Image Warping

15-463: Computational Photography
Alexei Efros, CMU, Fall 2011

Some slides from Steve Seitz

Image Transformations

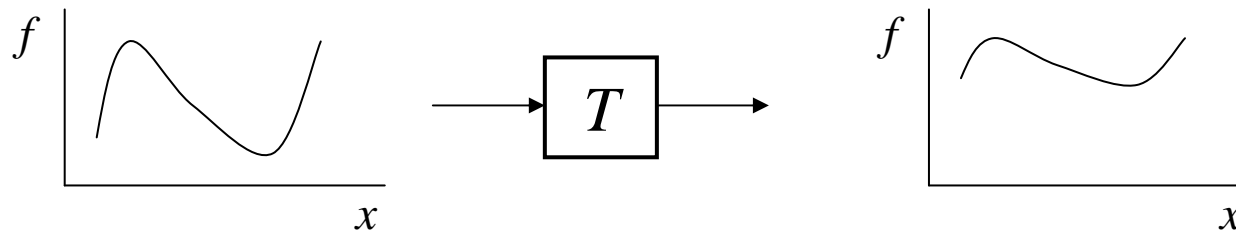image filtering: change **range** of image

$$g(x) = T(f(x))$$

image warping: change **domain** of image
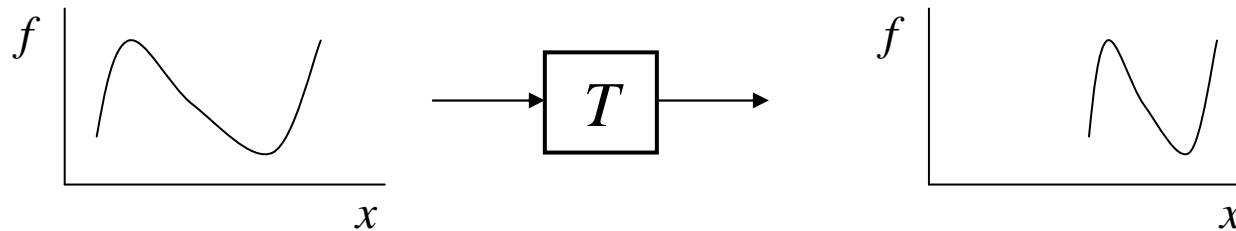
$$g(x) = f(T(x))$$

Image Transformations

image filtering: change **range** of image

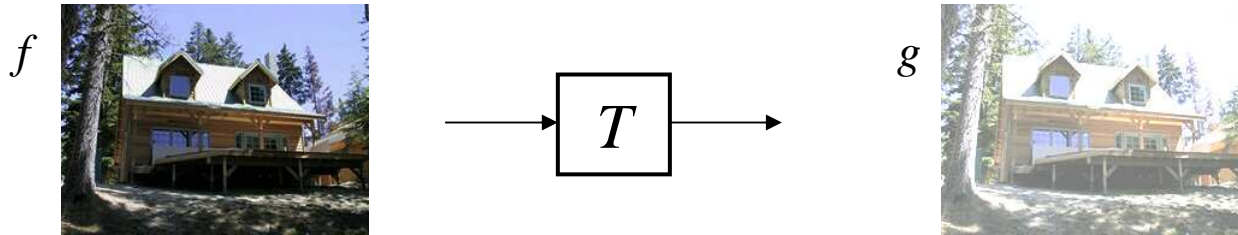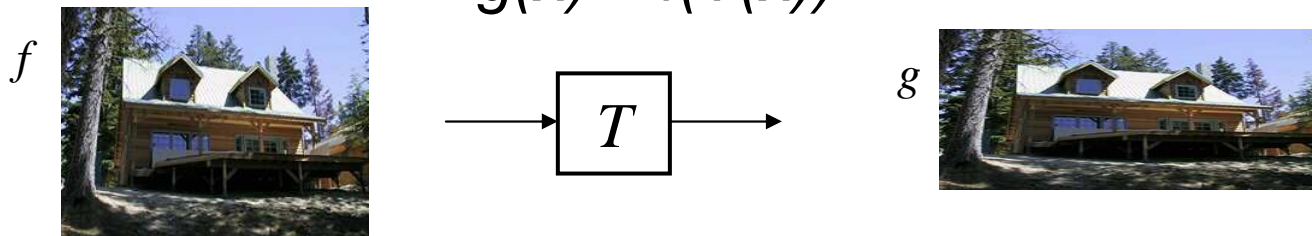$$g(x) = T(f(x))$$

image warping: change **domain** of image

$$g(x) = f(T(x))$$

Parametric (global) warping

Examples of parametric warps:

translation

rotation

aspect

affine

perspective

cylindrical

Parametric (global) warping
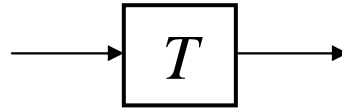
$\mathbf{p} = (x, y)$ $\mathbf{p'} = (x', y')$

Transformation T is a coordinate-changing machine:

$$p' = T(p)$$

What does it mean that T is global?

- Is the same for any point p
- can be described by just a few numbers (parameters)

Let's represent T as a matrix:
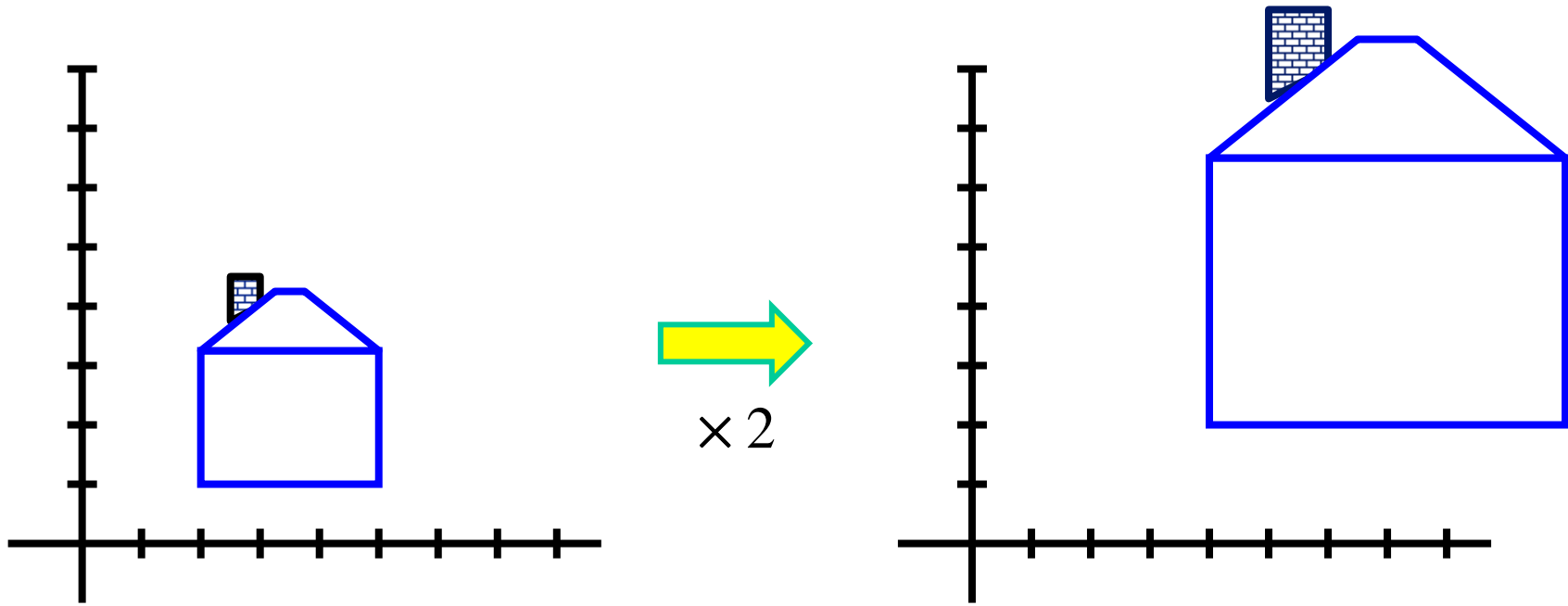
$$p' = \mathbf{M}p$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scaling

Scaling a coordinate means multiplying each of its components by a scalar

Uniform scaling means this scalar is the same for all components:

$\times 2$

Scaling

Non-uniform scaling: different scalars per component:

$$X \times 2,$$
$$Y \times 0.5$$
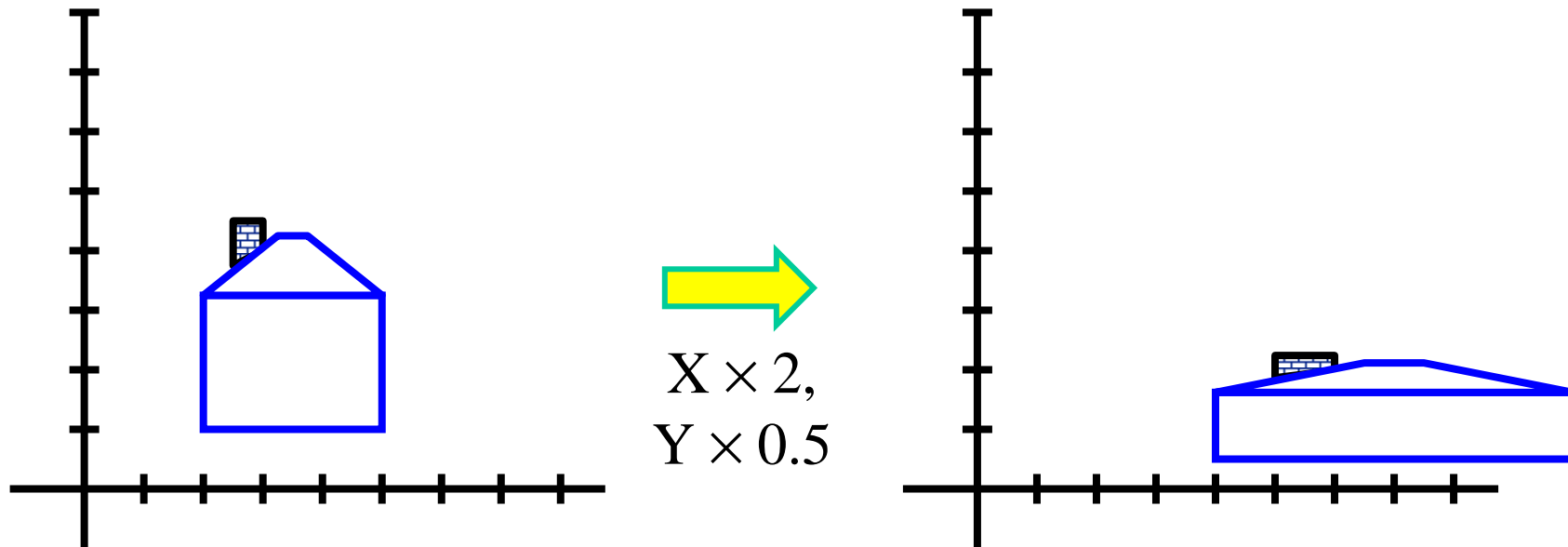
Scaling

Scaling operation:

$$x' = ax$$

$$y' = by$$

Or, in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{scaling matrix } S} \begin{bmatrix} x \\ y \end{bmatrix}$$

What's inverse of S?

2-D Rotation

$$x' = x\,\cos(\theta) - y\,\sin(\theta)$$
$$y' = x\,\sin(\theta) + y\,\cos(\theta)$$
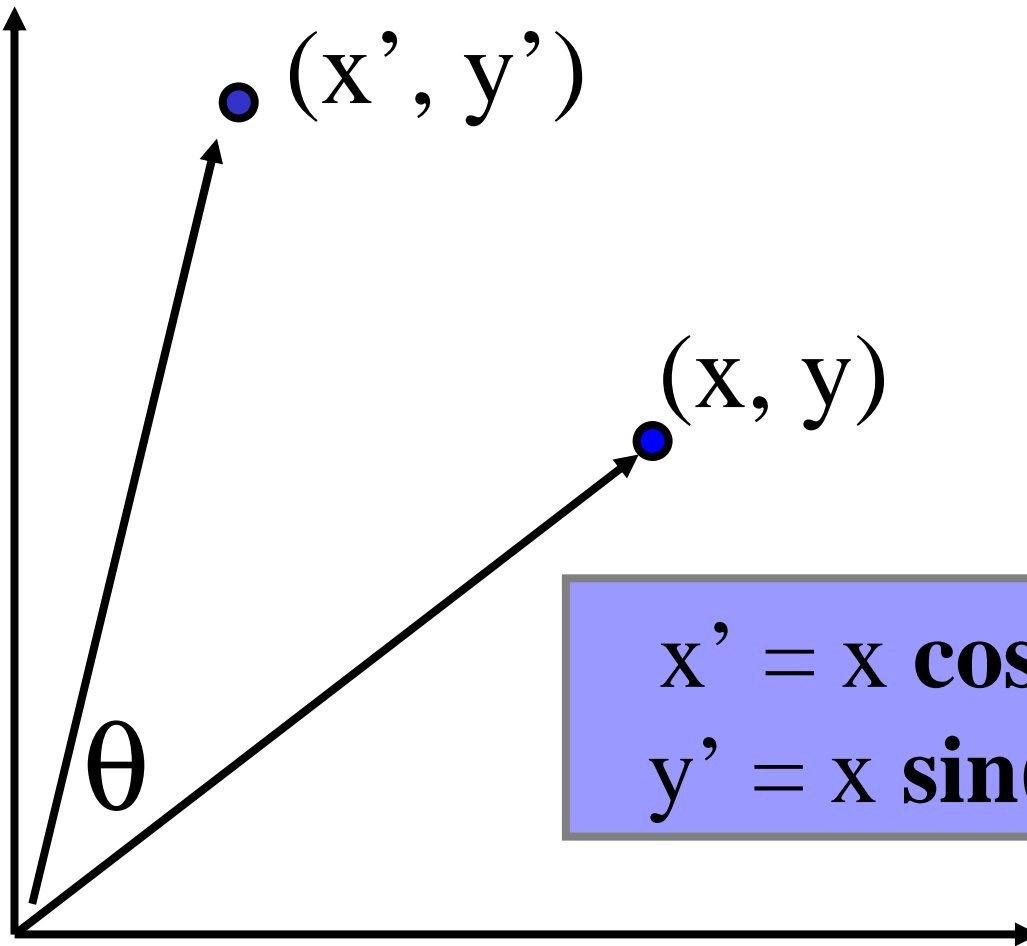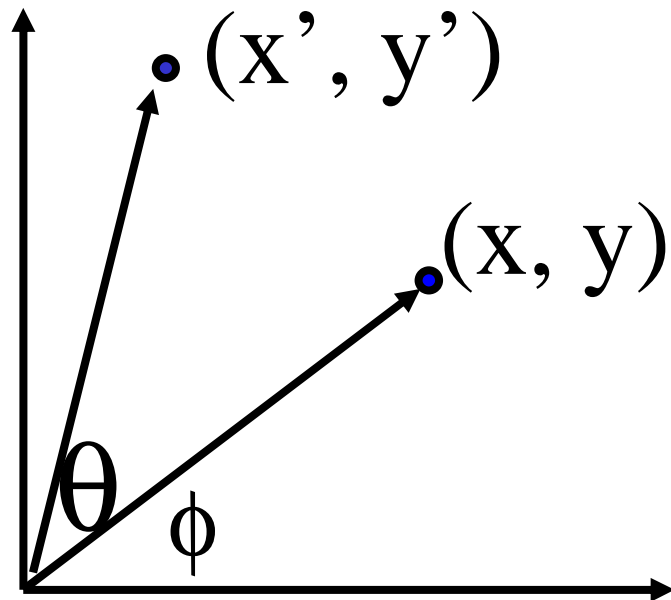
2-D Rotation

$x = r \cos(\phi)$

$y = r \sin(\phi)$

$x' = r \cos(\phi + \theta)$

$y' = r \sin(\phi + \theta)$

Trig Identity…

$x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta)$

$y' = r \sin(\phi) \cos(\theta) + r \cos(\phi) \sin(\theta)$

Substitute…

$x' = x \, \mathbf{cos}(\theta) - y \, \mathbf{sin}(\theta)$

$y' = x \, \mathbf{sin}(\theta) + y \, \mathbf{cos}(\theta)$

2-D Rotation

This is easy to capture in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Even though $\sin(\theta)$ and $\cos(\theta)$ are nonlinear functions of θ,

- *x' is a linear combination of x and y*
- *y' is a linear combination of x and y*

What is the inverse transformation?

- Rotation by $-\theta$
- For rotation matrices $\mathbf{R}^{-1} = \mathbf{R}^{T}$

2x2 Matrices

What types of transformations can be
represented with a 2x2 matrix?

2D Identity?

$$x' = x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Scale around (0,0)?

$$x' = s_x * x$$
$$y' = s_y * y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)?

$$x' = \cos \Theta * x - \sin \Theta * y$$
$$y' = \sin \Theta * x + \cos \Theta * y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Shear?

$$x' = x + sh_x * y$$
$$y' = sh_y * x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What types of transformations can be
   represented with a 2x2 matrix?

2D Mirror about Y axis?

$x' = -x$
$y' = y$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror over (0,0)?

$x' = -x$
$y' = -y$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

What types of transformations can be
represented with a 2x2 matrix?

2D Translation?

$$x' = x + t_x$$
$$y' = y + t_y$$

NO!

Only linear 2D transformations
can be represented with a 2x2 matrix

All 2D Linear Transformations

Linear transformations are combinations of …
- Scale,
- Rotation,
- Shear, and
- Mirror

Properties of linear transformations:
- Origin maps to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition
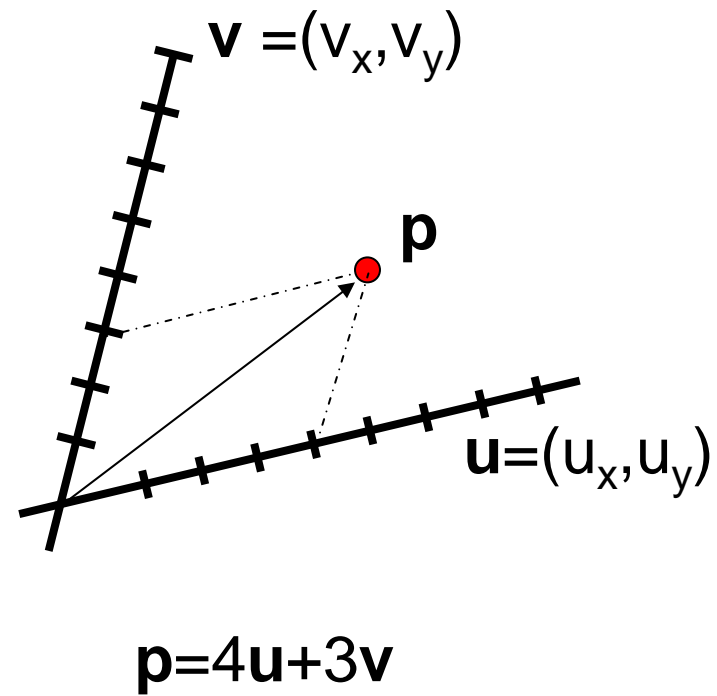
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} i & j \\ k & l \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Consider a different Basis

$\mathbf{j} = (0,1)$

$\mathbf{q}$

$\mathbf{i} = (1,0)$

$\mathbf{q} = 4\mathbf{i} + 3\mathbf{j} = (4,3)$

$\mathbf{v} = (v_x, v_y)$

$\mathbf{p}$

$\mathbf{u} = (u_x, u_y)$

$\mathbf{p} = 4\mathbf{u} + 3\mathbf{v}$
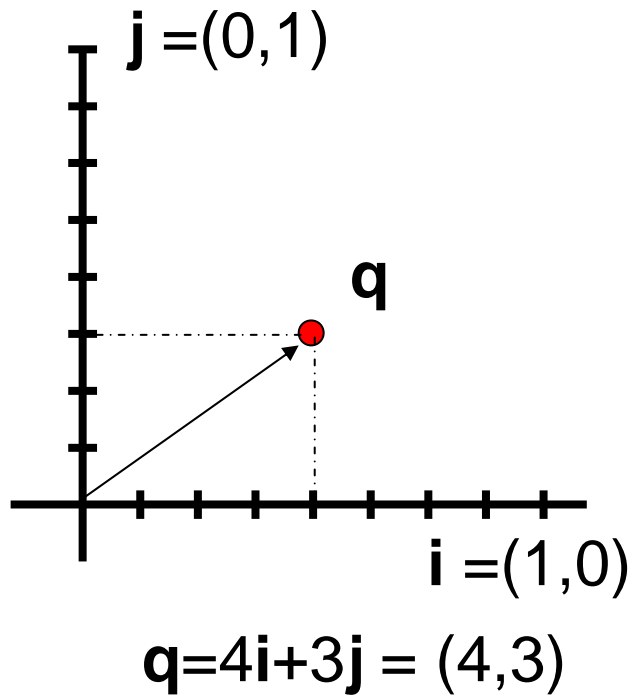
Linear Transformations as Change of Basis

$$\mathbf{v} = (v_x, v_y)$$

$$\mathbf{p^{uv}}$$

$$\mathbf{u} = (u_x, u_y)$$

$$\mathbf{j} = (0,1)$$

$$\mathbf{p^{ij}}$$

$$\mathbf{i} = (1,0)$$

$$\mathbf{p^{uv}} = (4,3)$$

$$\mathbf{p^{ij}} = 4\mathbf{u} + 3\mathbf{v}$$

$$p_x = 4u_x + 3v_x$$
$$p_y = 4u_y + 3v_y$$

$$\mathbf{p^{ij}} = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \mathbf{p^{uv}}$$
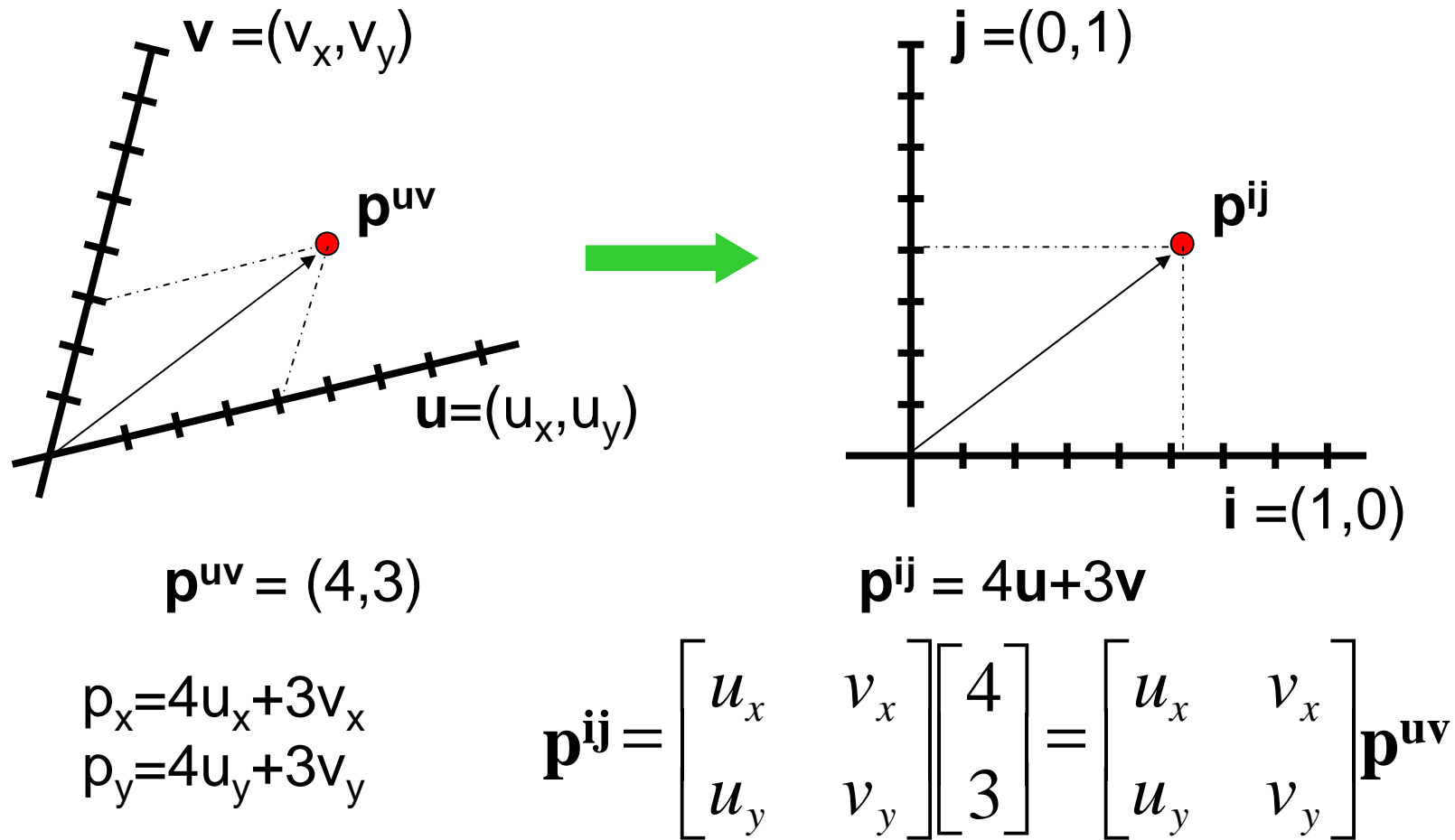
Any linear transformation is a basis!!!

What's the inverse transform?

$\mathbf{j} = (0,1)$

$\mathbf{p^{ij}}$

$\mathbf{i} = (1,0)$

$\mathbf{v} = (v_x, v_y)$

$\mathbf{p^{uv}}$

$\mathbf{u} = (u_x, u_y)$
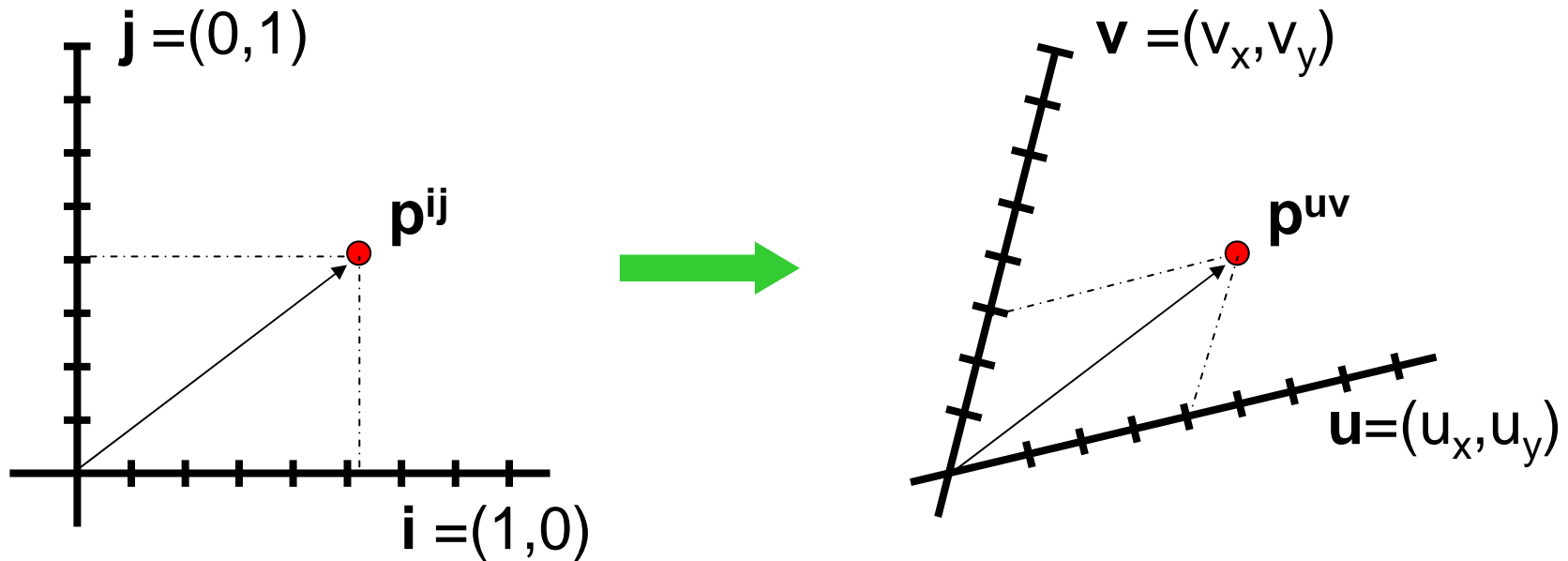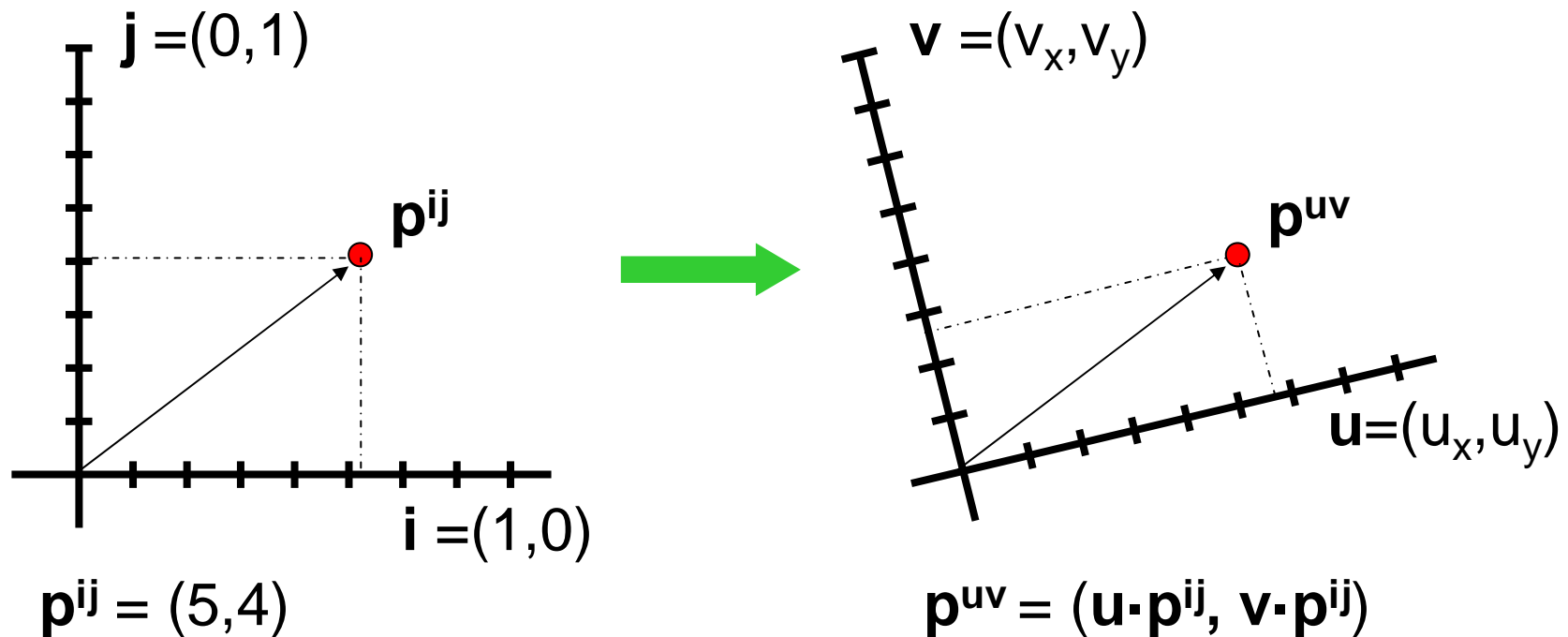
$\mathbf{p^{ij}} = (5,4) = p_x\mathbf{u} + p_y\mathbf{v}$

$\mathbf{p^{uv}} = (p_x, p_y) = ?$

$$\mathbf{p^{uv}} = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix}^{-1} \mathbf{p^{ij}}$$

- How can we change from any basis to any basis?
- What if the basis are orthogonal?

Projection onto orthogonal basis

$$\mathbf{p^{uv}} = \begin{bmatrix} u_x & u_x \\ v_y & v_y \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \mathbf{p^{ij}}$$

Homogeneous Coordinates

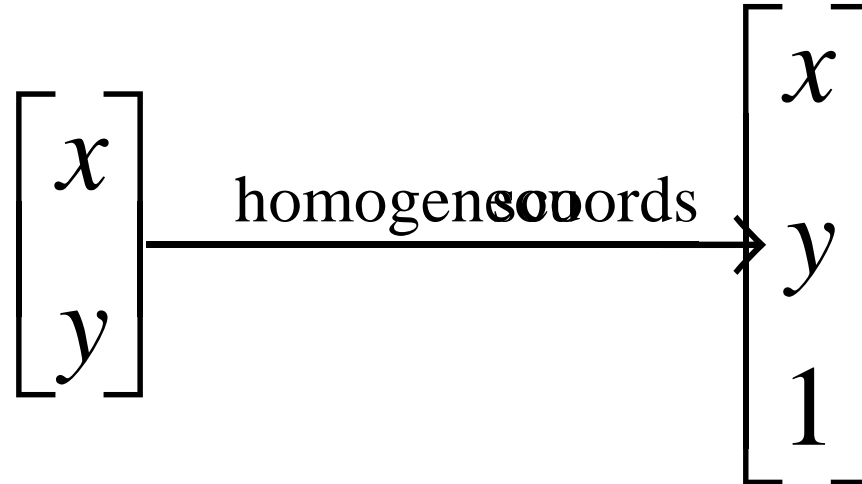**Q: How can we represent translation as a 3x3 matrix?**

$$x' = x + t_x$$

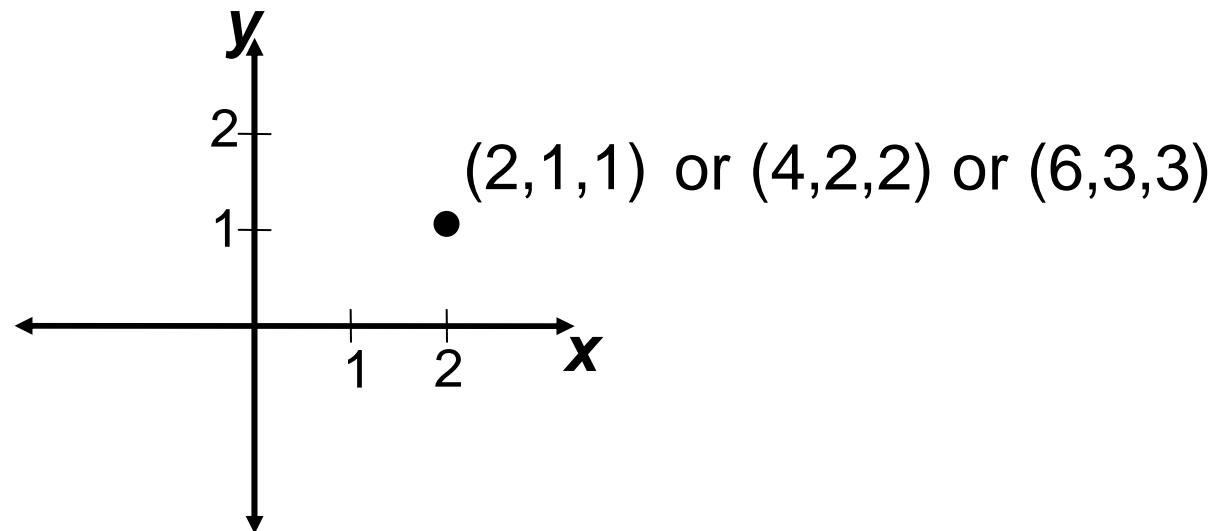$$y' = y + t_y$$

Homogeneous Coordinates

Homogeneous coordinates

- represent coordinates in 2 dimensions with a 3-vector

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{homogeneous coords}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

Add a 3rd coordinate to every 2D point

- (x, y, w) represents a point at location (x/w, y/w)
- (x, y, 0) represents a point at infinity
- (0, 0, 0) is not allowed

(2,1,1) or (4,2,2) or (6,3,3)

Convenient coordinate system to represent many useful transformations

Homogeneous Coordinates

Q: How can we represent translation as a 3x3 matrix?

$$x' = x + t_x$$

$$y' = y + t_y$$

A: Using the rightmost column:

$$Translation = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$
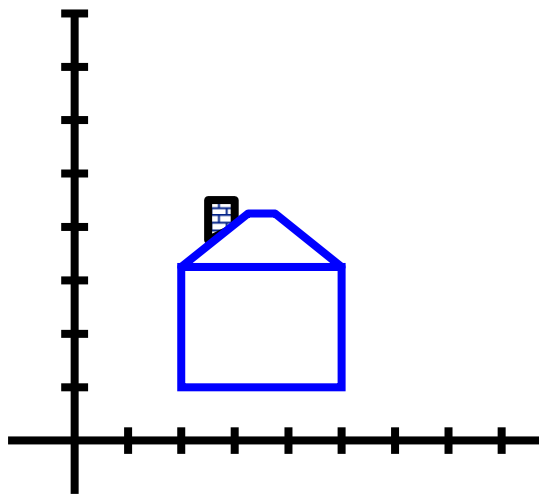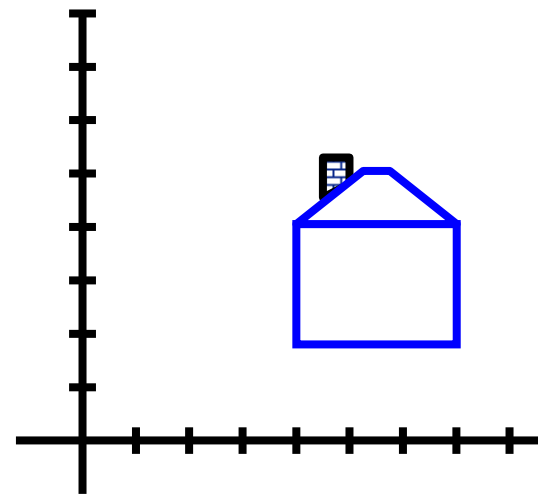
Translation

Example of translation

Homogeneous Coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

$t_x = 2$
$t_y = 1$

Basic 2D Transformations

Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

Matrix Composition

Transformations can be combined by
matrix multiplication

$$
\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}
$$

$\mathbf{p}'$ = $T(t_x, t_y)$ $R(\Theta)$ $S(s_x, s_y)$ $\mathbf{p}$

Affine Transformations

Affine transformations are combinations of …

- Linear transformations, and
- Translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of affine transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition
- Models change of basis

Will the last coordinate w always be 1?

Projective Transformations

Projective transformations …

- Affine transformations, and
- Projective warps

Properties of projective transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines do not necessarily remain parallel
- Ratios are not preserved
- Closed under composition
- Models change of basis

$$
\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}
$$

2D image transformations
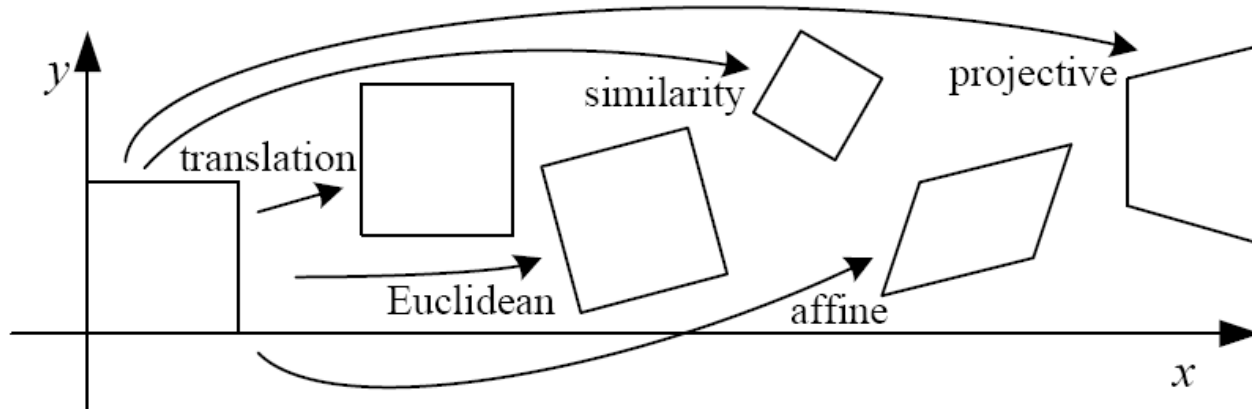
Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\left[\begin{array}{c\|c} I & t \end{array}\right]_{2\times3}$			
rigid (Euclidean)	$\left[\begin{array}{c\|c} R & t \end{array}\right]_{2\times3}$			
similarity	$\left[\begin{array}{c\|c} sR & t \end{array}\right]_{2\times3}$			
affine	$\left[\begin{array}{c} A \end{array}\right]_{2\times3}$			
projective	$\left[\begin{array}{c} \tilde{H} \end{array}\right]_{3\times3}$			

These transformations are a nested set of groups
   • Closed under composition and inverse is a member