

# Image Warping

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<http://www.jeffrey-martin.com>

15-463: Computational Photography  
Alexei Efros, CMU, Fall 2007

# Image Warping

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image filtering: change **range** of image

$$g(x) = T(f(x))$$

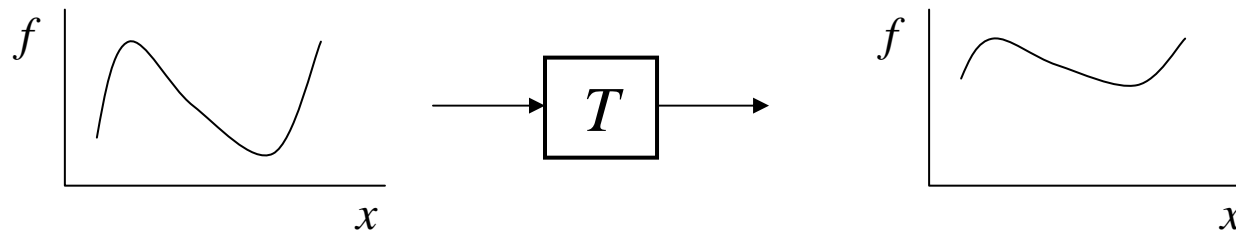
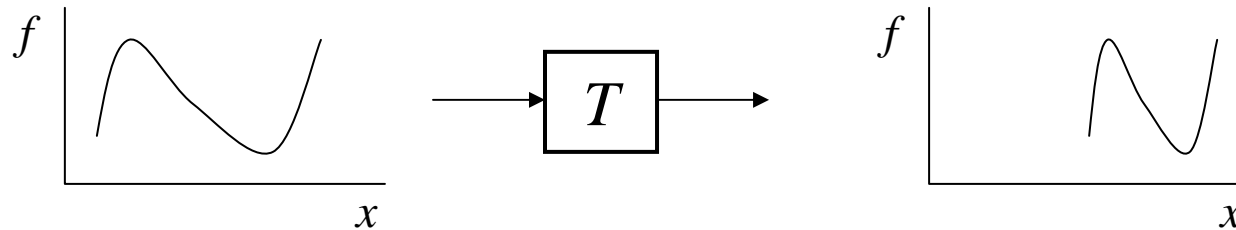


image warping: change **domain** of image

$$g(x) = f(T(x))$$



# Image Warping

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image filtering: change **range** of image

$$g(x) = h(T(x))$$

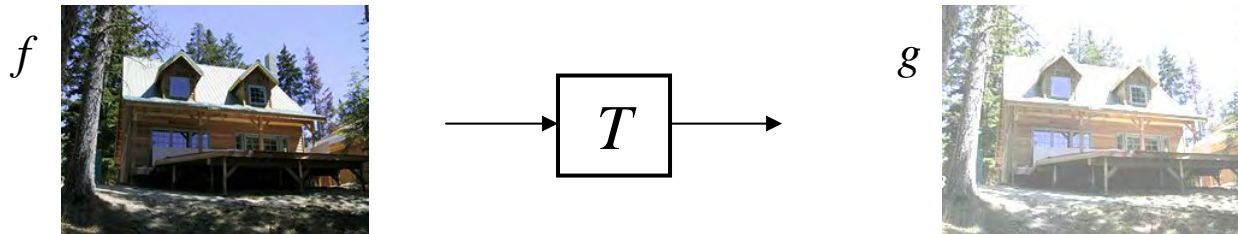
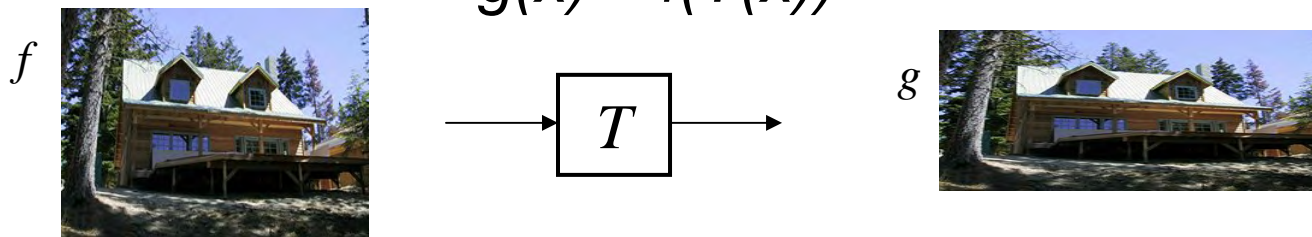


image warping: change **domain** of image

$$g(x) = f(T(x))$$



# Parametric (global) warping

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Examples of parametric warps:



translation



rotation



aspect



affine



perspective



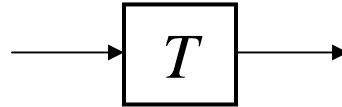
cylindrical

# Parametric (global) warping

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$$\mathbf{p} = (x, y)$$



$$\mathbf{p}' = (x', y')$$

Transformation  $T$  is a coordinate-changing machine:

$$\mathbf{p}' = T(\mathbf{p})$$

What does it mean that  $T$  is global?

- Is the same for any point  $\mathbf{p}$
- can be described by just a few numbers (parameters)

Let's represent  $T$  as a matrix:

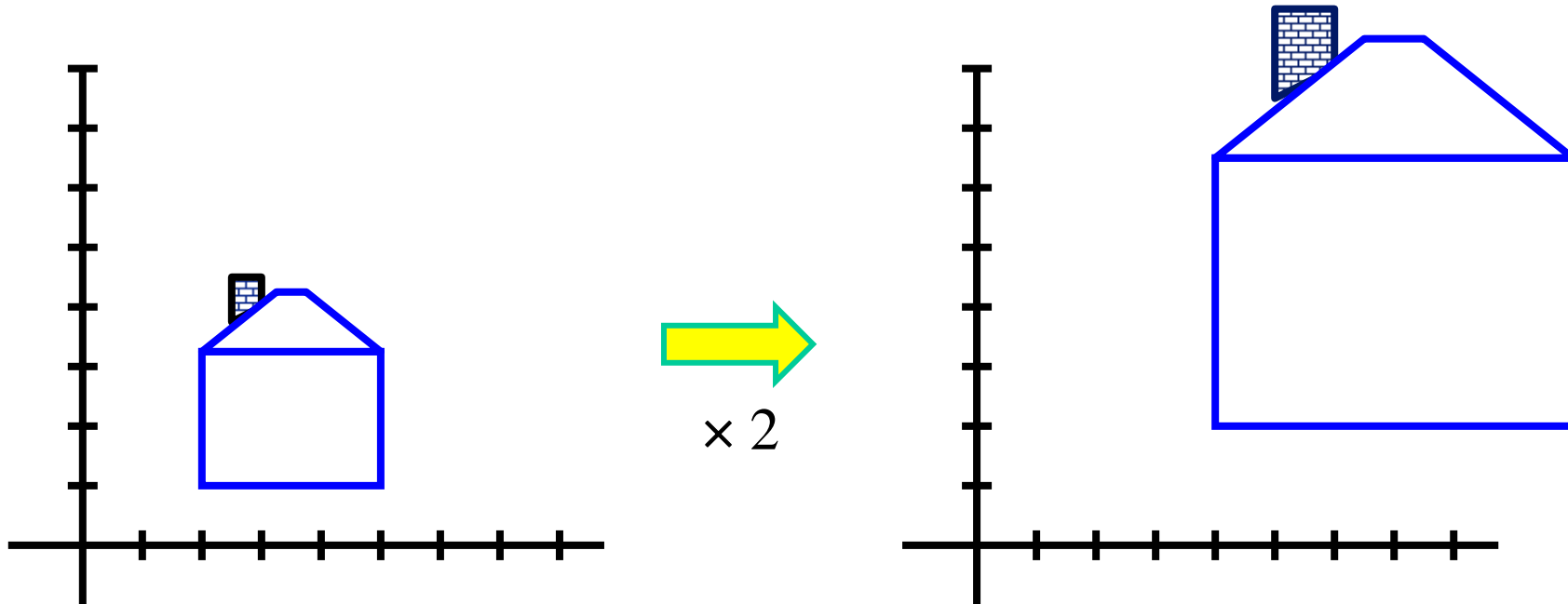
$$\mathbf{p}' = \mathbf{M}\mathbf{p}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

# Scaling

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*Scaling* a coordinate means multiplying each of its components by a scalar

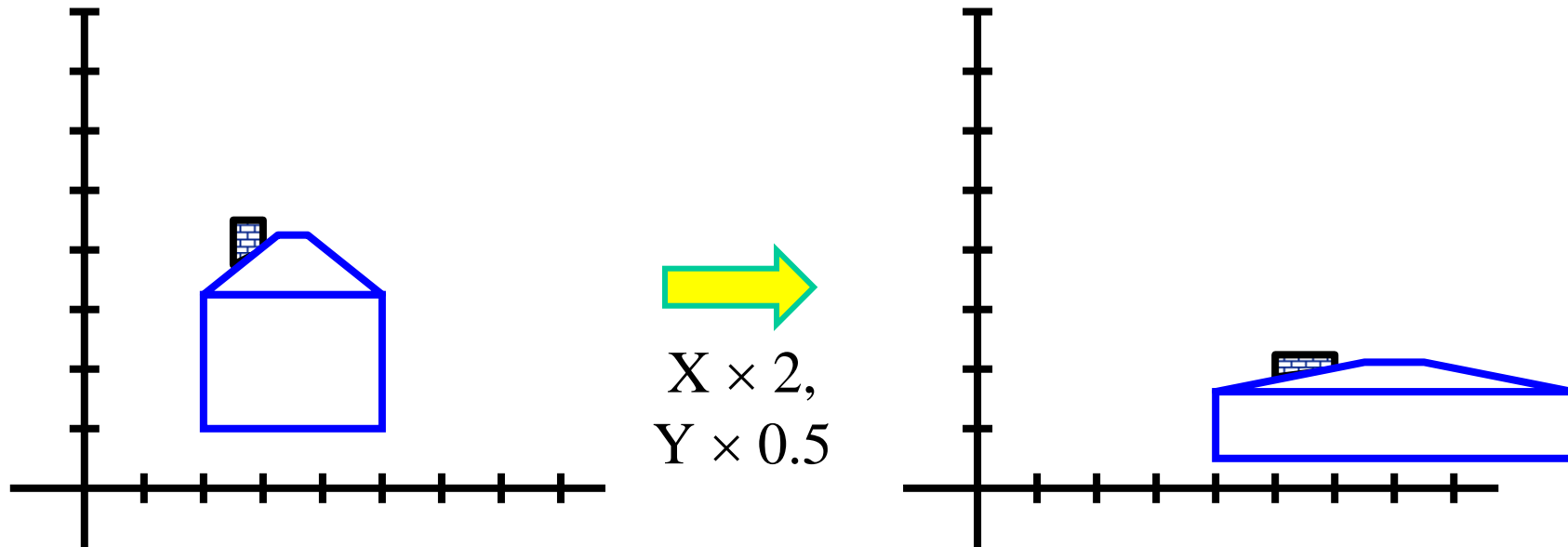
*Uniform scaling* means this scalar is the same for all components:



# Scaling

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*Non-uniform scaling*: different scalars per component:



# Scaling

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Scaling operation:

$$x' = ax$$

$$y' = by$$

Or, in matrix form:

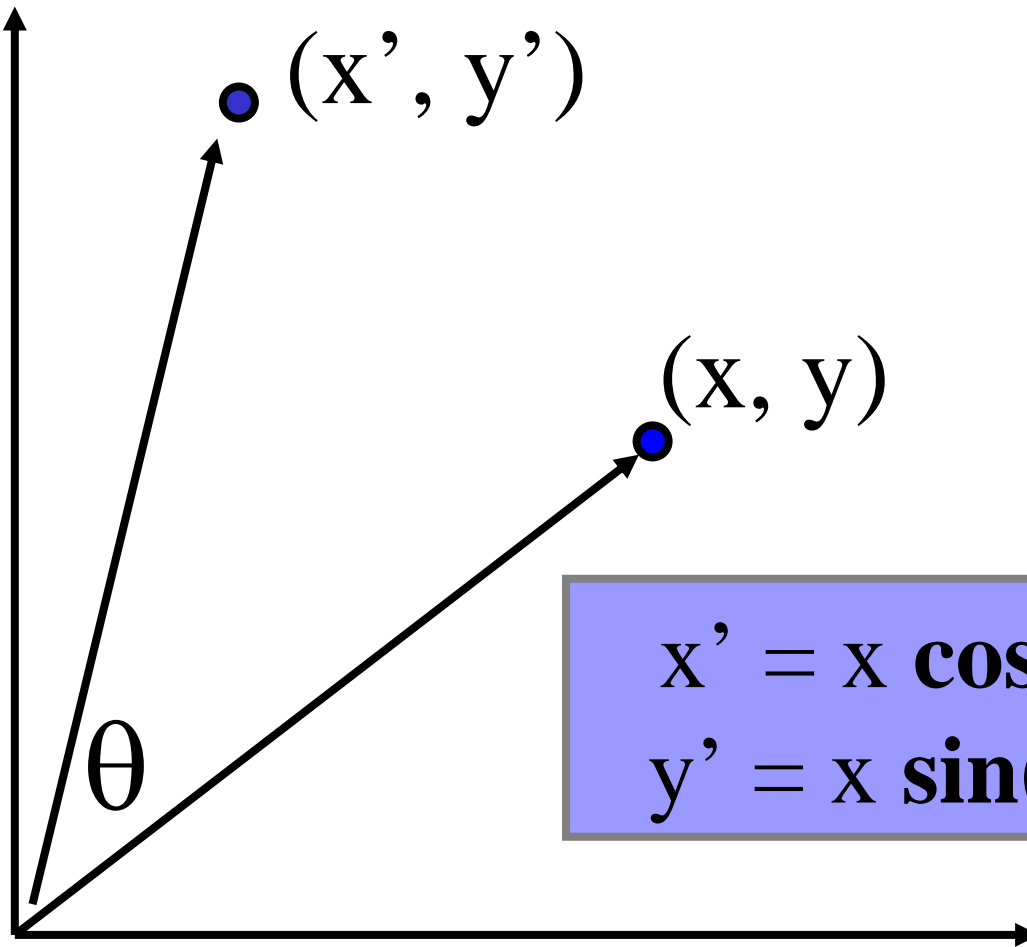
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{scaling matrix } S} \begin{bmatrix} x \\ y \end{bmatrix}$$

What's inverse of  $S$ ?



# 2-D Rotation

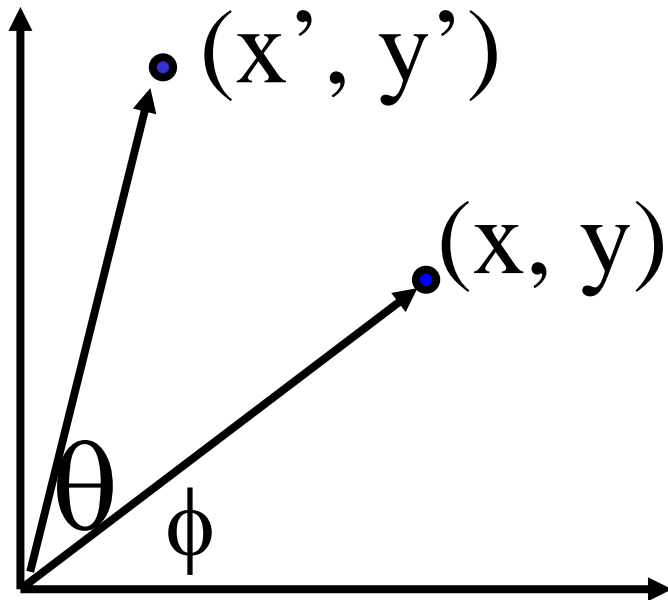
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$$\begin{aligned}x' &= x \cos(\theta) - y \sin(\theta) \\y' &= x \sin(\theta) + y \cos(\theta)\end{aligned}$$

# 2-D Rotation

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$$x = r \cos(\phi)$$

$$y = r \sin(\phi)$$

$$x' = r \cos(\phi + \theta)$$

$$y' = r \sin(\phi + \theta)$$

Trig Identity...

$$x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta)$$

$$y' = r \sin(\phi) \cos(\theta) + r \cos(\phi) \sin(\theta)$$

Substitute...

$$x' = x \cos(\theta) - y \sin(\theta)$$

$$y' = x \sin(\theta) + y \cos(\theta)$$

# 2-D Rotation

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This is easy to capture in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Even though  $\sin(\theta)$  and  $\cos(\theta)$  are nonlinear functions of  $\theta$ ,

- ***$x'$  is a linear combination of  $x$  and  $y$***
- ***$y'$  is a linear combination of  $x$  and  $y$***

What is the inverse transformation?

- Rotation by  $-\theta$
- For rotation matrices  **$\mathbf{R}^{-1} = \mathbf{R}^T$**

# 2x2 Matrices

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What types of transformations can be represented with a 2x2 matrix?

2D Identity?

$$\begin{aligned}x' &= x \\ y' &= y\end{aligned} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Scale around (0,0)?

$$\begin{aligned}\mathbf{x}' &= s_x * \mathbf{x} \\ \mathbf{y}' &= s_y * \mathbf{y}\end{aligned} \quad \begin{bmatrix} \mathbf{x}' \\ \mathbf{y}' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

# 2x2 Matrices

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What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)?

$$\begin{aligned}x' &= \cos \Theta * x - \sin \Theta * y \\y' &= \sin \Theta * x + \cos \Theta * y\end{aligned} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Shear?

$$\begin{aligned}x' &= x + sh_x * y \\y' &= sh_y * x + y\end{aligned} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

# 2x2 Matrices

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What types of transformations can be represented with a 2x2 matrix?

2D Mirror about Y axis?

$$\begin{aligned}x' &= -x \\ y' &= y\end{aligned}\quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror over (0,0)?

$$\begin{aligned}x' &= -x \\ y' &= -y\end{aligned}\quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

# 2x2 Matrices

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What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$\begin{aligned}x' &= x + t_x \\ y' &= y + t_y\end{aligned} \quad \text{NO!}$$

Only linear 2D transformations  
can be represented with a 2x2 matrix

# All 2D Linear Transformations

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Linear transformations are combinations of ...

- Scale,
- Rotation,
- Shear, and
- Mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

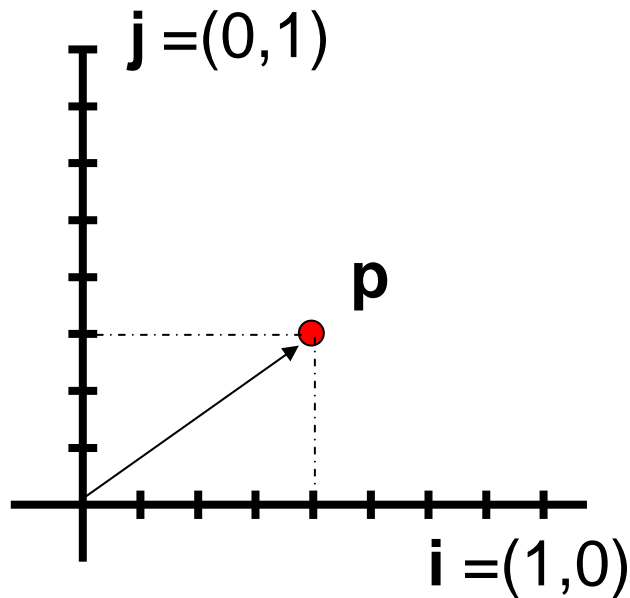
Properties of linear transformations:

- Origin maps to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition

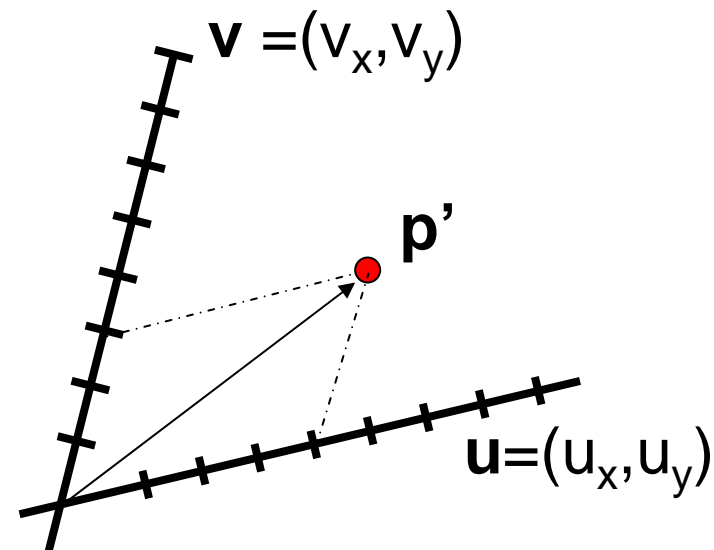
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



# Linear Transformations as Change of Basis



$$\mathbf{p} = 4\mathbf{i} + 3\mathbf{j} = (4, 3)$$



$$\mathbf{p}' = 4\mathbf{u} + 3\mathbf{v}$$

$$\begin{aligned} p'_x &= 4u_x + 3v_x \\ p'_y &= 4u_y + 3v_y \end{aligned}$$

$$\mathbf{p}' = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \mathbf{p}$$

Any linear transformation is a basis!!!

- What's the inverse transform?
- How can we change from any basis to any basis?
- What if the basis are orthogonal?

# Homogeneous Coordinates

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**Q: How can we represent translation as a 3x3 matrix?**

$$x' = x + t_x$$

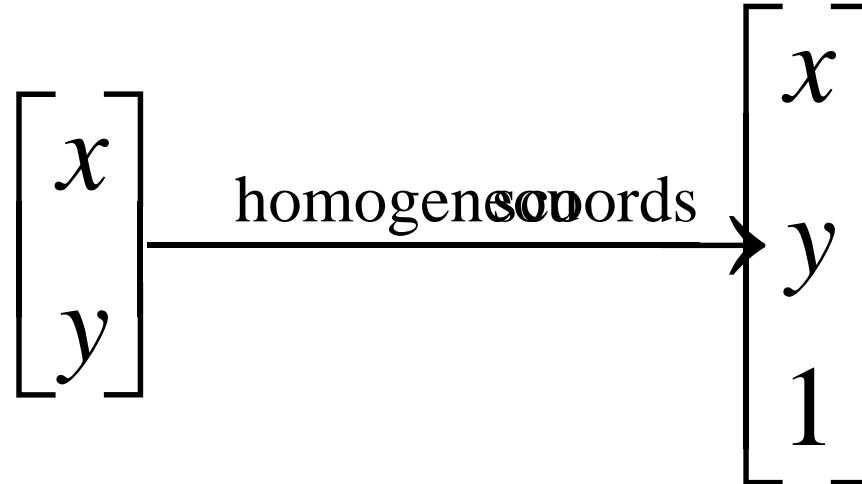
$$y' = y + t_y$$

# Homogeneous Coordinates

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## *Homogeneous coordinates*

- represent coordinates in 2 dimensions with a 3-vector

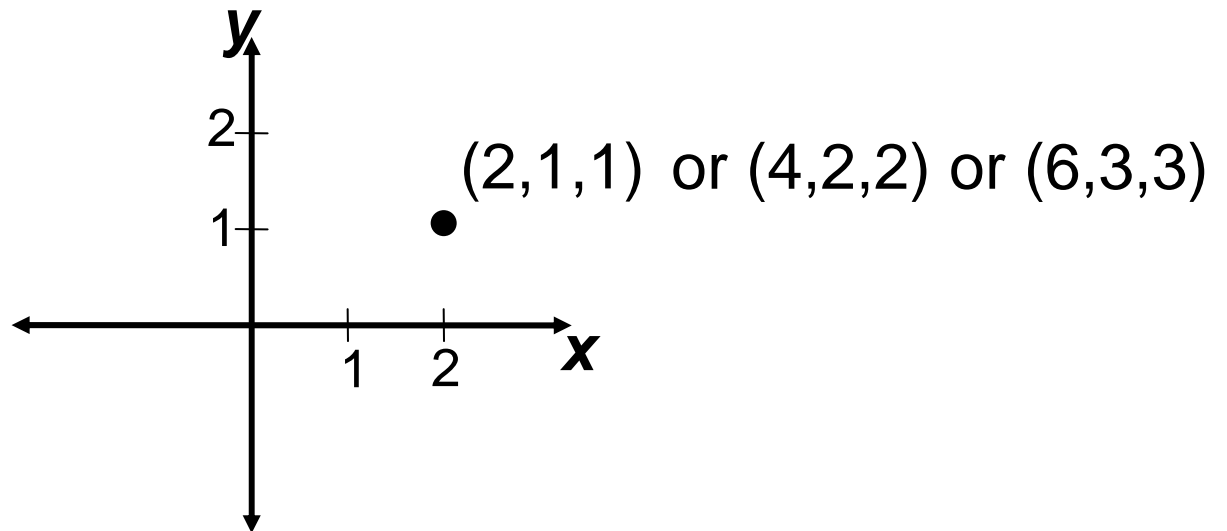


# Homogeneous Coordinates

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Add a 3rd coordinate to every 2D point

- $(x, y, w)$  represents a point at location  $(x/w, y/w)$
- $(x, y, 0)$  represents a point at infinity
- $(0, 0, 0)$  is not allowed



Convenient  
coordinate system to  
represent many  
useful  
transformations

# Homogeneous Coordinates

---

**Q: How can we represent translation as a 3x3 matrix?**

$$x' = x + t_x$$

$$y' = y + t_y$$

**A: Using the rightmost column:**

$$\mathbf{Translation} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

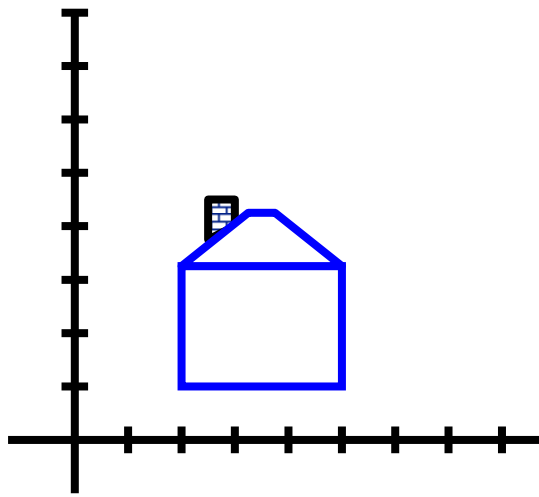
# Translation

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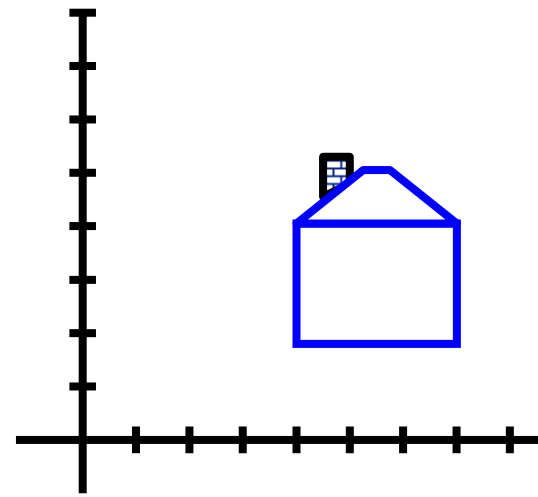
Example of translation

Homogeneous Coordinates

$$\begin{matrix} \Downarrow & & \Downarrow & & \Downarrow \\ \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+t_x \\ y+t_y \\ 1 \end{bmatrix} \end{matrix}$$



$$\begin{aligned} t_x &= 2 \\ t_y &= 1 \end{aligned}$$



# Basic 2D Transformations

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Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 \\ \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

# Affine Transformations

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Affine transformations are combinations of ...

- Linear transformations, and
- Translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of affine transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition
- Models change of basis



# Projective Transformations

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Projective transformations ...

- Affine transformations, and
- Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of projective transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines do not necessarily remain parallel
- Ratios are not preserved
- Closed under composition
- Models change of basis

# Matrix Composition

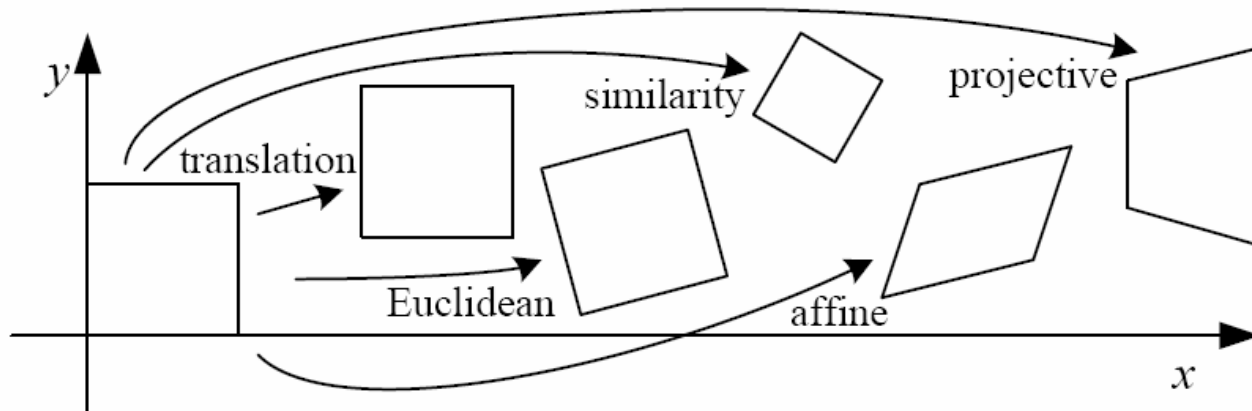
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Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left( \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 \\ \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\mathbf{p}' = \mathbf{T}(t_x, t_y) \mathbf{R}(\Theta) \mathbf{S}(s_x, s_y) \mathbf{p}$$

# 2D image transformations



| Name              | Matrix  | # D.O.F. | Preserves: | Icon |
|-------------------|---|----------|------------|------|
| translation       | $\begin{bmatrix} I &   & t \end{bmatrix}_{2 \times 3}$  |          |            |      |
| rigid (Euclidean) | $\begin{bmatrix} R &   & t \end{bmatrix}_{2 \times 3}$  |          |            |      |
| similarity        | $\begin{bmatrix} sR &   & t \end{bmatrix}_{2 \times 3}$ |          |            |      |
| affine            | $\begin{bmatrix} A \end{bmatrix}_{2 \times 3}$          |          |            |      |
| projective        | $\begin{bmatrix} \tilde{H} \end{bmatrix}_{3 \times 3}$  |          |            |      |

These transformations are a nested set of groups

- Closed under composition and inverse is a member