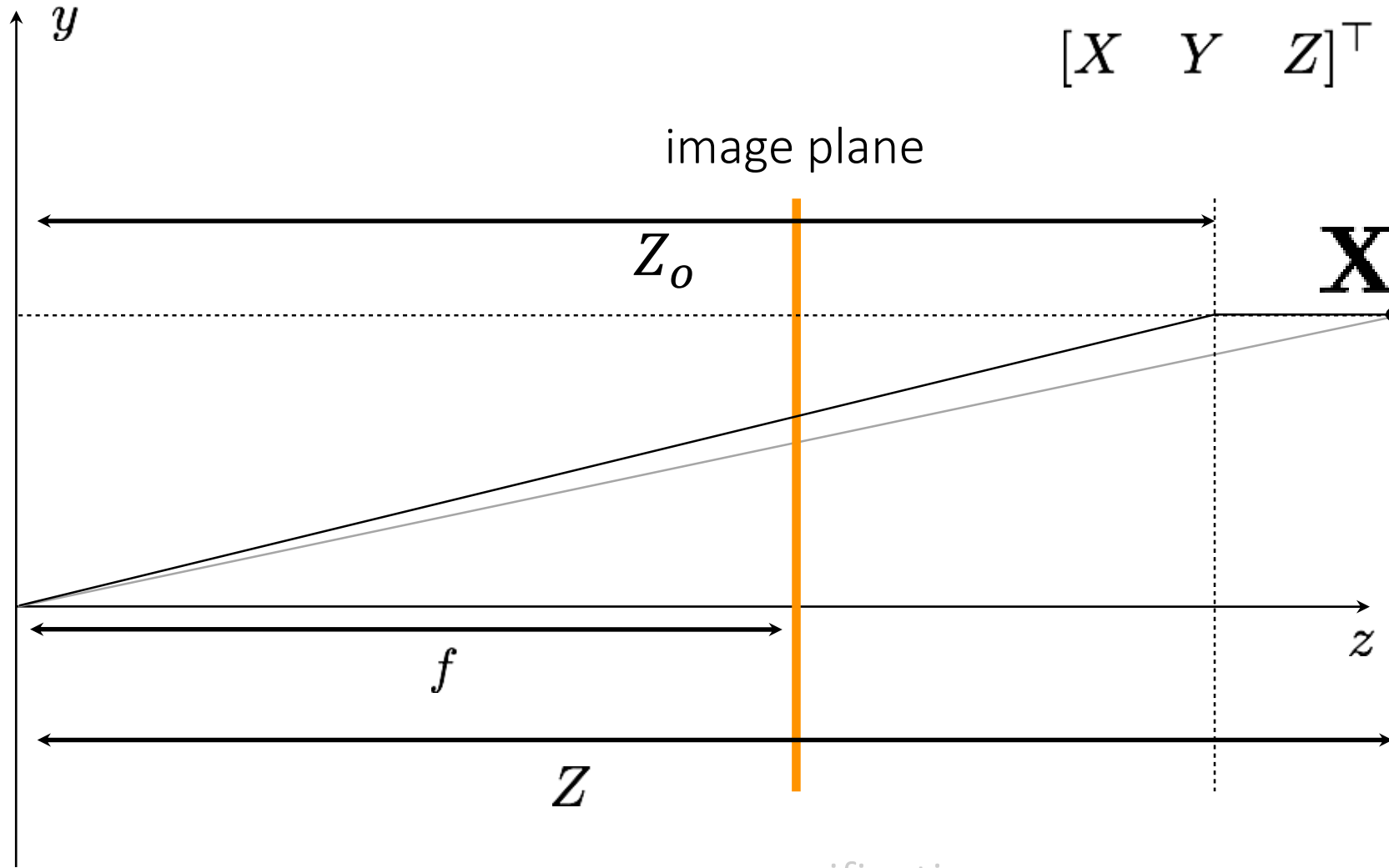


perspective camera



weak perspective camera

Weak perspective vs perspective camera



$$[X \quad Y \quad Z]^\top \mapsto [fX/Z_o \quad fY/Z_o]^\top$$

- magnification does not change with depth
- *constant* magnification depending on f and Z_o

magnification
changes with depth

$$[X \quad Y \quad Z]^\top \mapsto [fX/Z \quad fY/Z]^\top$$

Comparing camera matrices

Let's assume that the world and camera coordinate systems are the same.

- The *perspective* camera matrix can be written as:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- What would the matrix of the weak perspective camera look like?

Comparing camera matrices

Let's assume that the world and camera coordinate systems are the same.

- The *perspective* camera matrix can be written as:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- The *weak perspective* camera matrix can be written as:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & Z_o \end{bmatrix}$$

Comparing camera matrices

Let's assume that the world and camera coordinate systems are the same.

- The *finite projective* camera matrix can be written as:

$$\mathbf{P} = \mathbf{K} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where we now have the more general intrinsic matrix

- The *affine* camera matrix can be written as:

$$\mathbf{P} = \mathbf{K} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & Z_o \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

In both cameras, we can incorporate extrinsic parameters same as we did before.

When can we assume a weak perspective camera?

When can we assume a weak perspective camera?

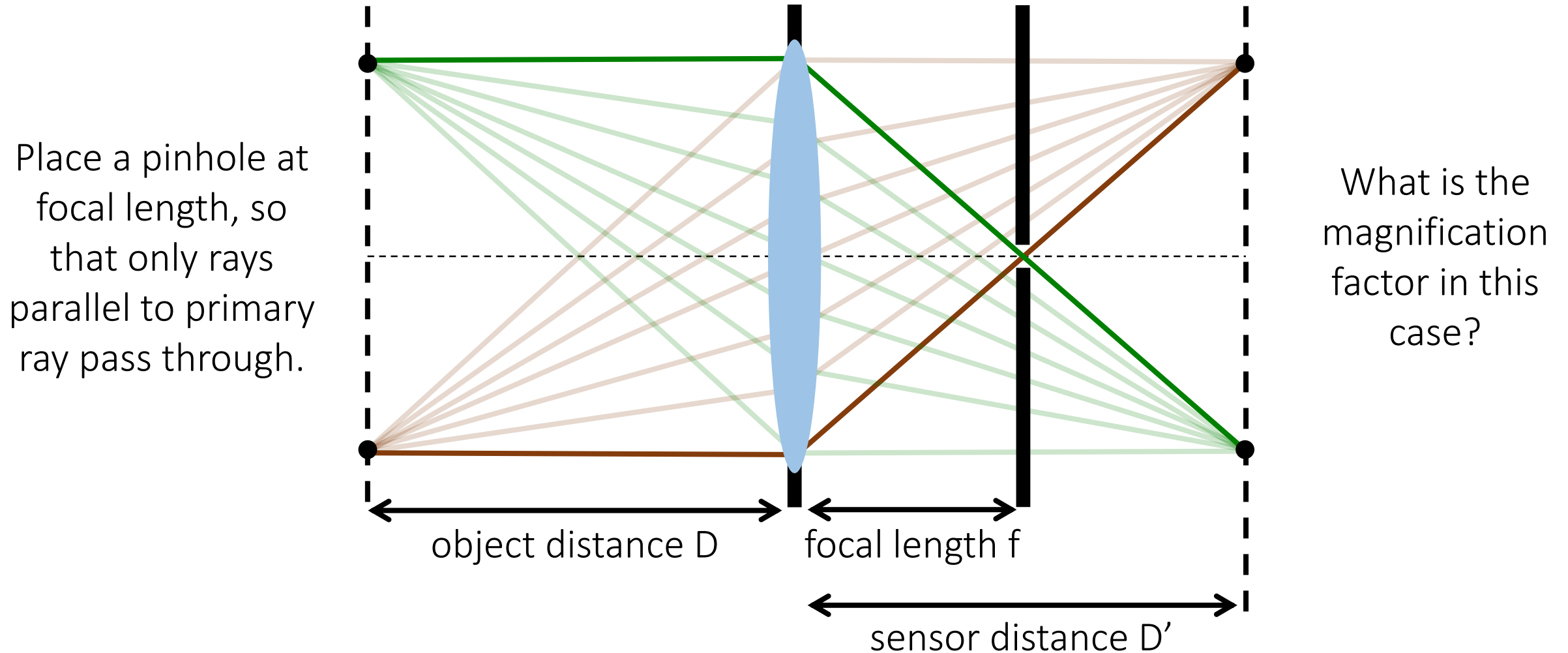
1. When the scene (or parts of it) is very far away.



Weak perspective projection applies to the mountains.

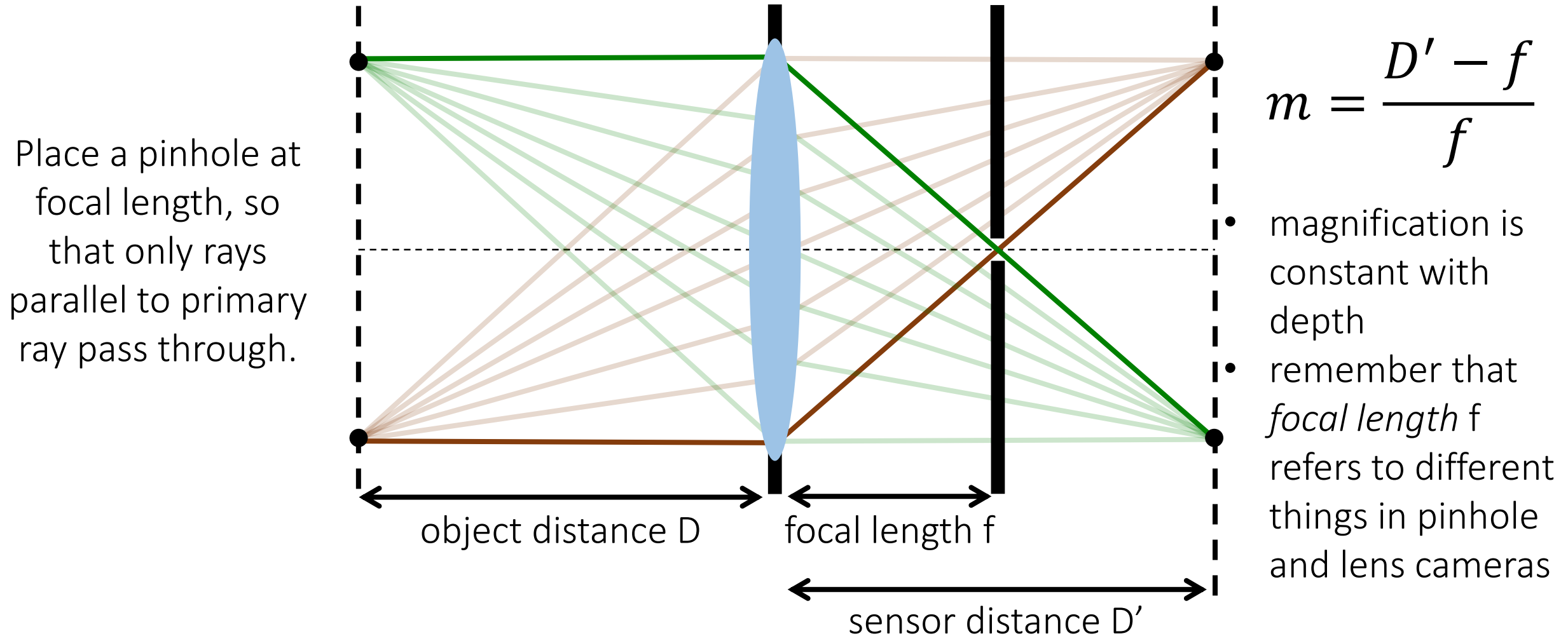
When can we assume a weak perspective camera?

2. When we use a telecentric lens.



When can we assume a weak perspective camera?

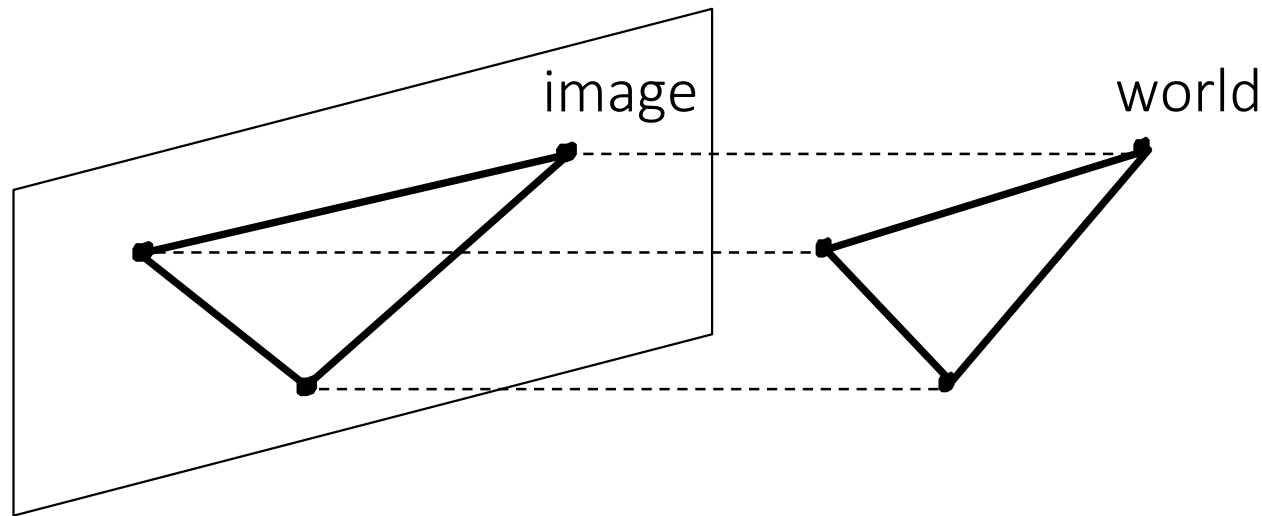
2. When we use a telecentric lens.



Orthographic camera

Special case of weak perspective camera where:

- constant magnification is equal to 1.
- there is no shift between camera and image origins.
- the world and camera coordinate systems are the same.

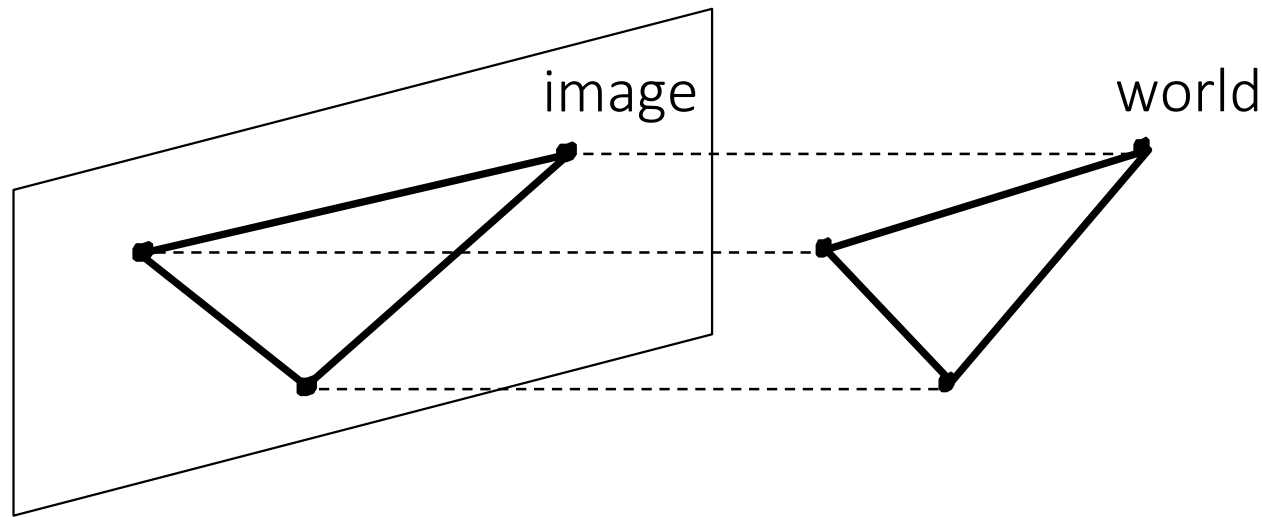


What is the camera matrix in this case?

Orthographic camera

Special case of weak perspective camera where:

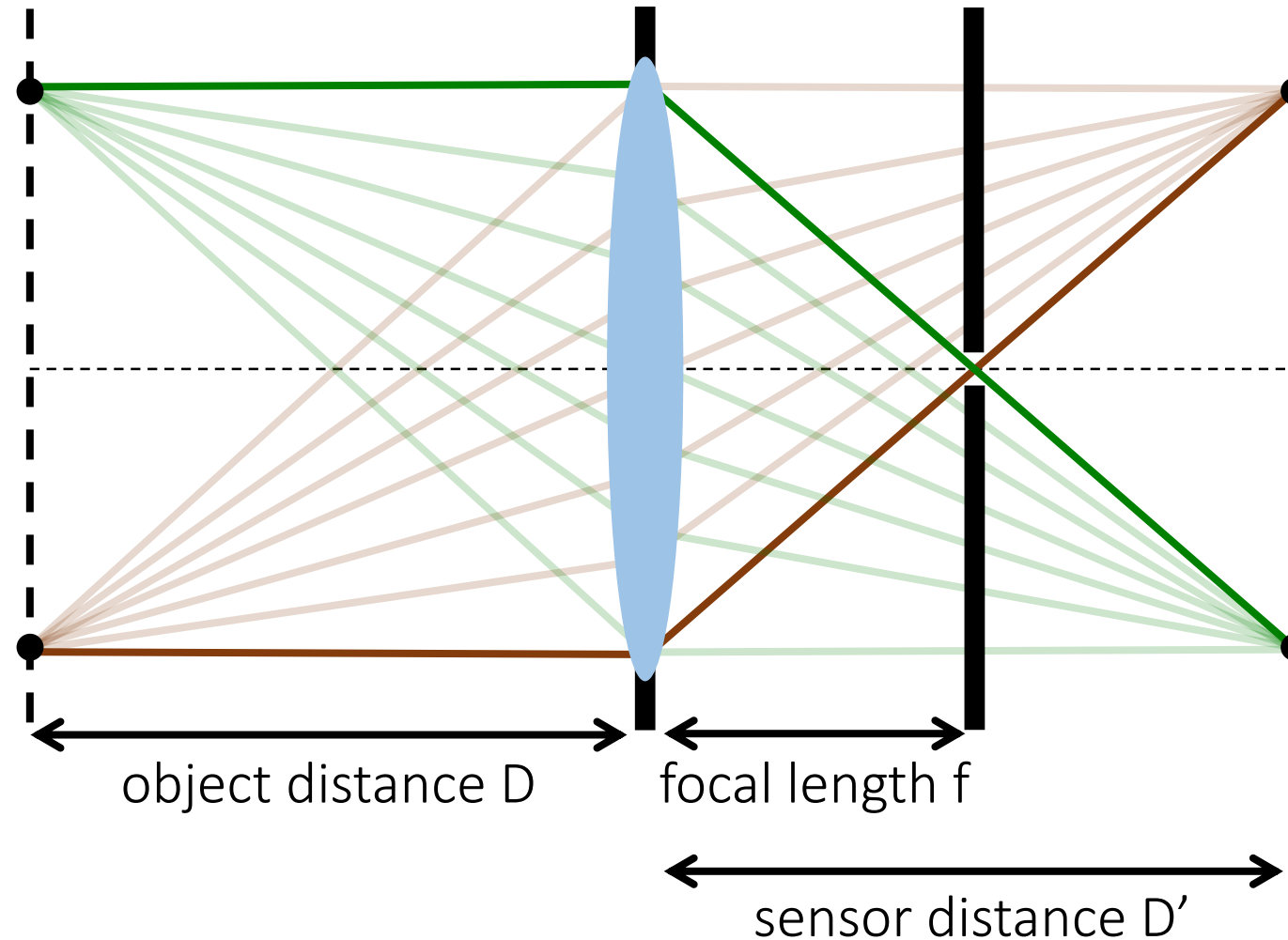
- constant magnification is equal to 1.
- there is no shift between camera and image origins.
- the world and camera coordinate systems are the same.



$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

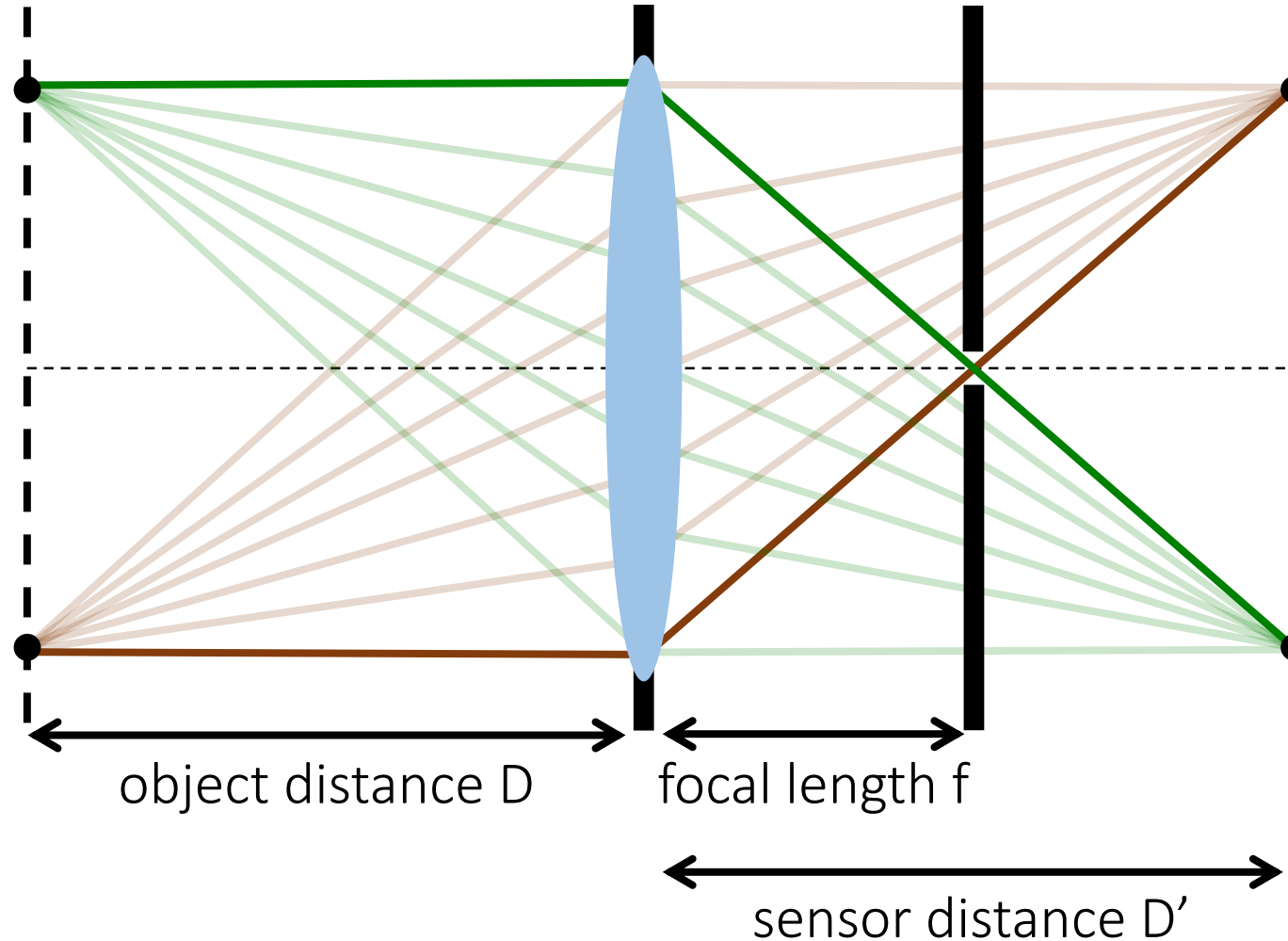
Orthographic projection using a telecentric lens

How do we make the telecentric lens act as an orthographic camera?



Orthographic projection using a telecentric lens

How do we make the telecentric lens act as an orthographic camera?

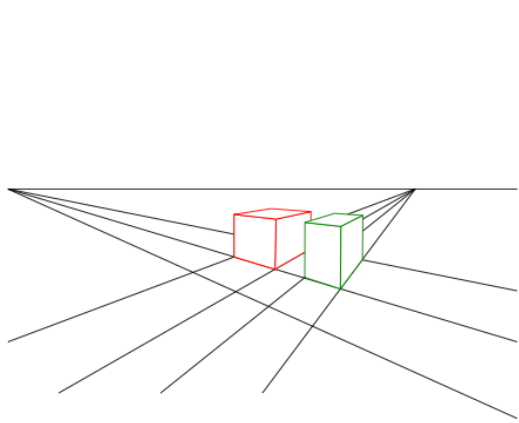


We set the sensor distance as:

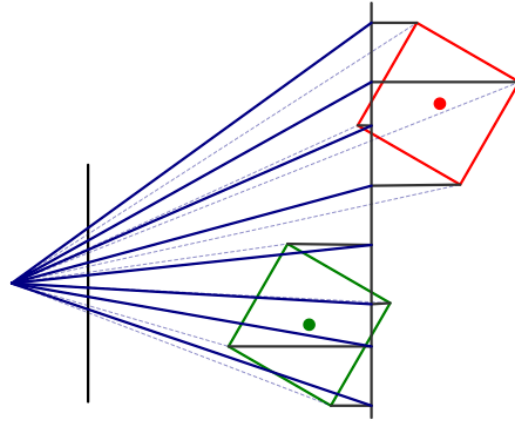
$$D' = 2f$$

in order to achieve unit magnification.

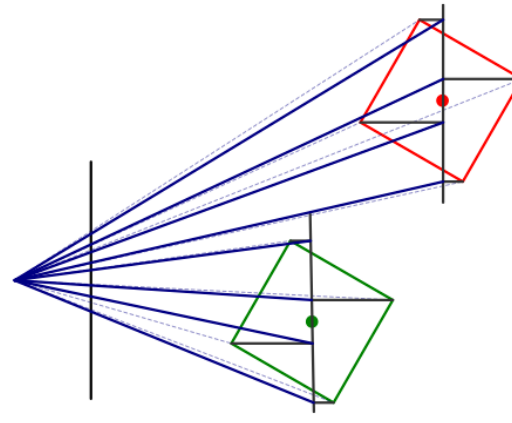
Many other types of cameras



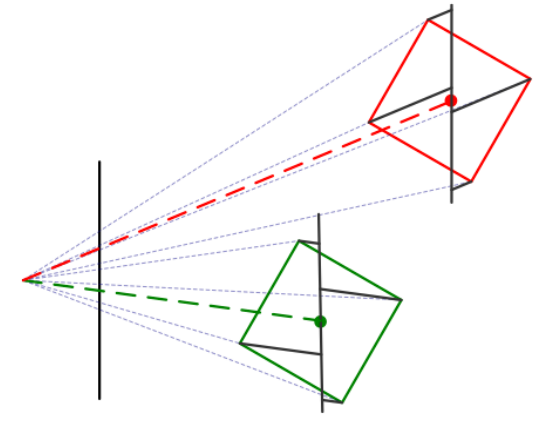
(a) 3D view



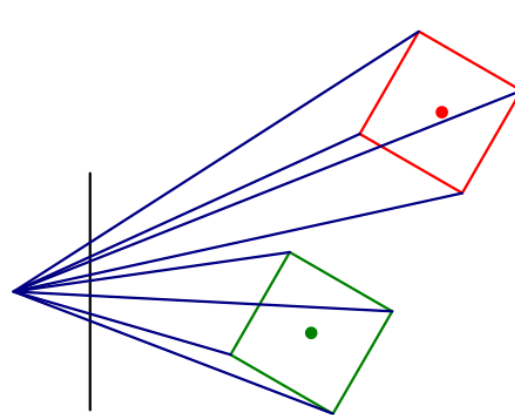
(b) orthography



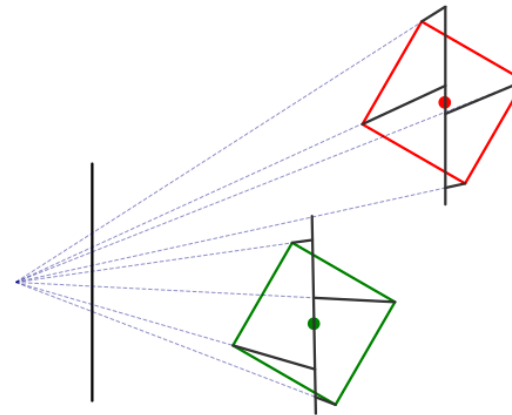
(c) scaled orthography



(d) para-perspective



(e) perspective



(f) object-centered

Geometric camera calibration

Geometric camera calibration

Given a set of matched points

$$\{\mathbf{X}_i, \mathbf{x}_i\}$$

point in 3D
space

point in the
image

and camera model

$$\mathbf{x} = \mathbf{f}(\mathbf{X}; \mathbf{p}) = \mathbf{P}\mathbf{X}$$

projection
model

parameters

Camera
matrix

Find the (pose) estimate of

P

We'll use a **perspective** camera
model for pose estimation

Same setup as homography estimation
(slightly different derivation here)

Where did we see homography estimation in this class?

Mapping between 3D point and image points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What are the unknowns?

Mapping between 3D point and image points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \text{---} & \mathbf{p}_1^\top & \text{---} \\ \text{---} & \mathbf{p}_2^\top & \text{---} \\ \text{---} & \mathbf{p}_3^\top & \text{---} \end{bmatrix} \begin{bmatrix} | \\ \mathbf{X} \\ | \end{bmatrix}$$

Heterogeneous coordinates

$$x' = \frac{\mathbf{p}_1^\top \mathbf{X}}{\mathbf{p}_3^\top \mathbf{X}} \quad y' = \frac{\mathbf{p}_2^\top \mathbf{X}}{\mathbf{p}_3^\top \mathbf{X}}$$

(non-linear relation between coordinates)

How can we make these relations linear?

How can we make these relations linear?

$$x' = \frac{\mathbf{p}_1^\top \mathbf{X}}{\mathbf{p}_3^\top \mathbf{X}} \quad y' = \frac{\mathbf{p}_2^\top \mathbf{X}}{\mathbf{p}_3^\top \mathbf{X}}$$

Make them linear with algebraic manipulation...

$$\mathbf{p}_2^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} y' = 0$$

$$\mathbf{p}_1^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} x' = 0$$

Now we can setup a system of linear equations with multiple point correspondences

$$\mathbf{p}_2^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} y' = 0$$

$$\mathbf{p}_1^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} x' = 0$$

How do we proceed?

$$\mathbf{p}_2^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} y' = 0$$

$$\mathbf{p}_1^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} x' = 0$$

In matrix form ...

$$\begin{bmatrix} \mathbf{X}^\top & \mathbf{0} & -x' \mathbf{X}^\top \\ \mathbf{0} & \mathbf{X}^\top & -y' \mathbf{X}^\top \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{0}$$

How do we proceed?

$$\mathbf{p}_2^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} y' = 0$$

$$\mathbf{p}_1^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} x' = 0$$

In matrix form ...

$$\begin{bmatrix} \mathbf{X}^\top & \mathbf{0} & -x' \mathbf{X}^\top \\ \mathbf{0} & \mathbf{X}^\top & -y' \mathbf{X}^\top \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \mathbf{0}$$

For N points ...

$$\begin{bmatrix} \mathbf{X}_1^\top & \mathbf{0} & -x' \mathbf{X}_1^\top \\ \mathbf{0} & \mathbf{X}_1^\top & -y' \mathbf{X}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{X}_N^\top & \mathbf{0} & -x' \mathbf{X}_N^\top \\ \mathbf{0} & \mathbf{X}_N^\top & -y' \mathbf{X}_N^\top \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \mathbf{0}$$

*How do we solve
this system?*

Solve for camera matrix by

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|^2 \text{ subject to } \|\mathbf{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{0} & -x' \mathbf{X}_1^\top \\ \mathbf{0} & \mathbf{X}_1^\top & -y' \mathbf{X}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{X}_N^\top & \mathbf{0} & -x' \mathbf{X}_N^\top \\ \mathbf{0} & \mathbf{X}_N^\top & -y' \mathbf{X}_N^\top \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

SVD!

Solve for camera matrix by

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|^2 \text{ subject to } \|\mathbf{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{0} & -x' \mathbf{X}_1^\top \\ \mathbf{0} & \mathbf{X}_1^\top & -y' \mathbf{X}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{X}_N^\top & \mathbf{0} & -x' \mathbf{X}_N^\top \\ \mathbf{0} & \mathbf{X}_N^\top & -y' \mathbf{X}_N^\top \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Solution $\mathbf{x}$ is the column of $\mathbf{V}$
corresponding to smallest singular
value of

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$$

Solve for camera matrix by

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|^2 \text{ subject to } \|\mathbf{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{0} & -x' \mathbf{X}_1^\top \\ \mathbf{0} & \mathbf{X}_1^\top & -y' \mathbf{X}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{X}_N^\top & \mathbf{0} & -x' \mathbf{X}_N^\top \\ \mathbf{0} & \mathbf{X}_N^\top & -y' \mathbf{X}_N^\top \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Equivalently, solution $\mathbf{x}$ is the Eigenvector corresponding to smallest Eigenvalue of

$$\mathbf{A}^\top \mathbf{A}$$

Now we have:

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix}$$

Are we done?

Almost there ...

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix}$$

How do you get the intrinsic and extrinsic parameters from the projection matrix?

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}| -\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M}| -\mathbf{M}\mathbf{c}] \end{aligned}$$

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R} | -\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M} | -\mathbf{M}\mathbf{c}] \end{aligned}$$

Find the camera center $\mathbf{C}$

What is the projection of the camera center?

Find intrinsic $\mathbf{K}$ and rotation $\mathbf{R}$

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R} | -\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M} | -\mathbf{M}\mathbf{c}] \end{aligned}$$

Find the camera center $\mathbf{C}$

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

How do we compute the camera center from this?

Find intrinsic $\mathbf{K}$ and rotation $\mathbf{R}$

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

$$\begin{aligned}\mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R} | -\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M} | -\mathbf{M}\mathbf{c}]\end{aligned}$$

Find the camera center $\mathbf{c}$

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

SVD of $\mathbf{P}$!

$\mathbf{c}$ is the Eigenvector corresponding to
smallest Eigenvalue

Find intrinsic $\mathbf{K}$ and rotation $\mathbf{R}$

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

$$\begin{aligned}\mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R} | -\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M} | -\mathbf{M}\mathbf{c}]\end{aligned}$$

Find the camera center $\mathbf{c}$

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

SVD of $\mathbf{P}$!

$\mathbf{c}$ is the Eigenvector corresponding to
smallest Eigenvalue

Find intrinsic $\mathbf{K}$ and rotation $\mathbf{R}$

$$\mathbf{M} = \mathbf{K}\mathbf{R}$$

*Any useful properties of K
and R we can use?*

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

$$\begin{aligned}\mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R} | -\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M} | -\mathbf{M}\mathbf{c}]\end{aligned}$$

Find the camera center $\mathbf{c}$

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

SVD of $\mathbf{P}$!

$\mathbf{c}$ is the Eigenvector corresponding to
smallest Eigenvalue

Find intrinsic $\mathbf{K}$ and rotation $\mathbf{R}$

$$\mathbf{M} = \mathbf{K}\mathbf{R}$$

right upper
triangle

orthogonal

*How do we find K
and R ?*

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

$$\begin{aligned}\mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R} | -\mathbf{R}\mathbf{c}] \\ &= [\mathbf{M} | -\mathbf{M}\mathbf{c}]\end{aligned}$$

Find the camera center $\mathbf{c}$

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

SVD of $\mathbf{P}$!

$\mathbf{c}$ is the *Eigenvector* corresponding to
smallest *Eigenvalue*

Find intrinsic $\mathbf{K}$ and rotation $\mathbf{R}$

$$\mathbf{M} = \mathbf{K}\mathbf{R}$$

QR decomposition

Geometric camera calibration

Given a set of matched points

$$\{\mathbf{X}_i, \mathbf{x}_i\}$$

point in 3D
space

point in the
image

*Where do we get these
matched points from?*

and camera model

$$\mathbf{x} = \mathbf{f}(\mathbf{X}; \mathbf{p}) = \mathbf{P}\mathbf{X}$$

projection
model

parameters

Camera
matrix

Find the (pose) estimate of

P

We'll use a **perspective** camera
model for pose estimation

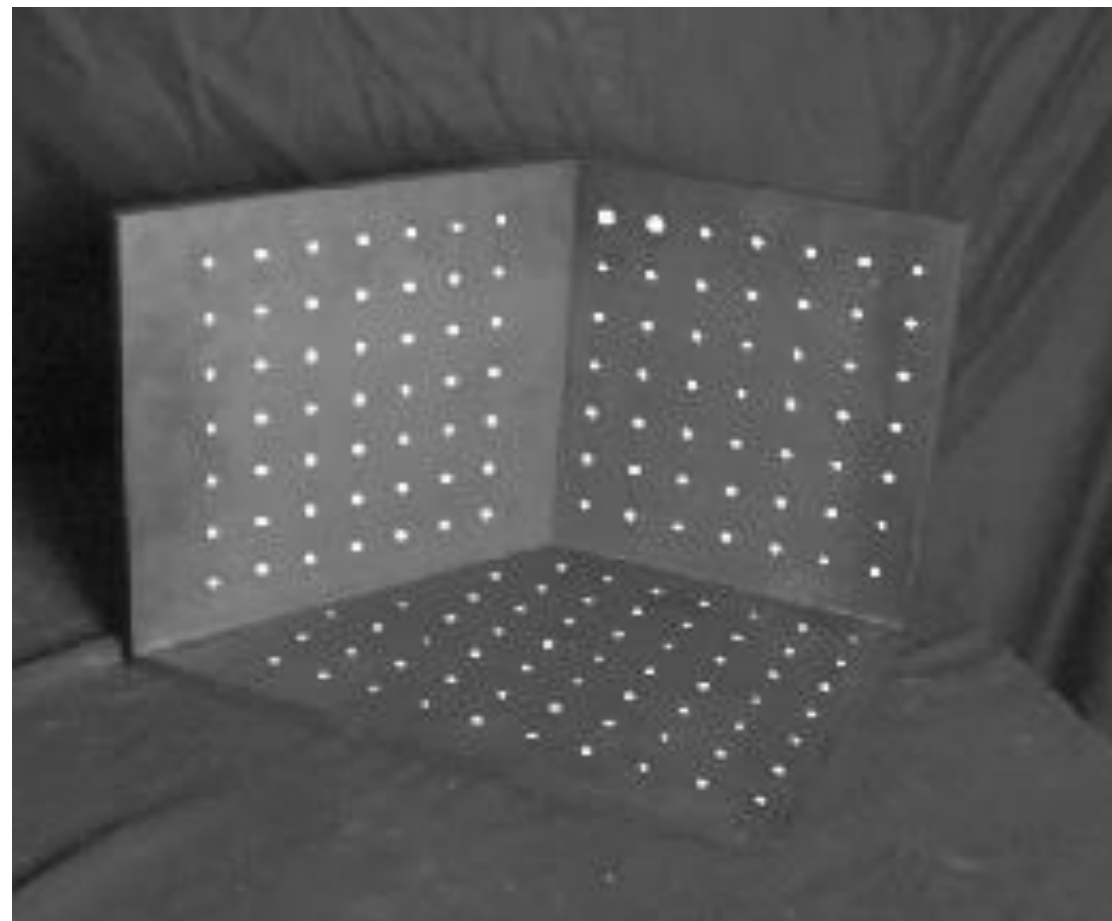
Calibration using a reference object

Place a known object in the scene:

- identify correspondences between image and scene
- compute mapping from scene to image

Issues:

- must know geometry very accurately
- must know 3D->2D correspondence



Geometric camera calibration

Advantages:

- Very simple to formulate.
- Analytical solution.

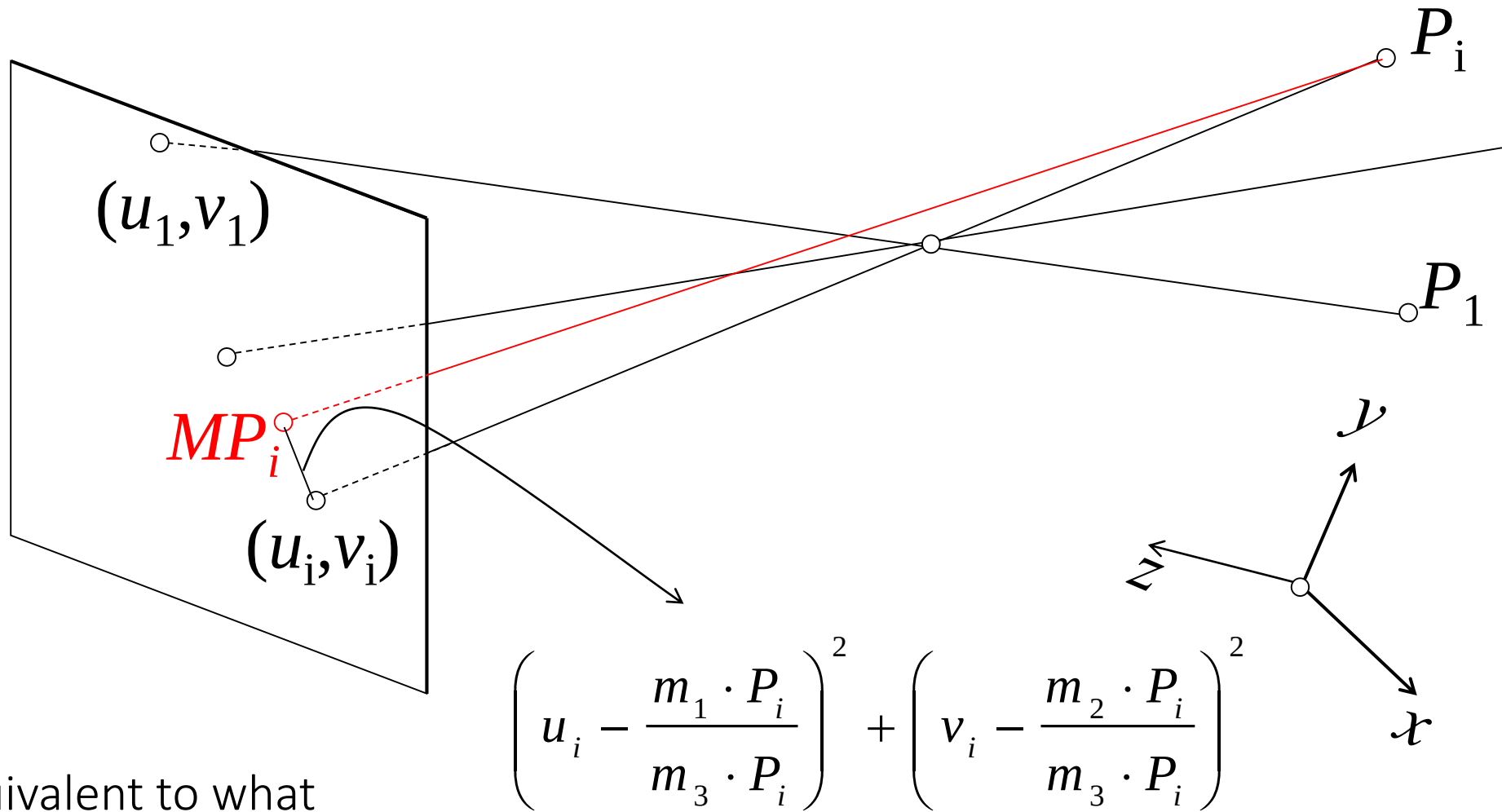
Disadvantages:

- Doesn't model radial distortion.
- Hard to impose constraints (e.g., known f).
- Doesn't minimize the correct error function.

For these reasons, *nonlinear methods* are preferred

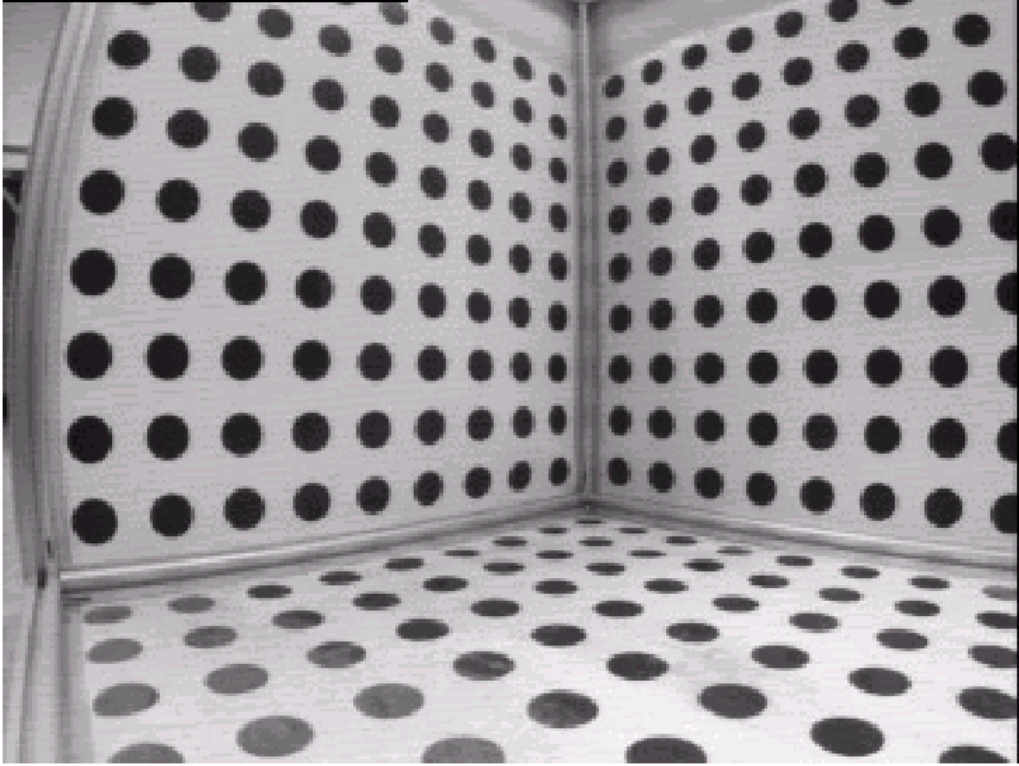
- Define error function E between projected 3D points and image positions
 - E is nonlinear function of intrinsics, extrinsics, radial distortion
- Minimize E using nonlinear optimization techniques

Minimizing reprojection error

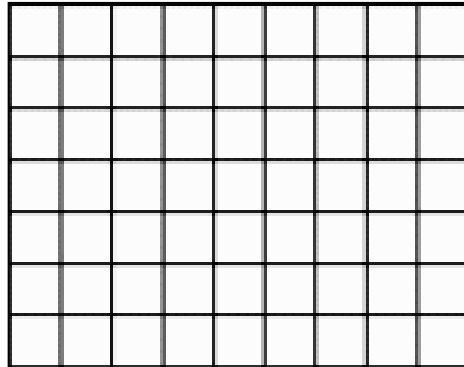


Is this equivalent to what we were doing previously?

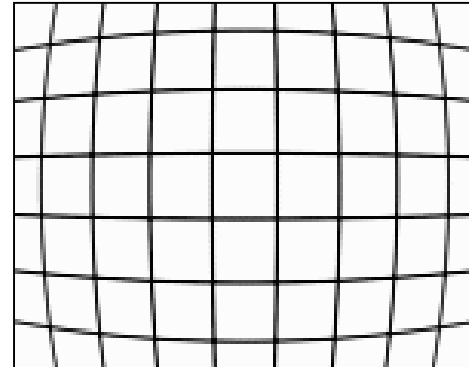
Radial distortion



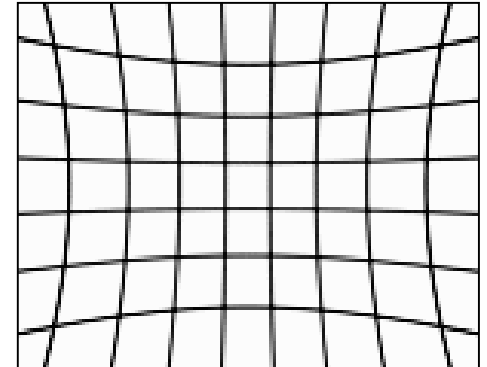
What causes this distortion?



no distortion

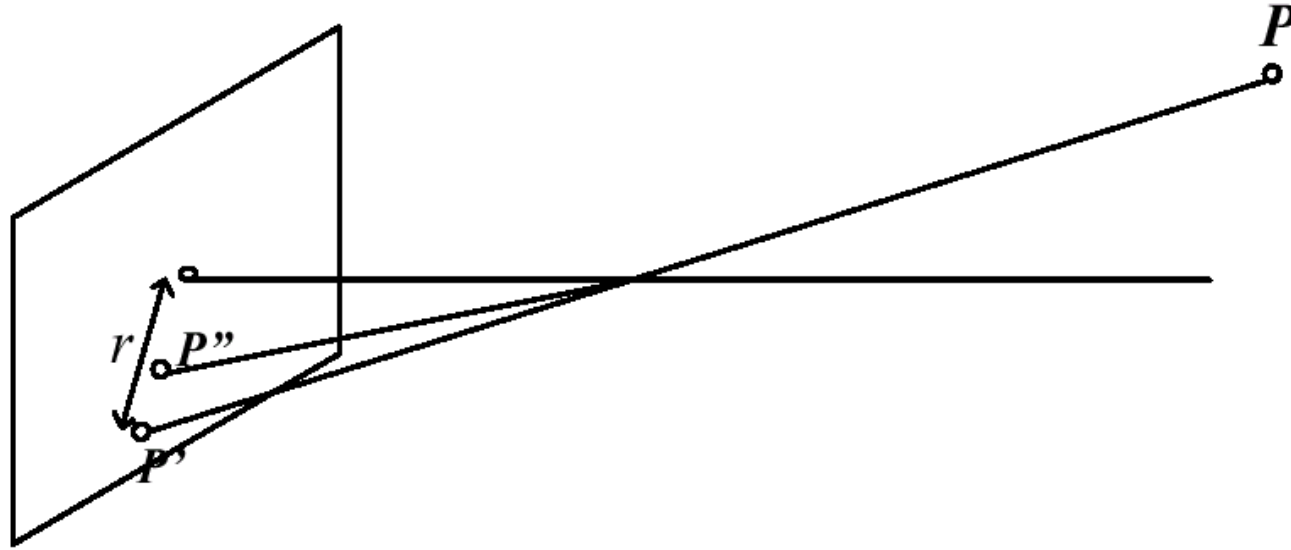


barrel distortion



pincushion distortion

Radial distortion model



Ideal:

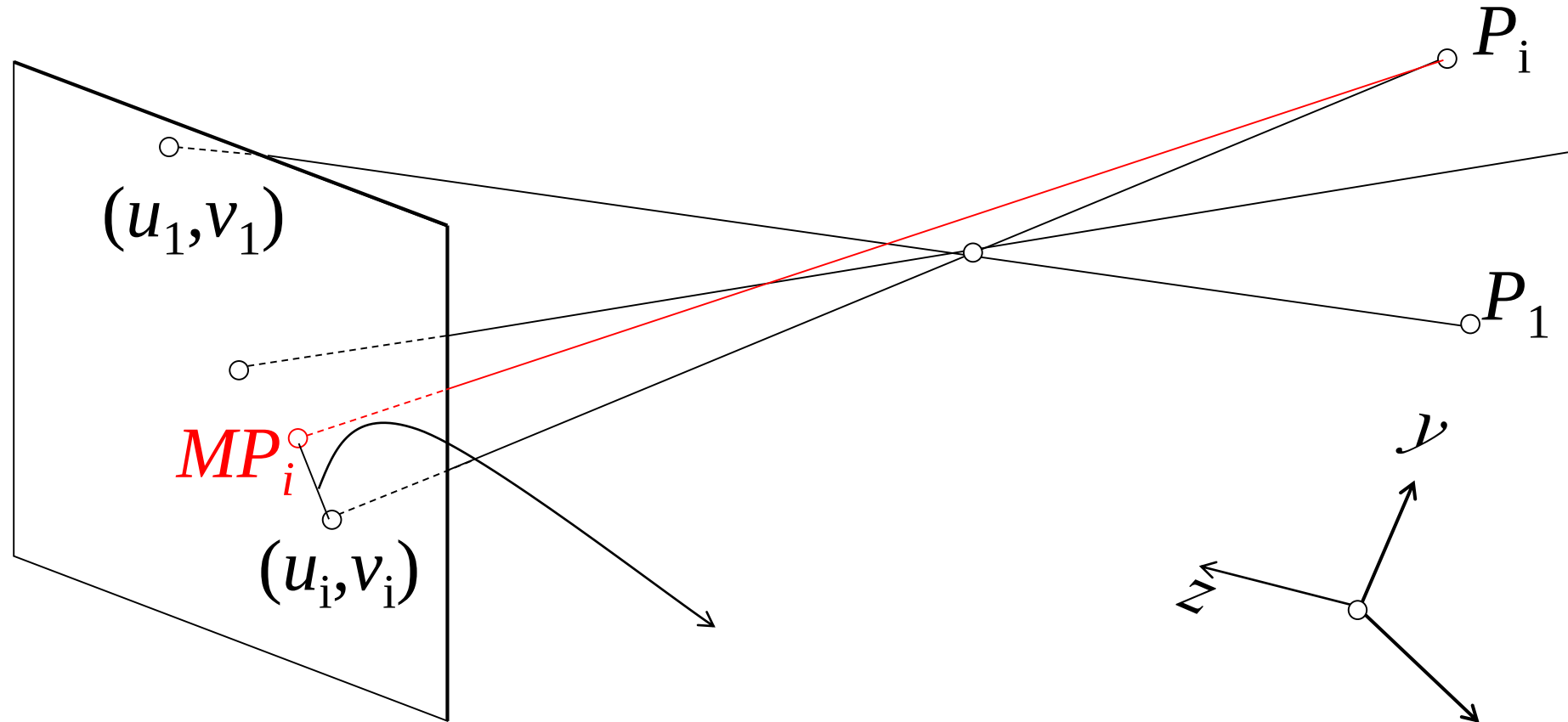
$$\begin{aligned}x' &= f \frac{x}{z} \\ y' &= f \frac{y}{z}\end{aligned}$$

Distorted:

$$\begin{aligned}x'' &= \frac{1}{\lambda} x, \\y'' &= \frac{1}{\lambda} y,\end{aligned}$$

$$\lambda = 1 + k_1 r^2 + k_2 r^4 + \dots$$

Minimizing reprojection error with radial distortion



Add distortions to
reprojection error:

$$\left(u_i - \frac{1}{\lambda} \frac{m_1 \cdot P_i}{m_3 \cdot P_i}\right)^2 + \left(v_i - \frac{1}{\lambda} \frac{m_2 \cdot P_i}{m_3 \cdot P_i}\right)^2$$

Correcting radial distortion

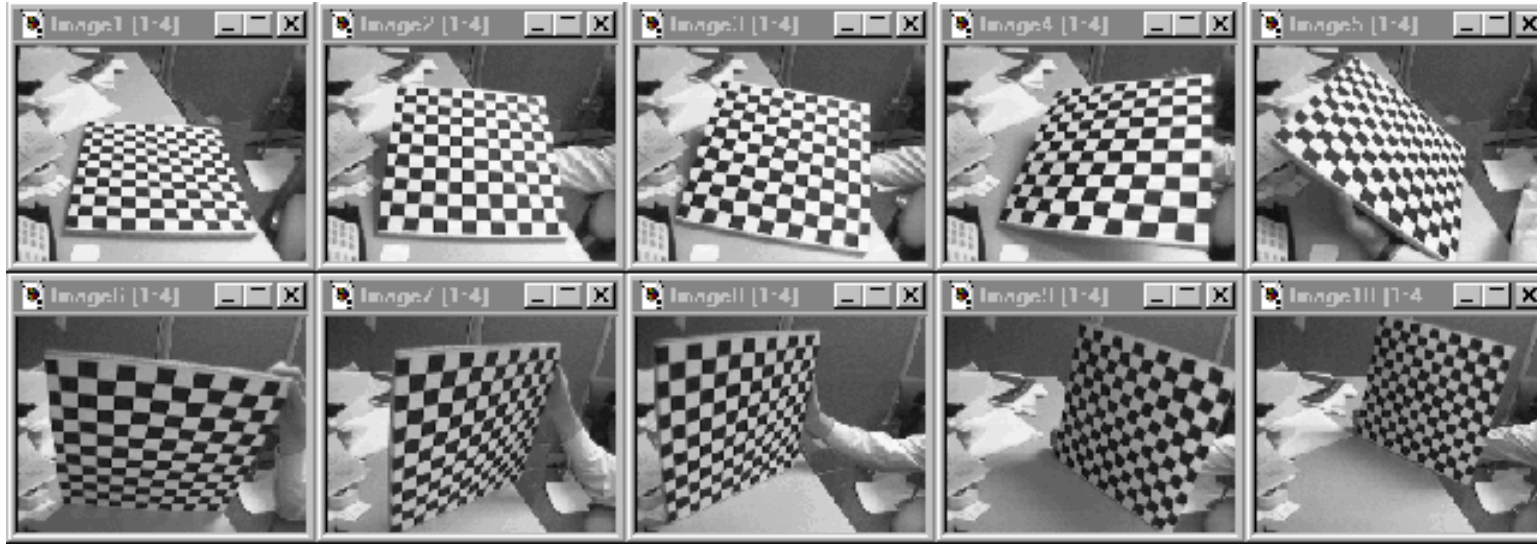


before



after

Alternative: Multi-plane calibration



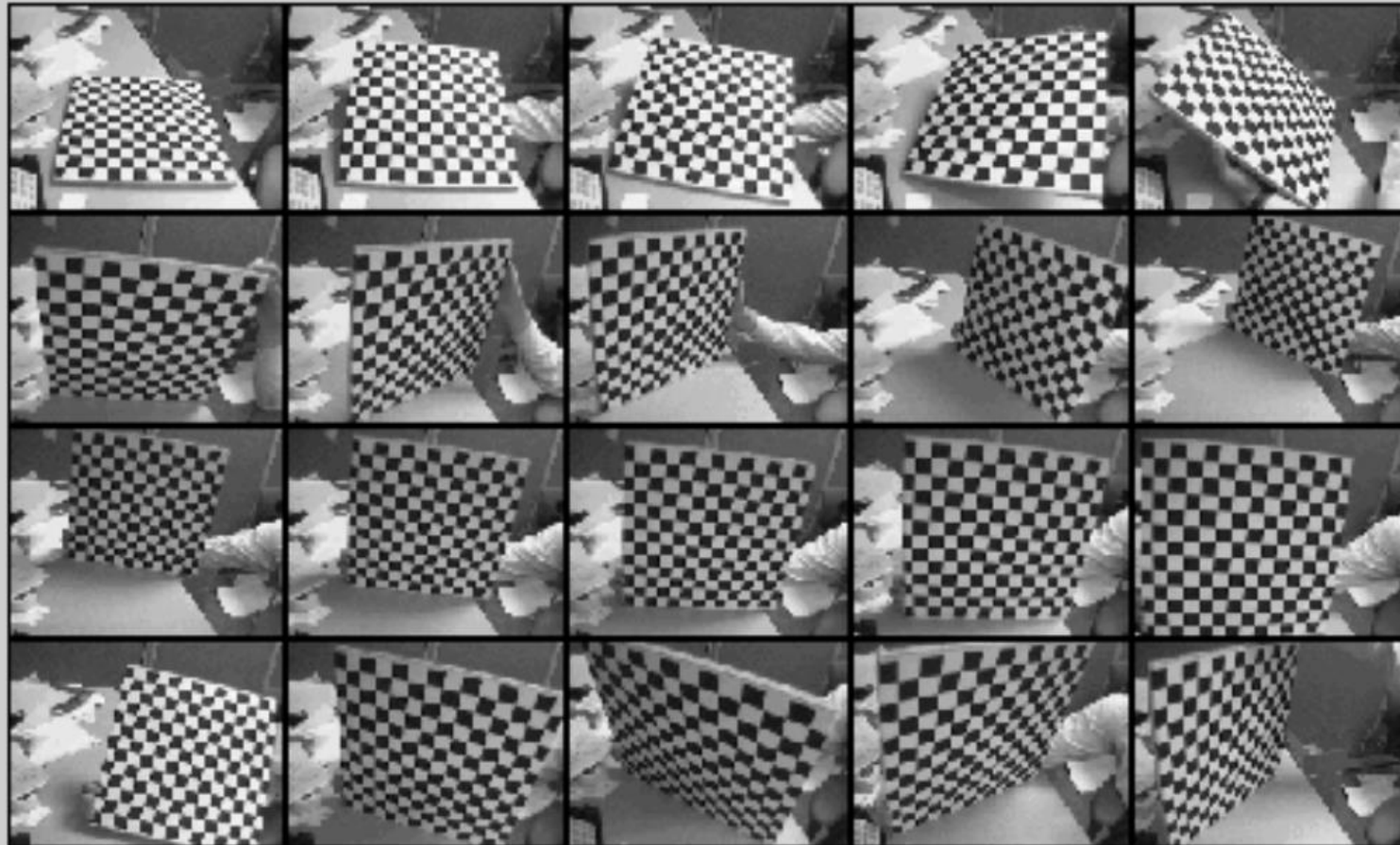
Advantages:

- Only requires a plane
- Don't have to know positions/orientations
- Great code available online!
 - Matlab version: http://www.vision.caltech.edu/bouguetj/calib_doc/index.html
 - Also available on OpenCV.

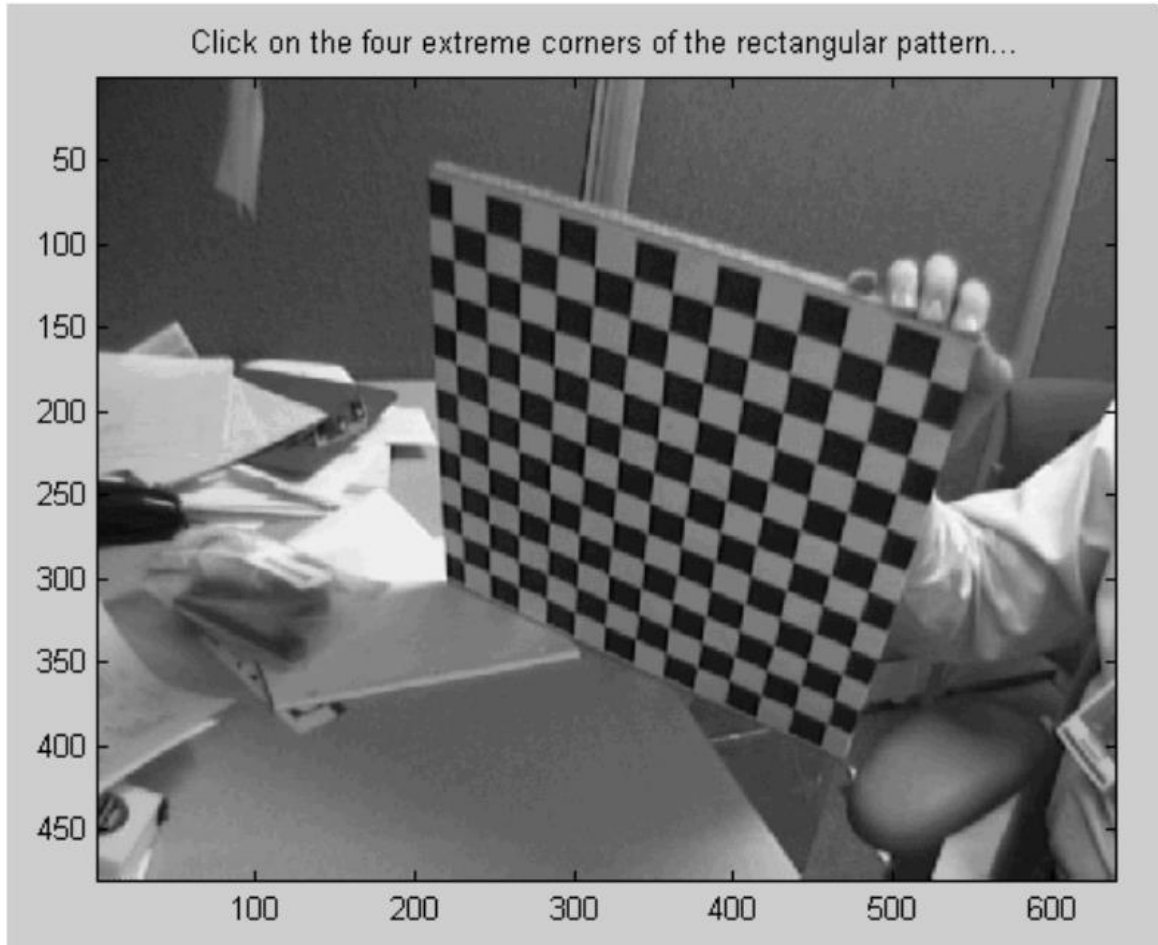
Disadvantage: Need to solve non-linear optimization problem.

Step-by-step demonstration

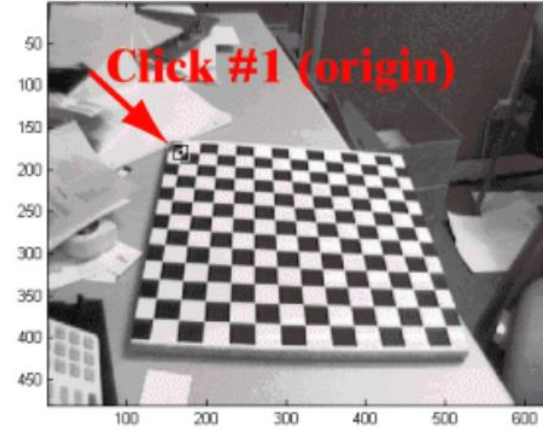
Calibration images



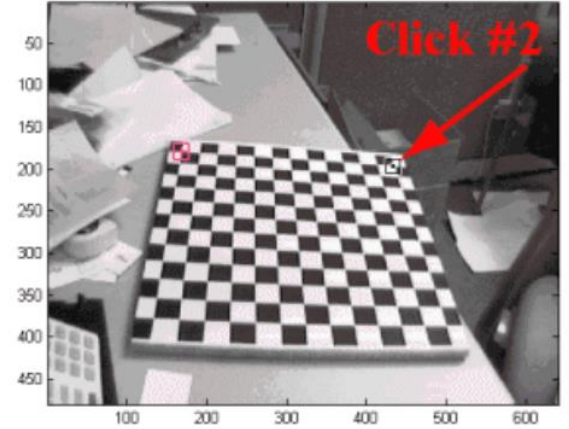
Step-by-step demonstration



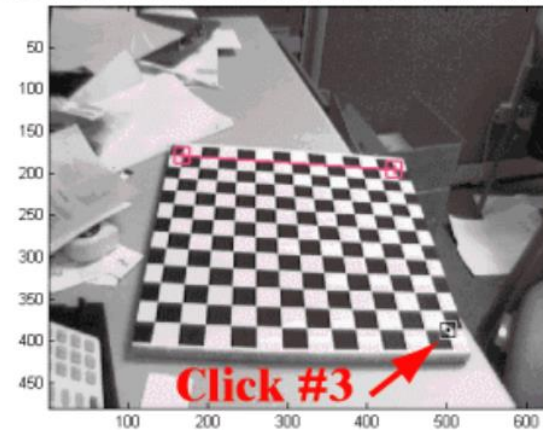
Click on the four extreme corners of the rectangular pattern (first corner = origin)... Image 1



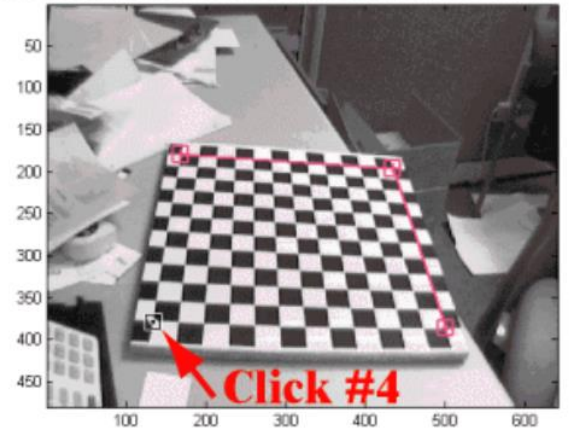
Click on the four extreme corners of the rectangular pattern (first corner = origin)... Image 1



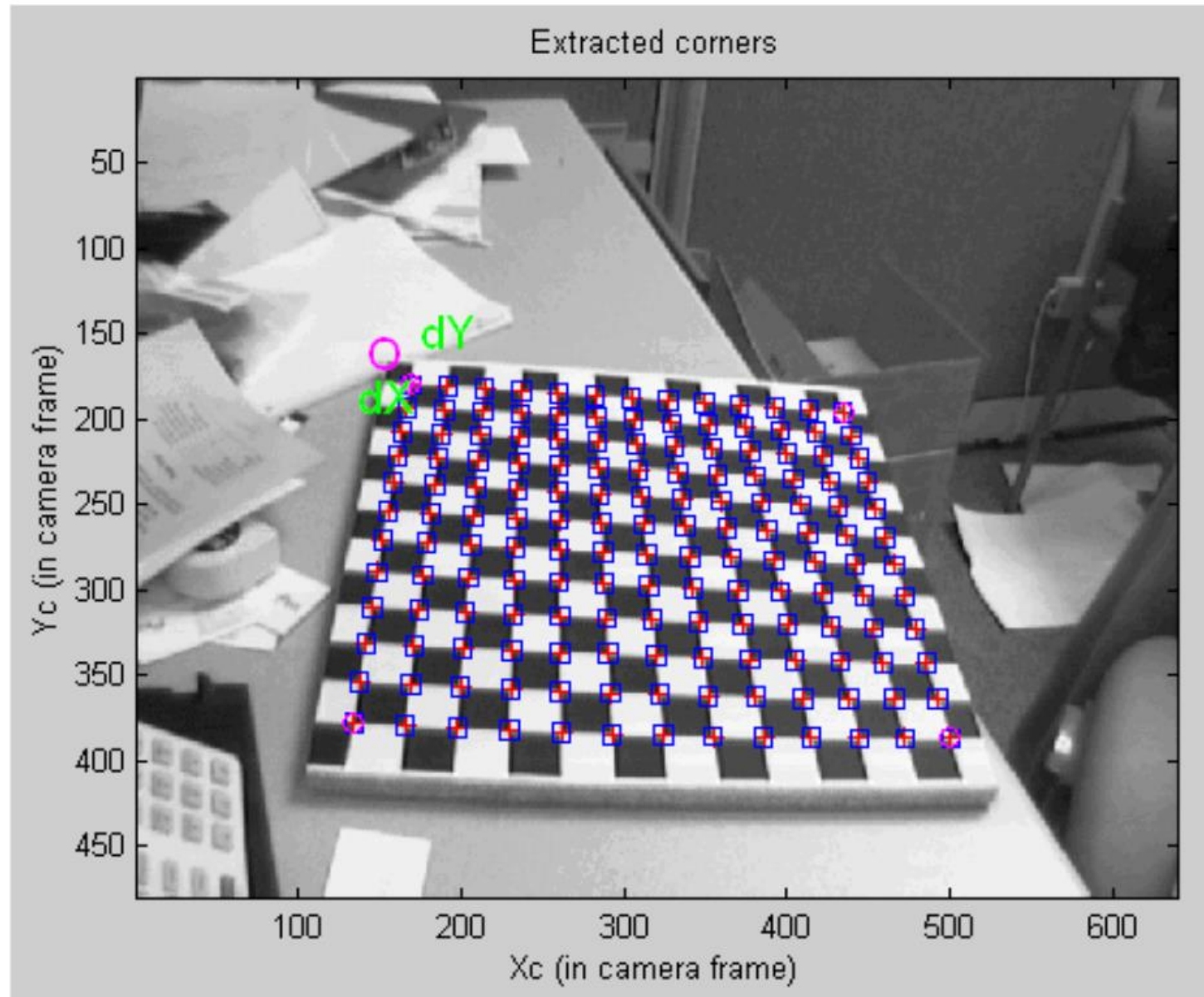
Click on the four extreme corners of the rectangular pattern (first corner = origin)... Image 1



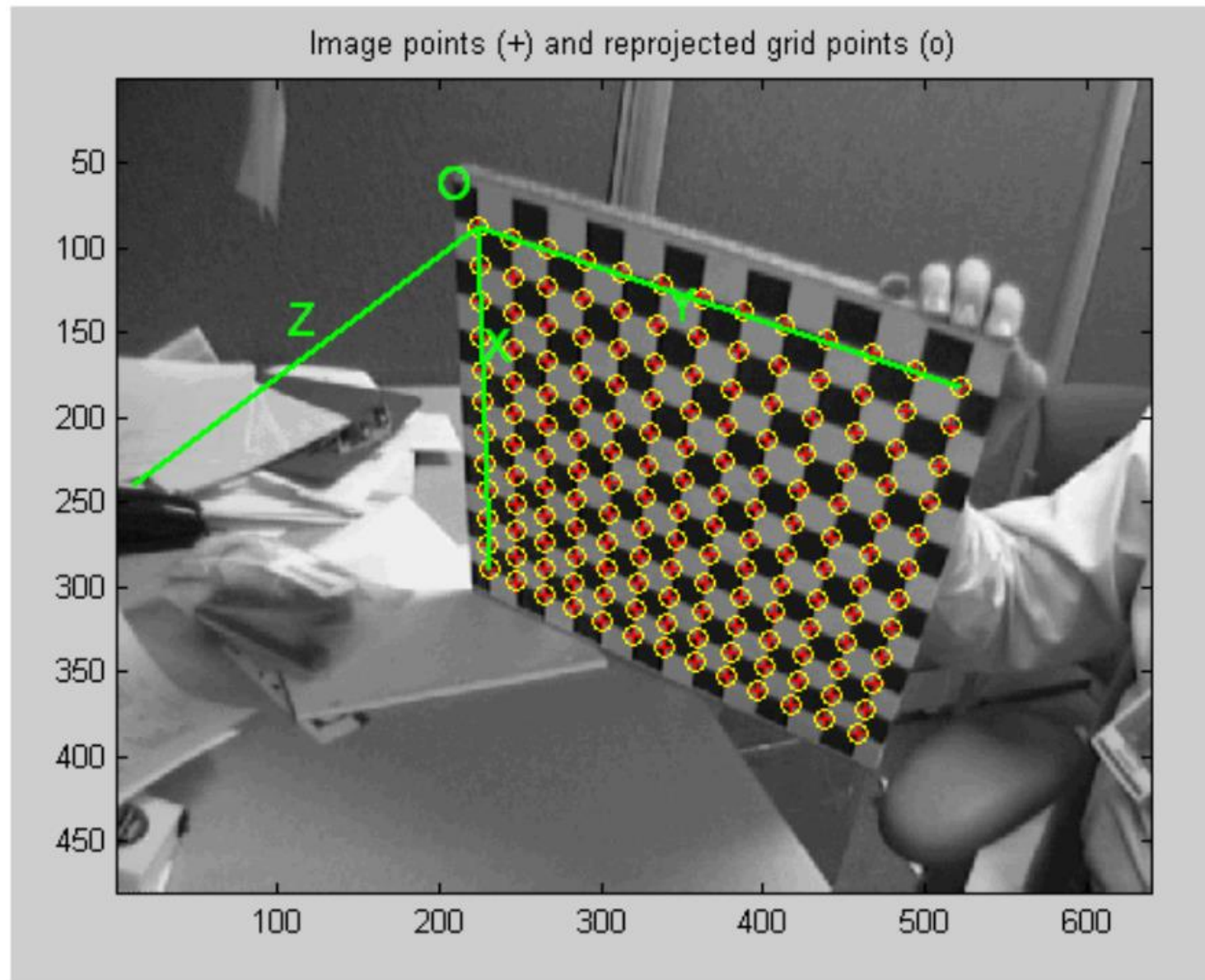
Click on the four extreme corners of the rectangular pattern (first corner = origin)... Image 1



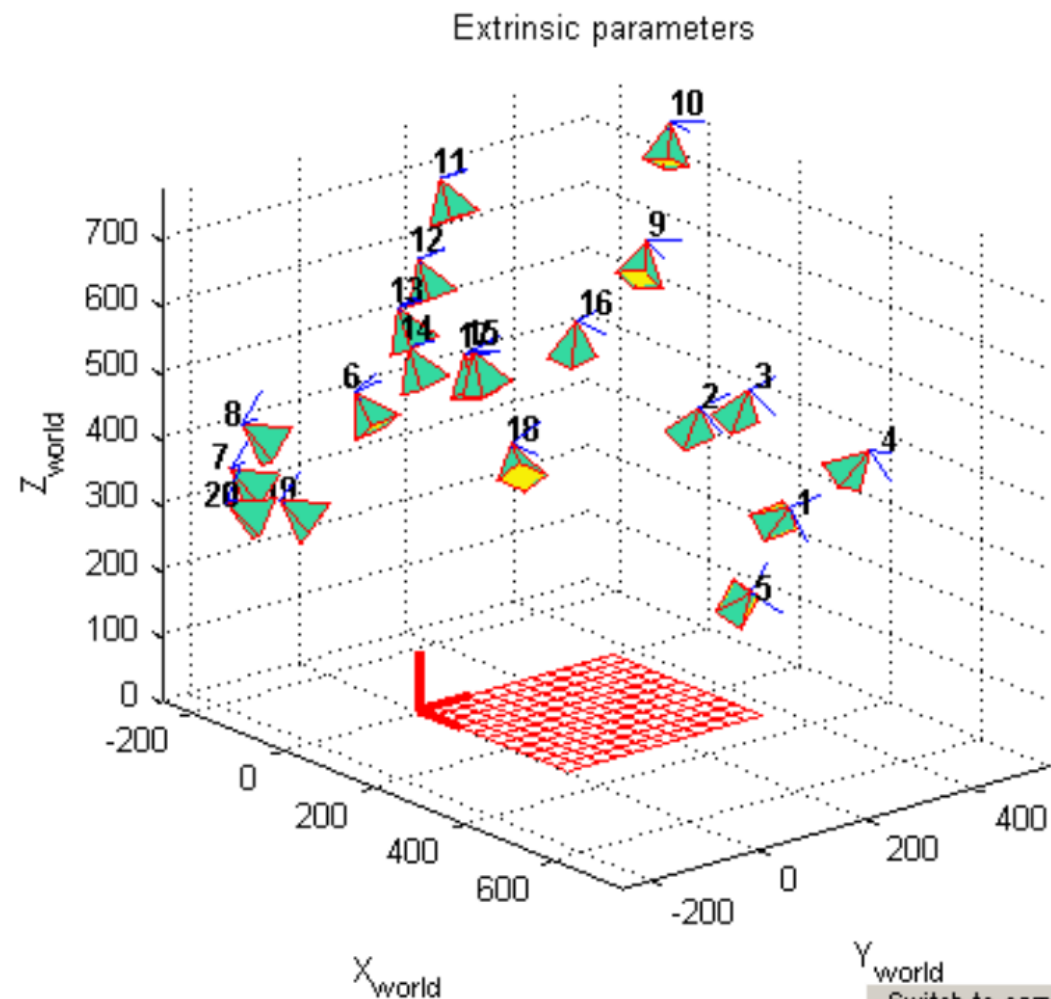
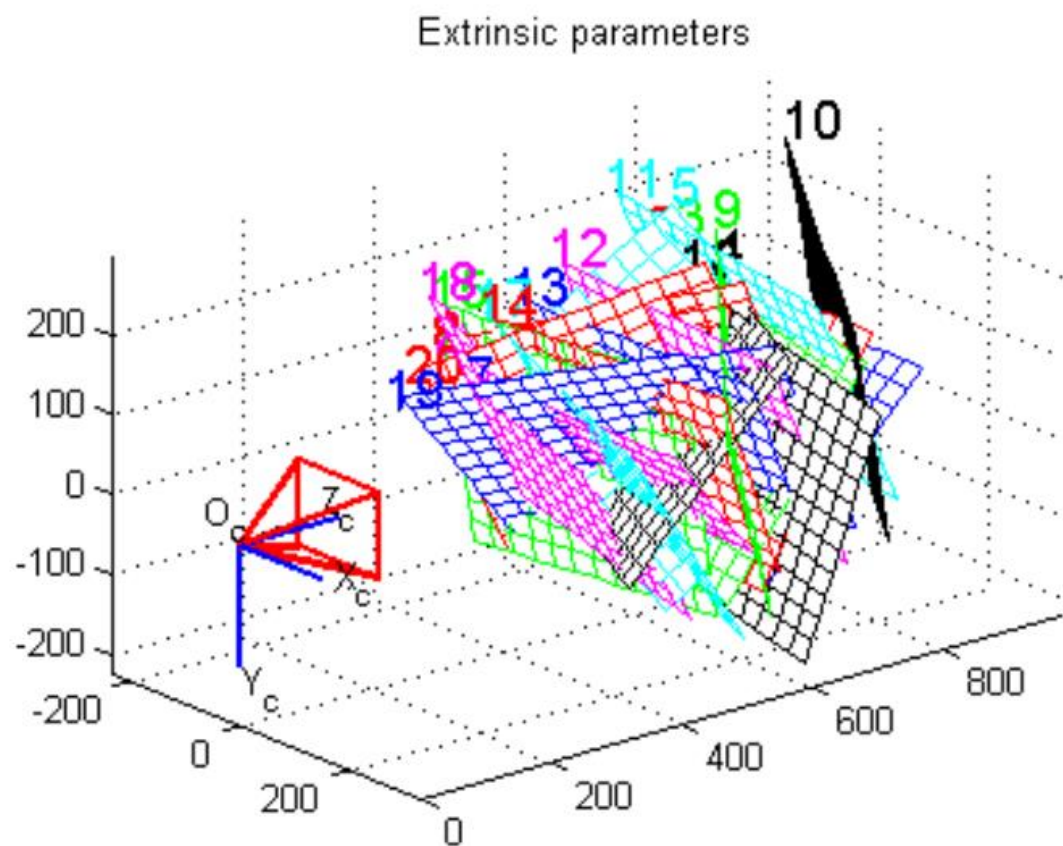
Step-by-step demonstration



Step-by-step demonstration



Step-by-step demonstration



Switch to camera-centered view

What does it mean to “calibrate a camera”?

What does it mean to “calibrate a camera”?

Many different ways to calibrate a camera:

- Radiometric calibration.  lecture 5
- Color calibration.  lecture 6
- Geometric calibration.  lecture 13
- Noise calibration.  later lecture
- Lens (or aberration) calibration.  lecture 12, (maybe) later lecture

References

Basic reading:

- Szeliski textbook, Section 2.1.5, 6.2.
- Bouguet, “Camera calibration toolbox for Matlab,” available at http://www.vision.caltech.edu/bouguetj/calib_doc/

The main resource for camera calibration in Matlab, where the screenshots in this lecture were taken from. It also has a detailed of the camera calibration algorithm and an extensive reference section.

Additional reading:

- Hartley and Zisserman, “Multiple View Geometry in Computer Vision,” Cambridge University Press 2004.
Chapter 6 of this book has a very thorough treatment of camera models.
- Gortler, “Foundations of 3D Computer Graphics,” MIT Press 2012.
Chapter 10 of this book has a nice discussion of pinhole cameras from a graphics point of view.
- Zhang, “A flexible new technique for camera calibration,” PAMI 2000.
The paper that introduced camera calibration from multiple views of a planar target.