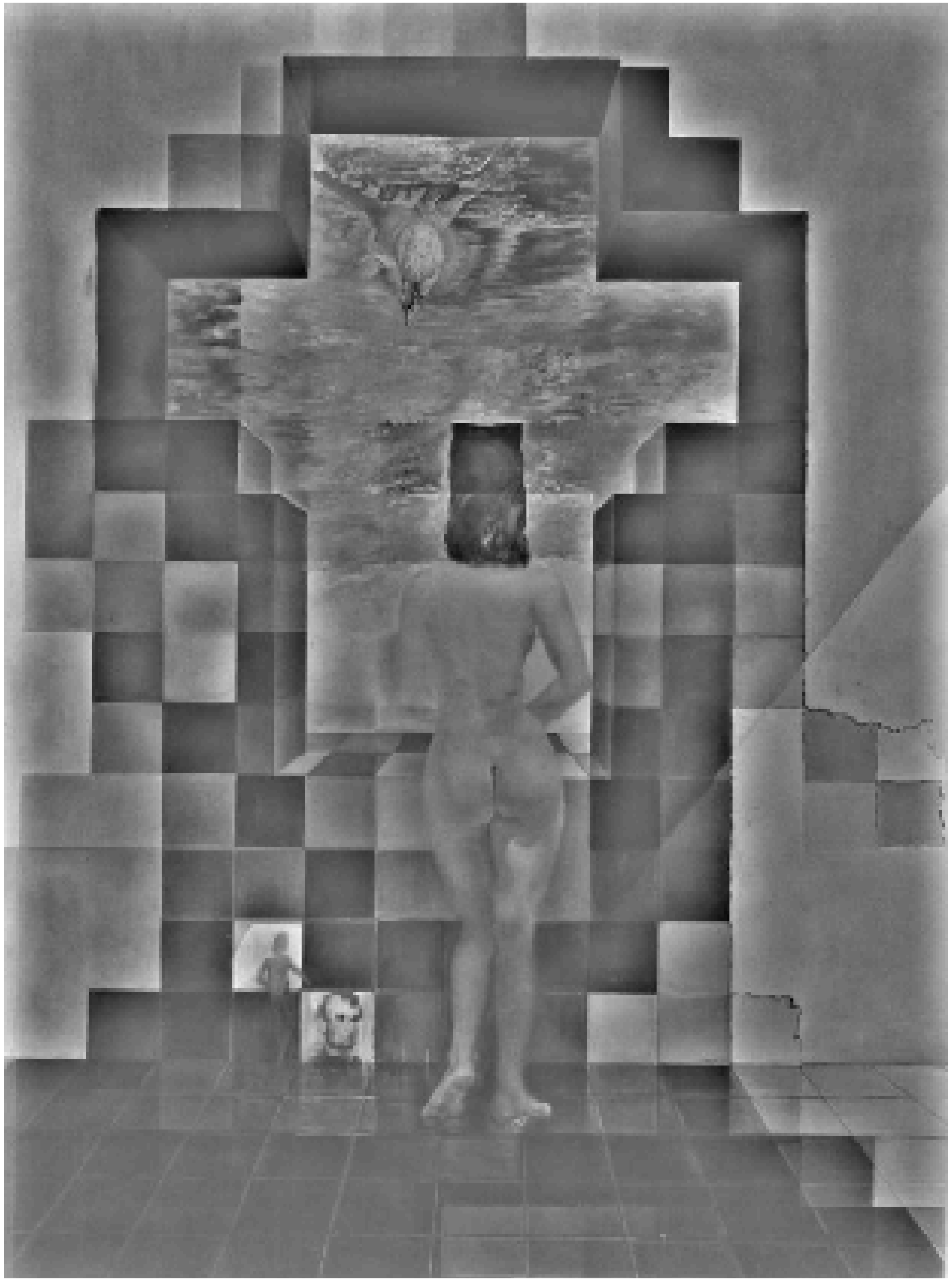
The Frequency Domain

Somewhere in Cinque Terre, May 2005

Many slides borrowed from Steve Seitz

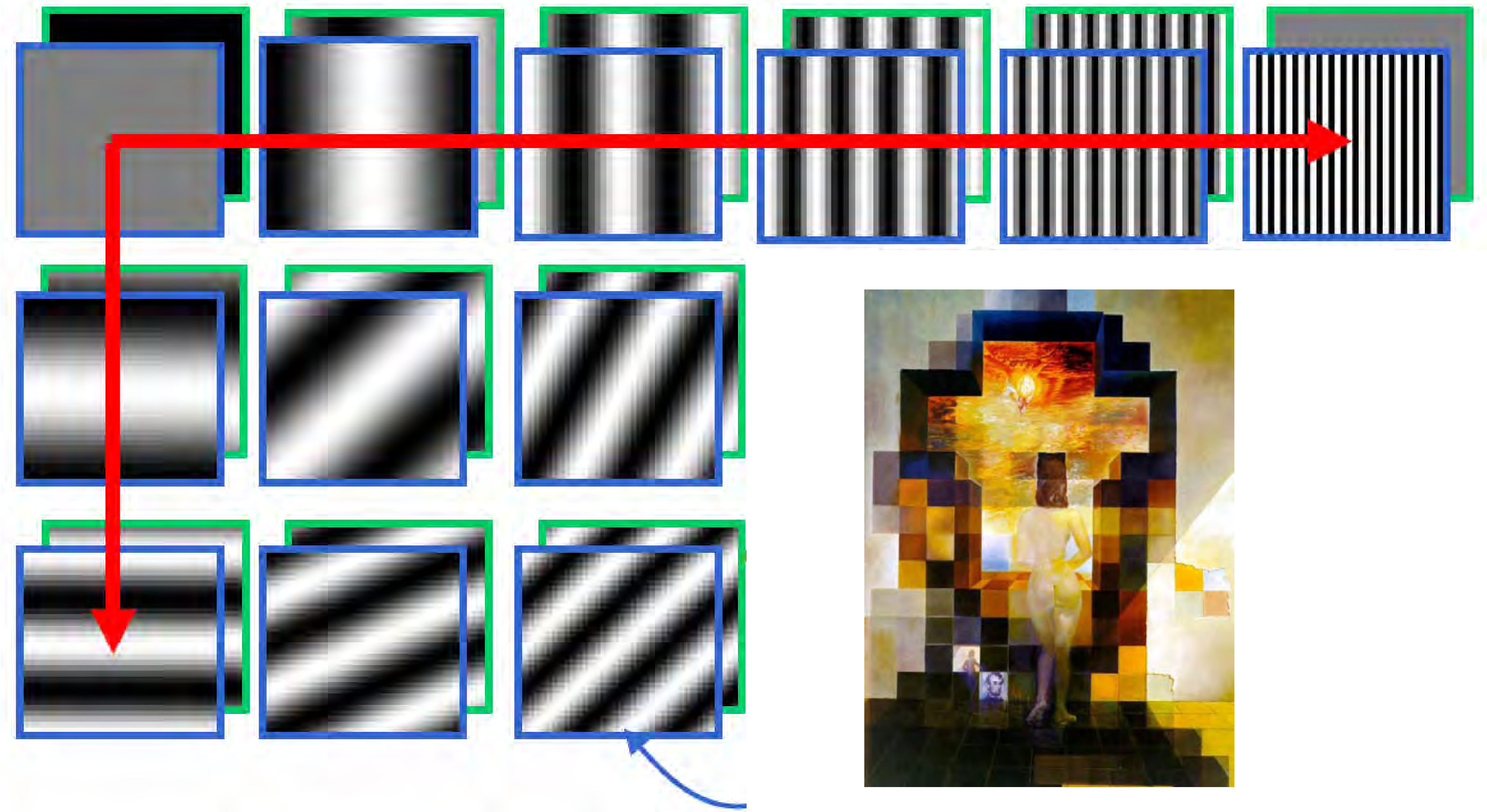
15-463: Computational Photography
Alexei Efros, CMU, Fall 2007

Salvador Dali
“Gala Contemplating the Mediterranean Sea, which at 30 meters becomes the portrait of Abraham Lincoln”, 1976



A nice set of basis

Teases away fast vs. slow changes in the image.



This change of basis has a special name…

Jean Baptiste Joseph Fourier (1768-1830)

had crazy idea (1807):

> ***Any*** *periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies.*

Don't believe it?

- Neither did Lagrange, Laplace, Poisson and other big wigs
- Not translated into English until 1878!

But it's true!

- called Fourier Series

A sum of sines

Our building block:

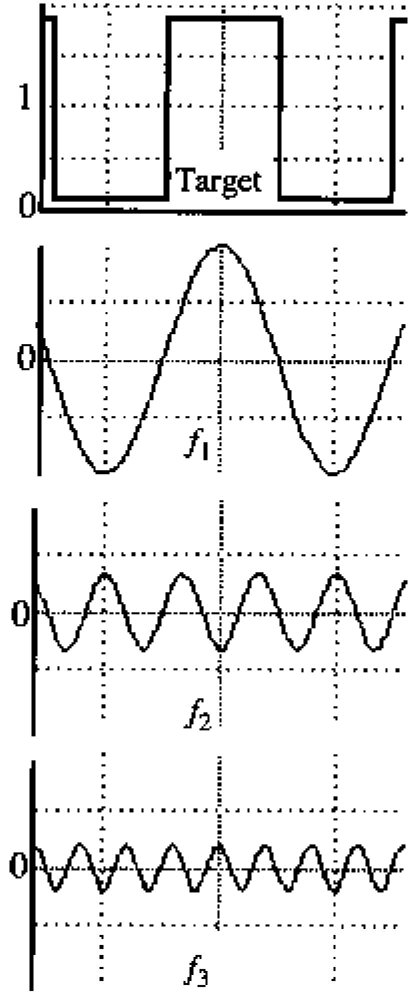
$$A\sin(\omega x + \phi)$$

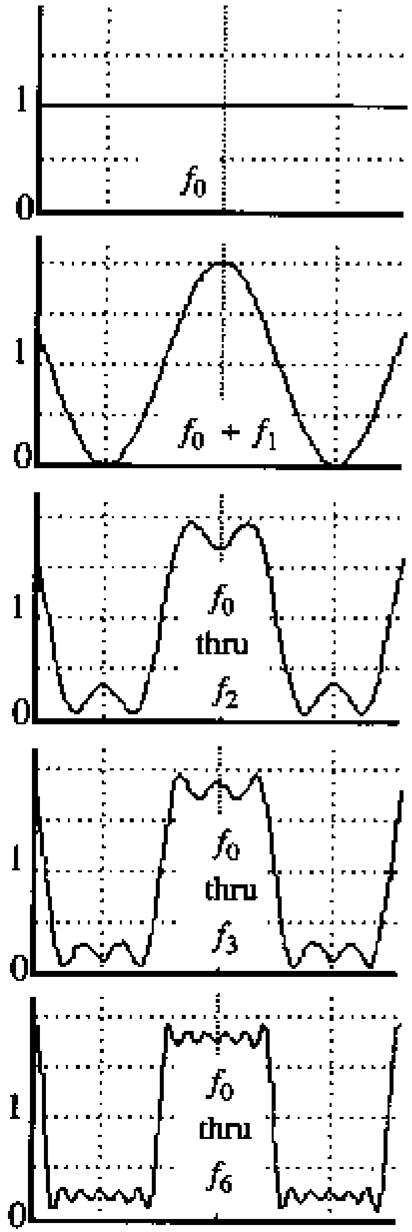
Add enough of them to get any signal *f(x)* you want!

How many degrees of freedom?

What does each control?

Which one encodes the coarse vs. fine structure of the signal?





f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

Fourier Transform

We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of *x*:

$$\boldsymbol{f(x)} \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow \boldsymbol{F(\omega)}$$

For every ω from 0 to inf, ***F(ω)*** holds the amplitude *A* and phase ϕ of the corresponding sine $A\sin(\omega x + \phi)$

- How can *F* hold both? Complex number trick!

$$F(\omega) = R(\omega) + iI(\omega)$$

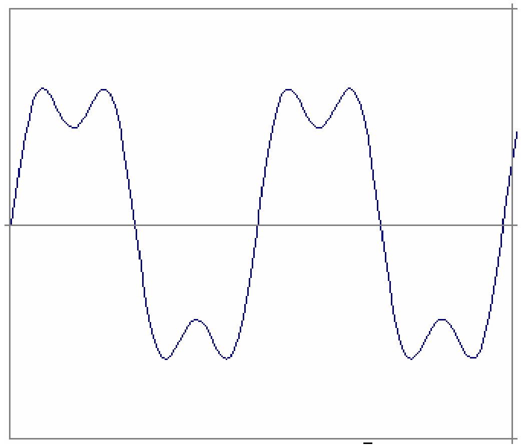
$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad\qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

We can always go back:

$$\boldsymbol{F(\omega)} \longrightarrow \boxed{\text{Inverse Fourier Transform}} \longrightarrow \boldsymbol{f(x)}$$

Time and Frequency

example : $g(t) = \sin(2pf\,t) + (1/3)\sin(2p(3f)\,t)$



Time and Frequency

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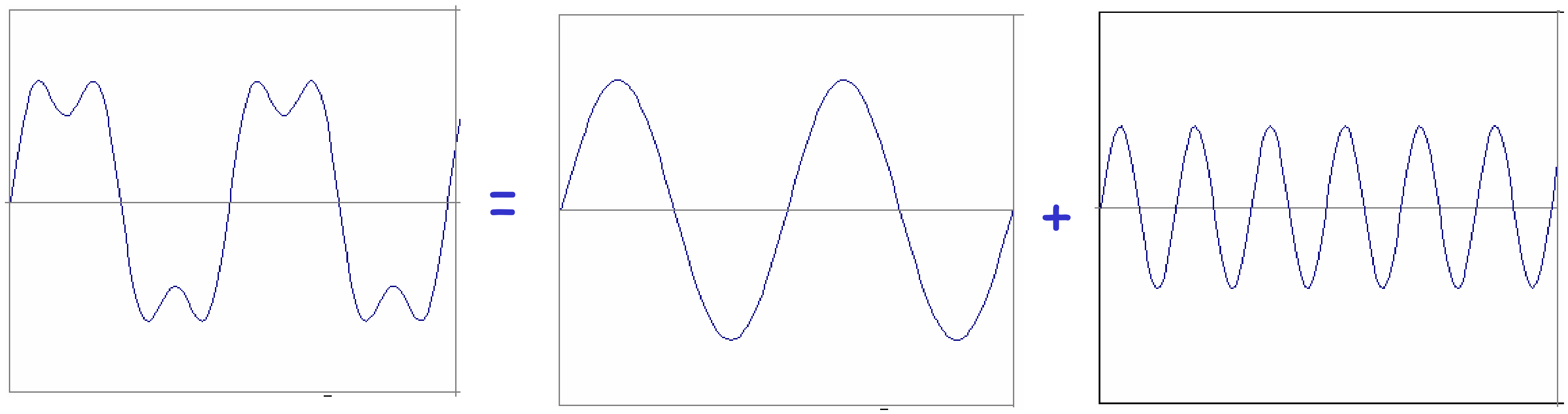
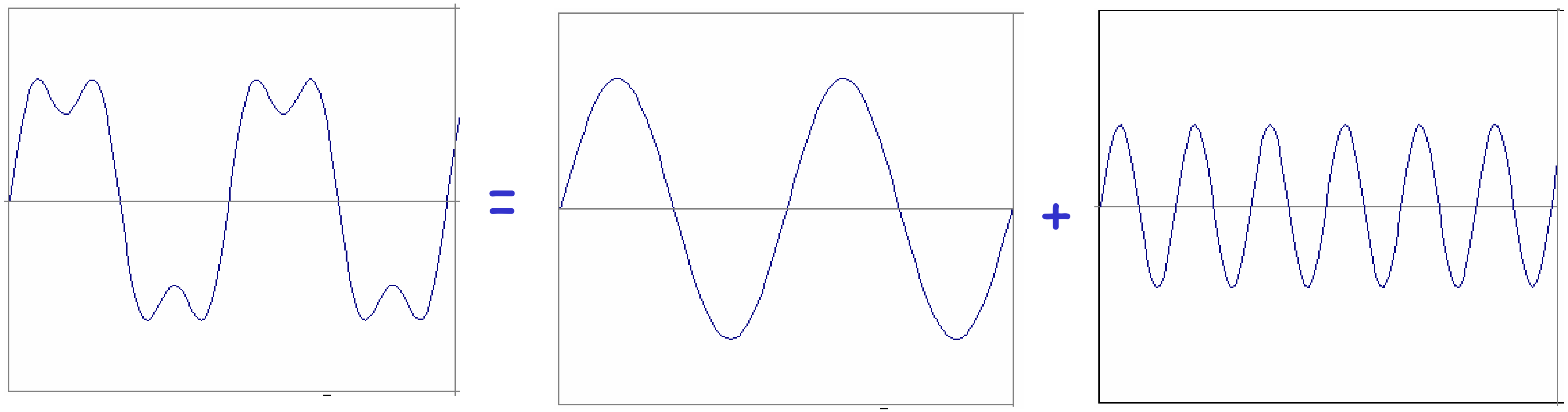
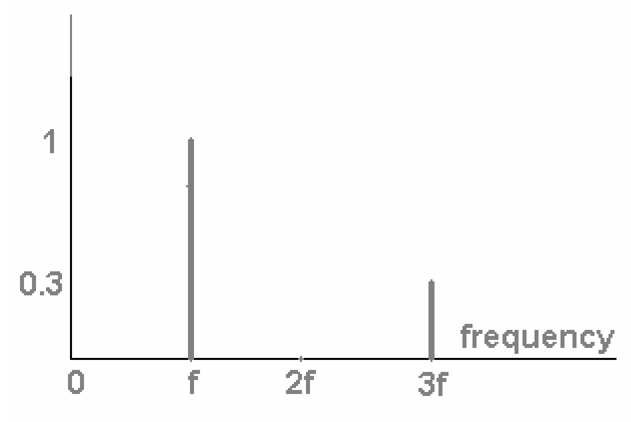


Frequency Spectra

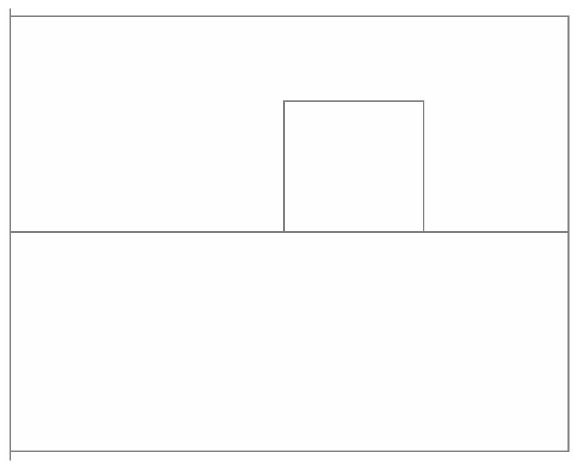
example : $g(t) = \sin(2pf\ t) + (1/3)\sin(2p(3f)\ t)$



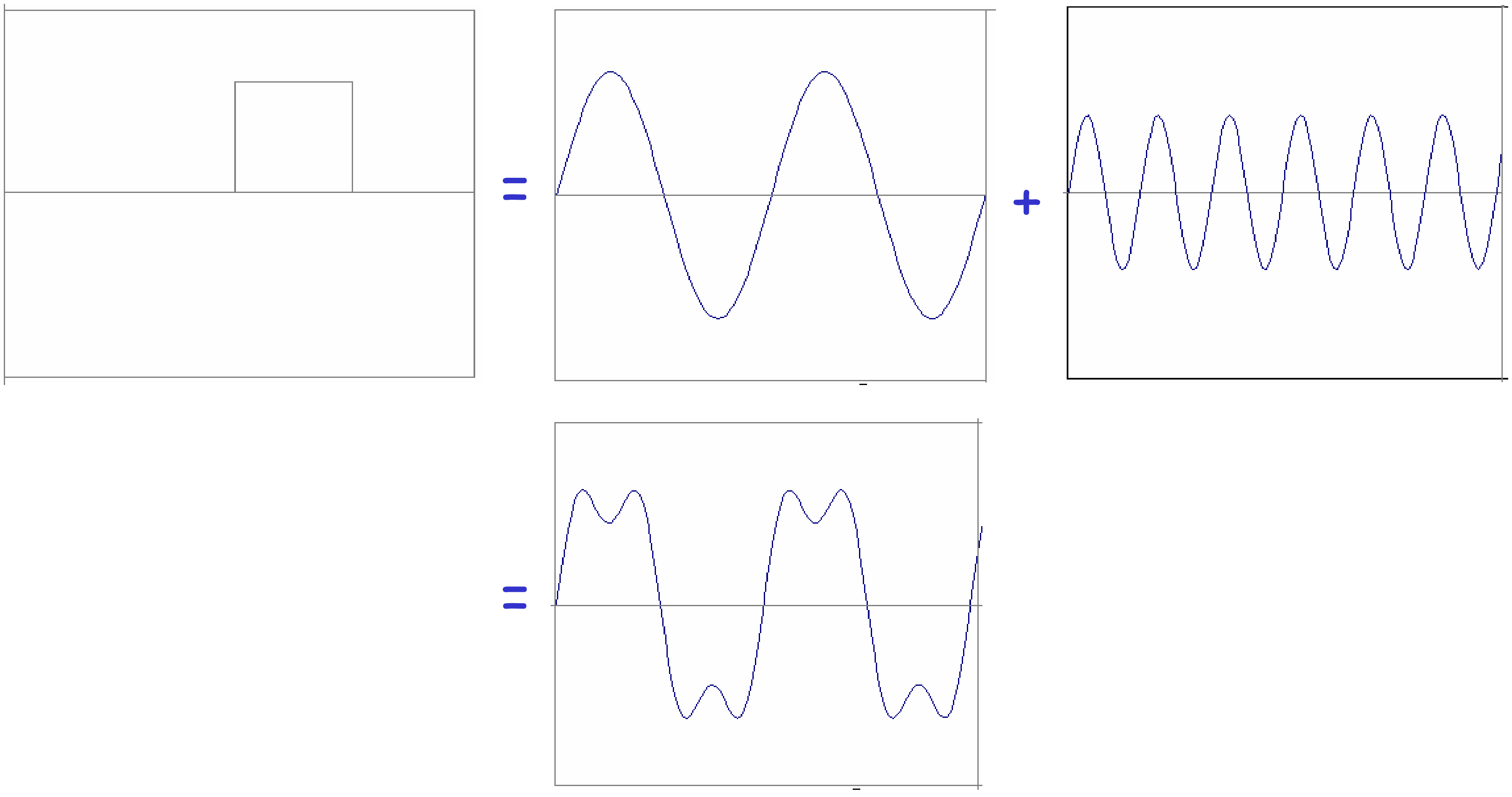


Frequency Spectra

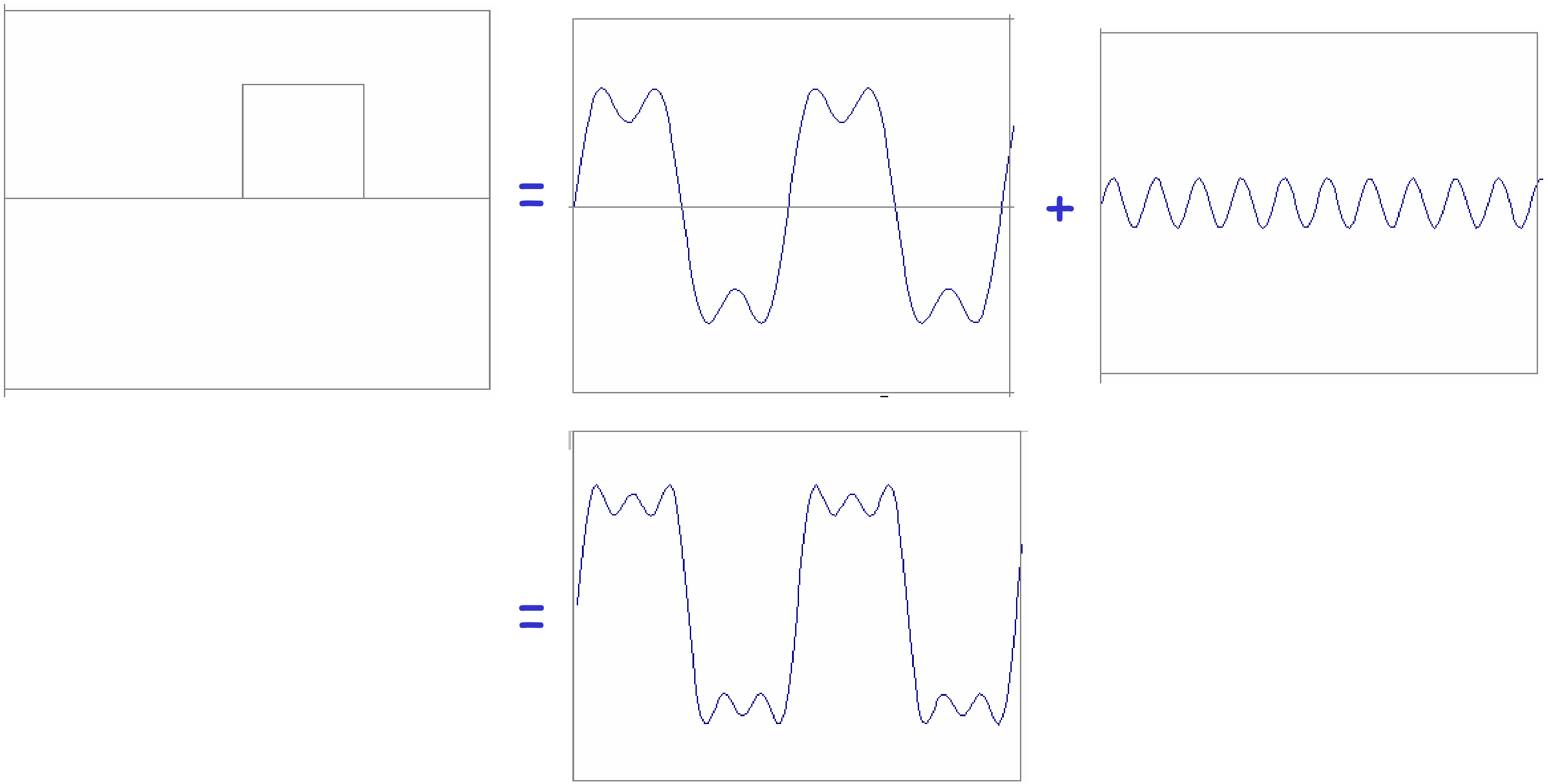
Usually, frequency is more interesting than the phase



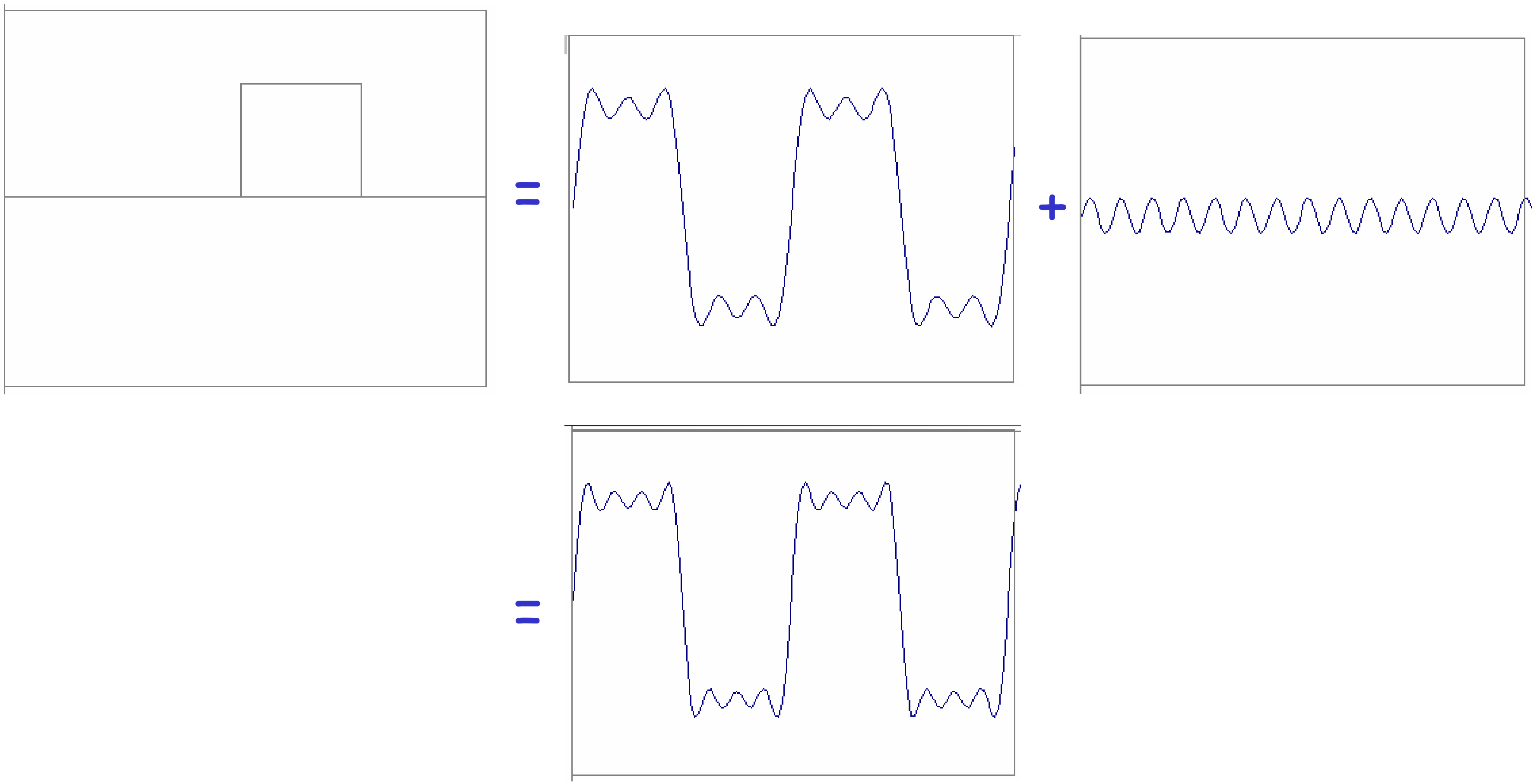
Frequency Spectra



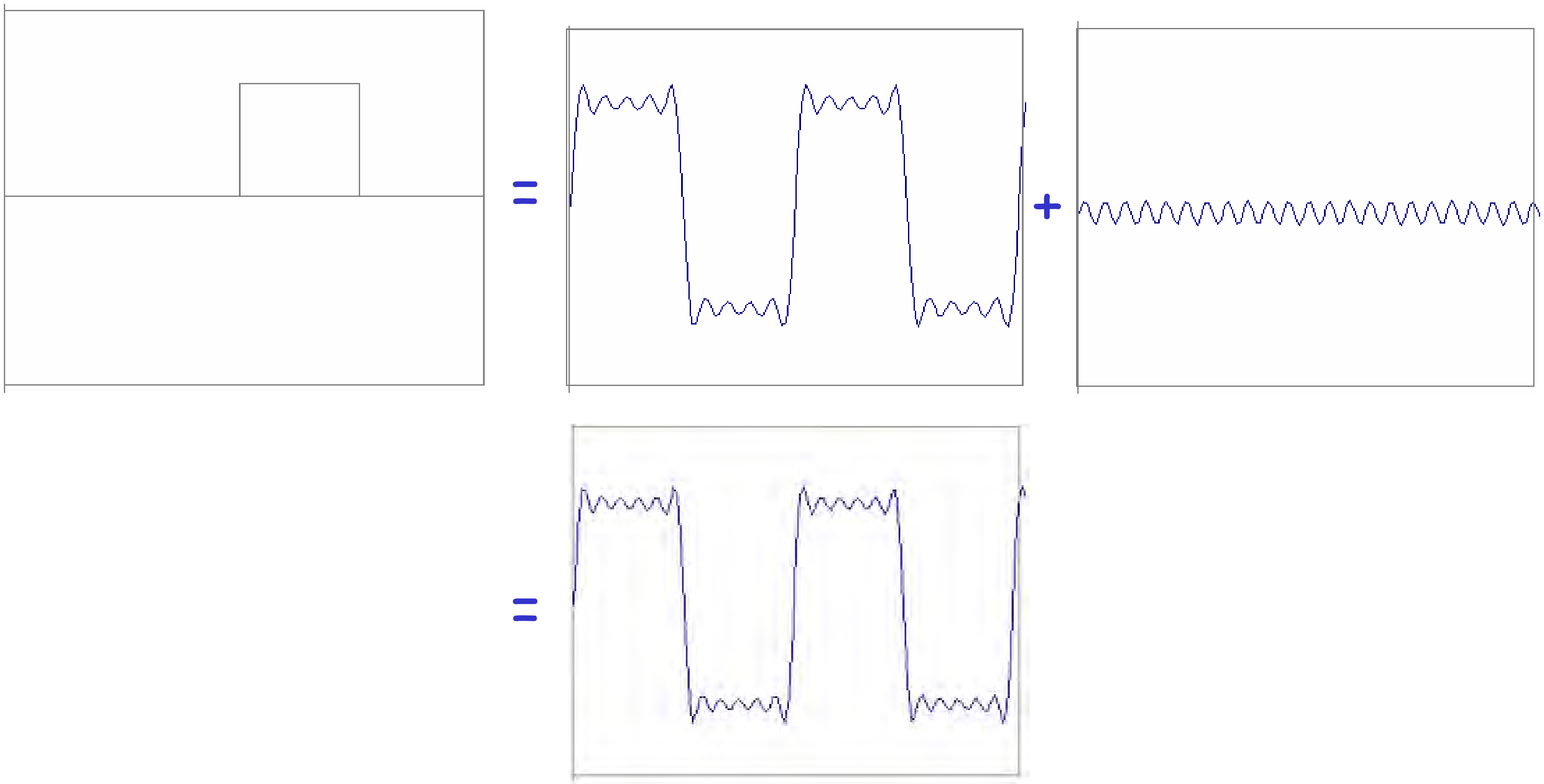
Frequency Spectra



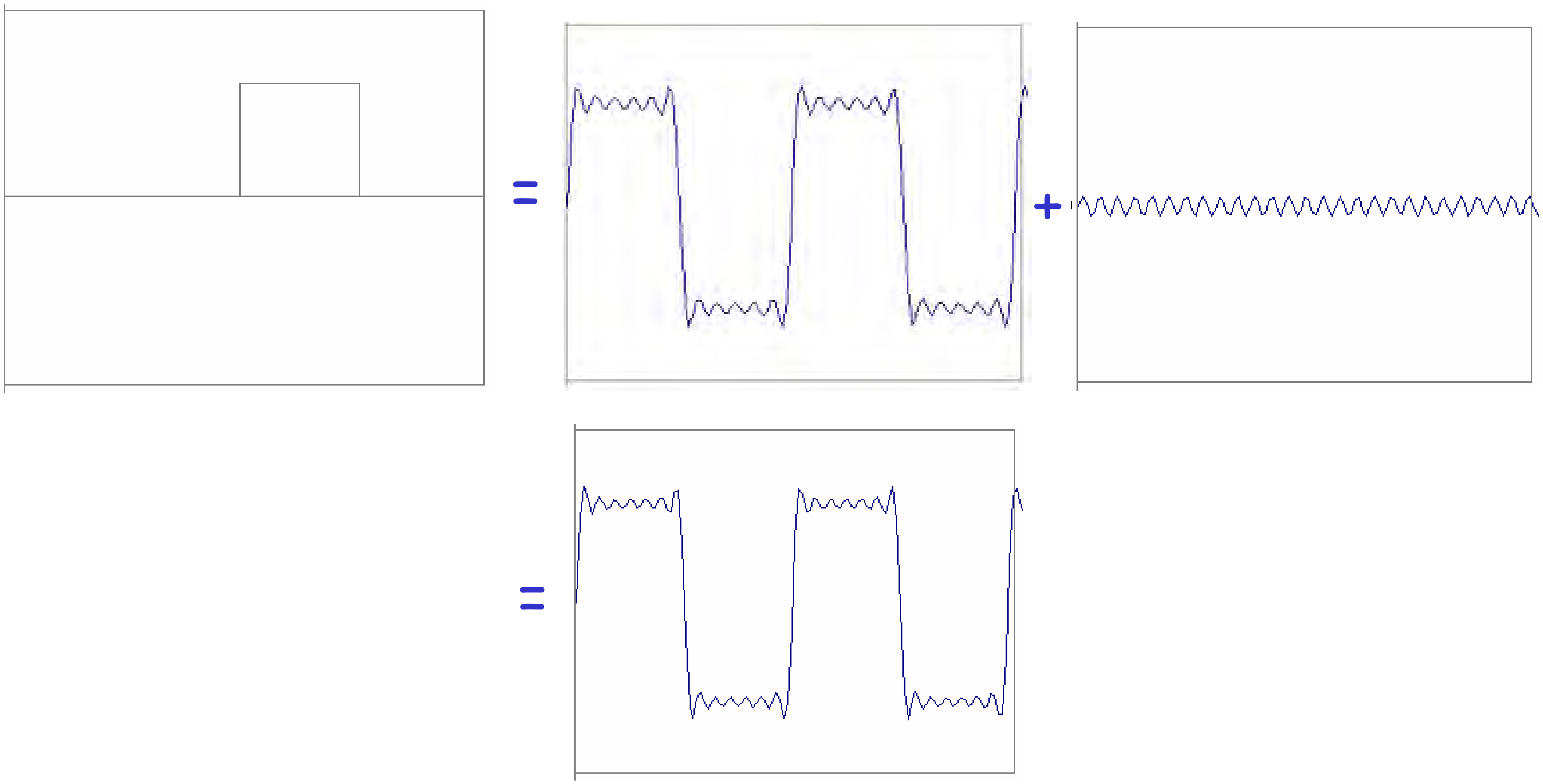
Frequency Spectra



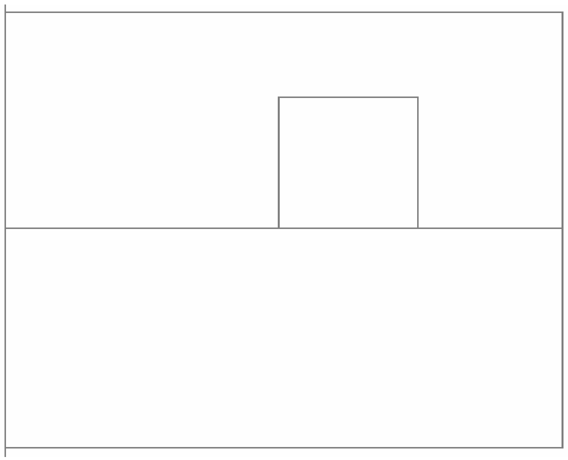
Frequency Spectra



Frequency Spectra

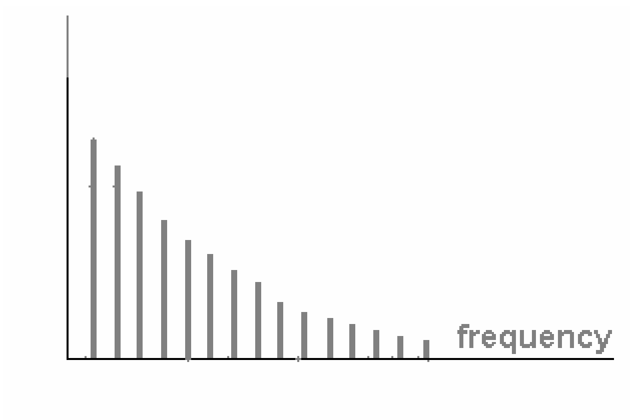


Frequency Spectra

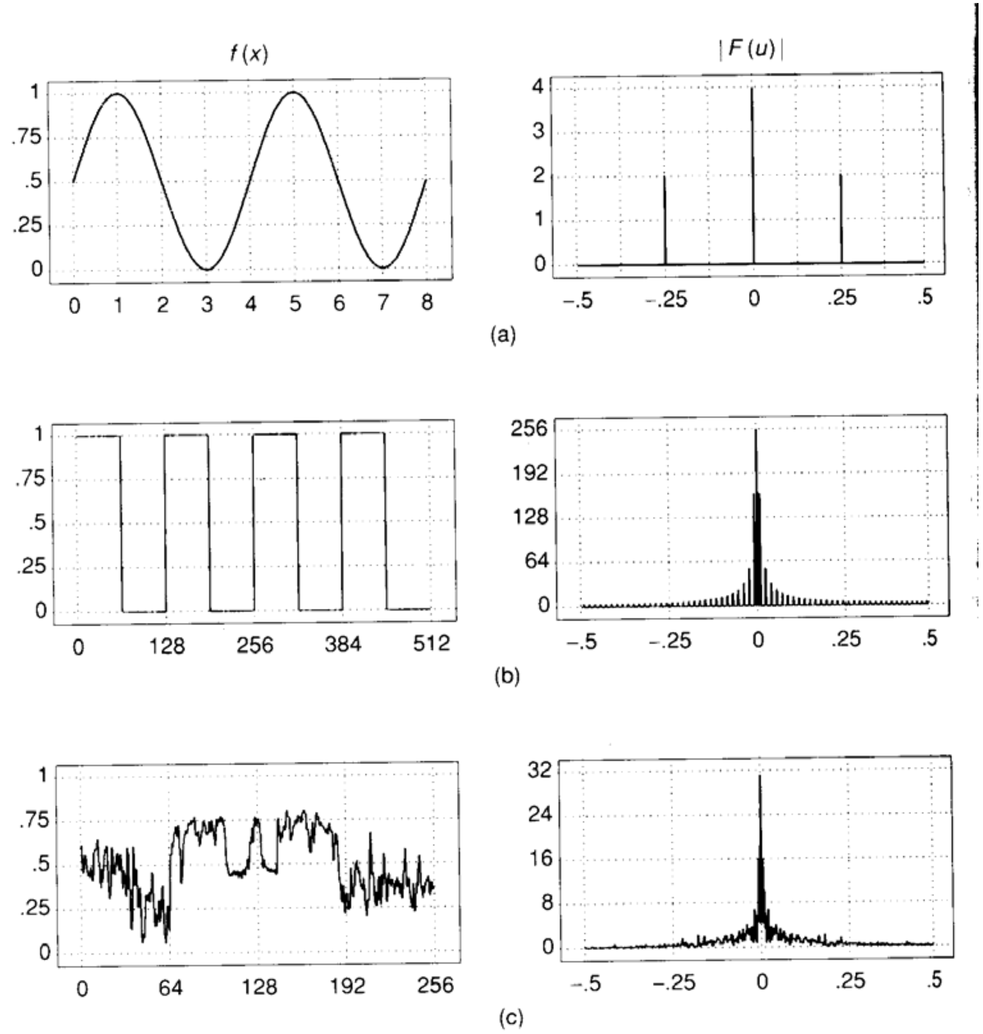


=

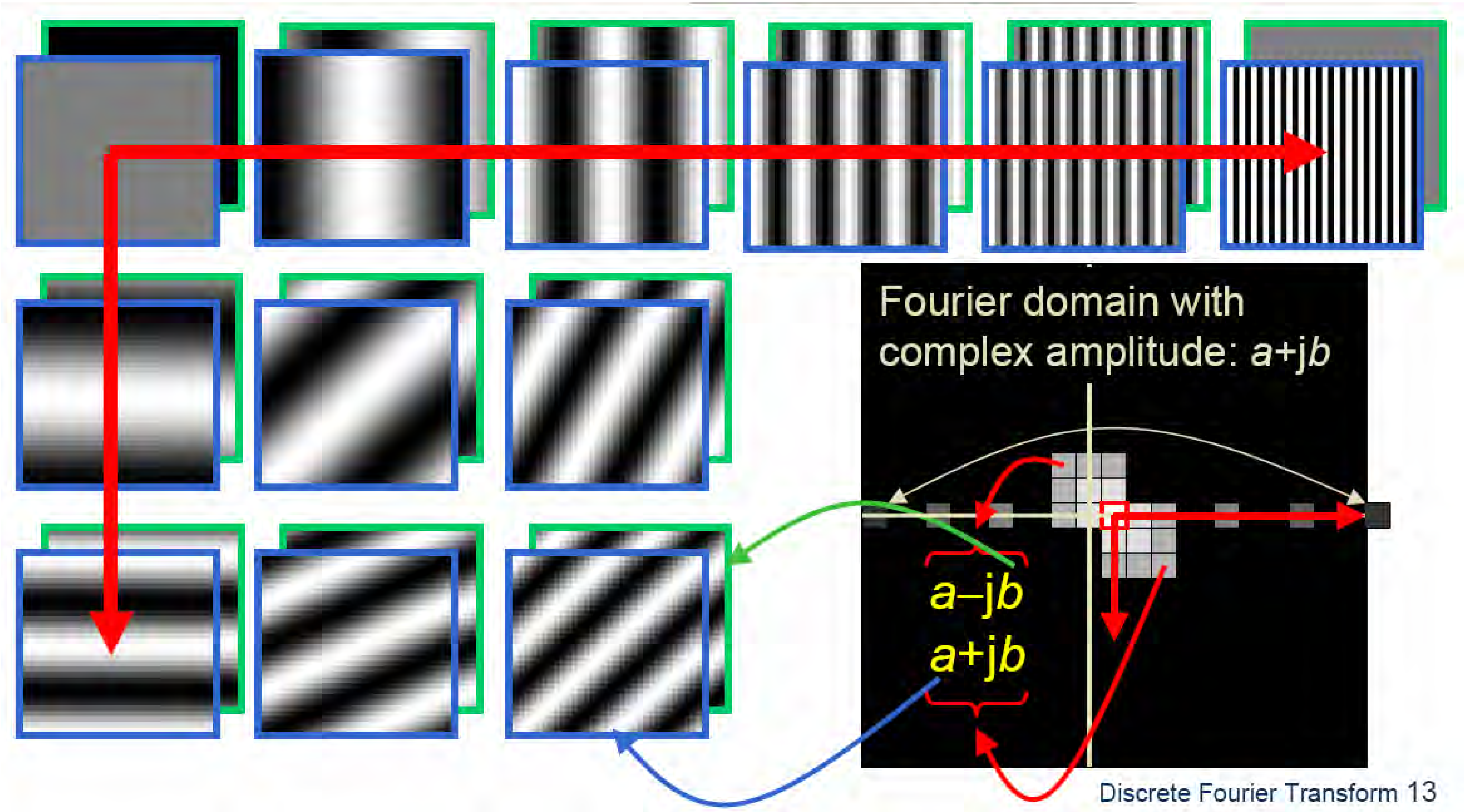
$$A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)$$



Frequency Spectra

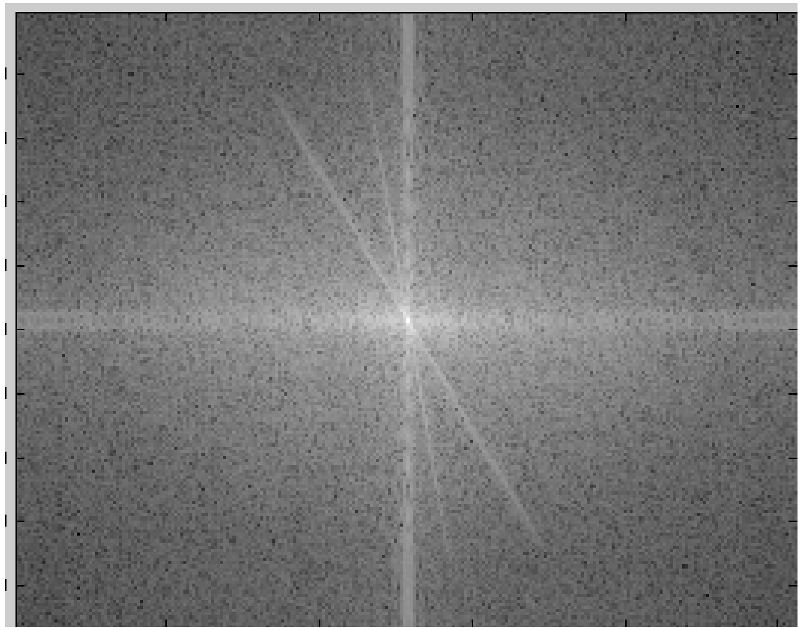


Extension to 2D



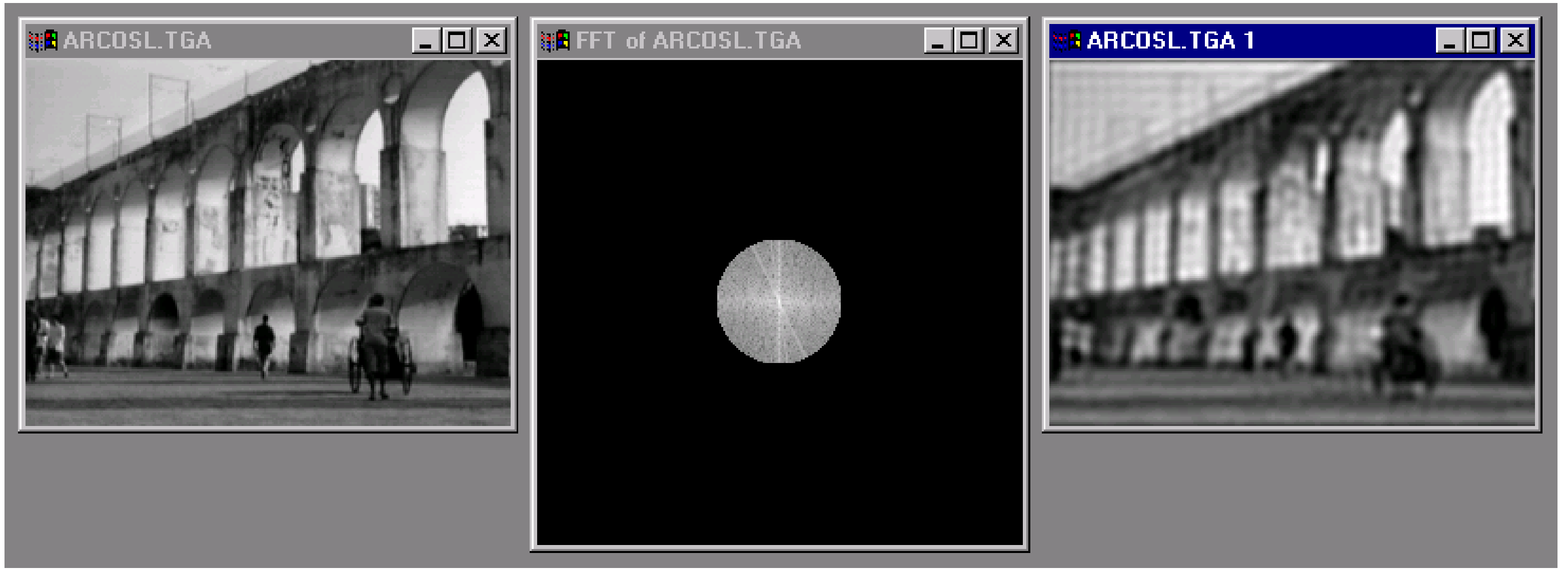
in Matlab, check out: imagesc(log(abs(fftshift(fft2(im)))));

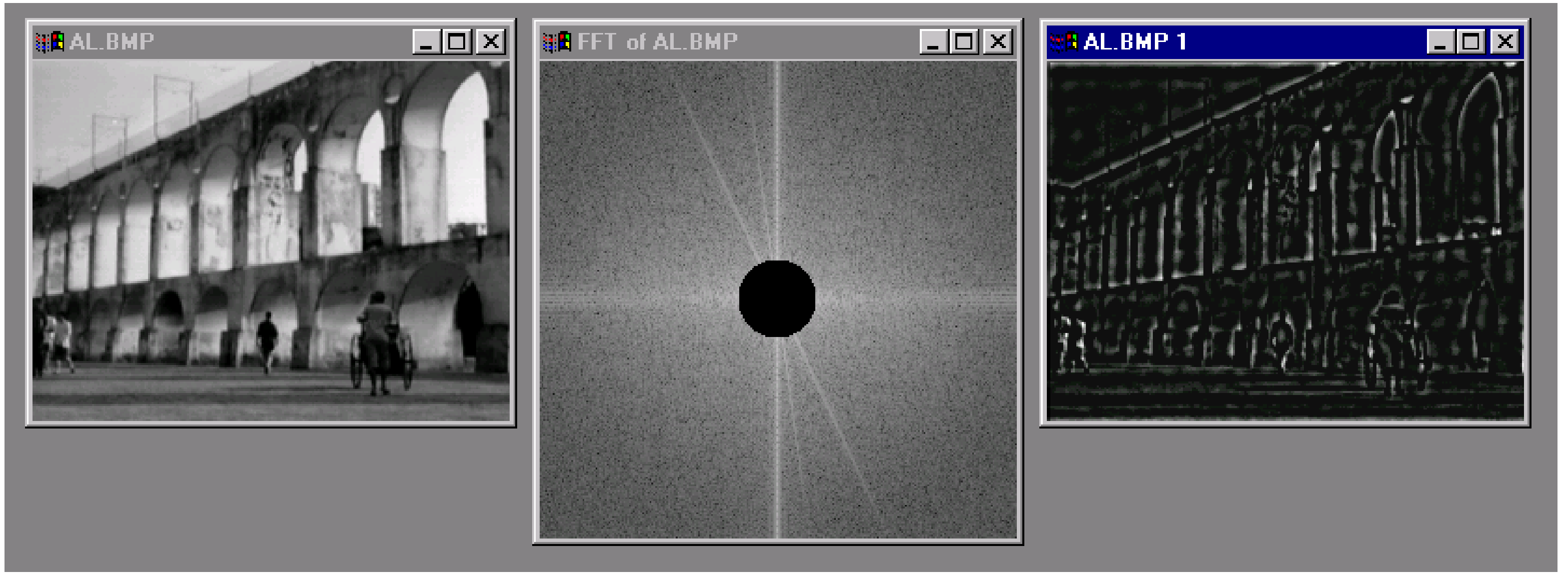
Man-made Scene



Can change spectrum, then reconstruct

Low and High Pass filtering





The Convolution Theorem

The greatest thing since sliced (banana) bread!

- The Fourier transform of the convolution of two functions is the product of their Fourier transforms

$$\mathrm{F}[g * h] = \mathrm{F}[g]\mathrm{F}[h]$$

- The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms

$$\mathrm{F}^{-1}[gh] = \mathrm{F}^{-1}[g] * \mathrm{F}^{-1}[h]$$

- **Convolution** in spatial domain is equivalent to **multiplication** in frequency domain!

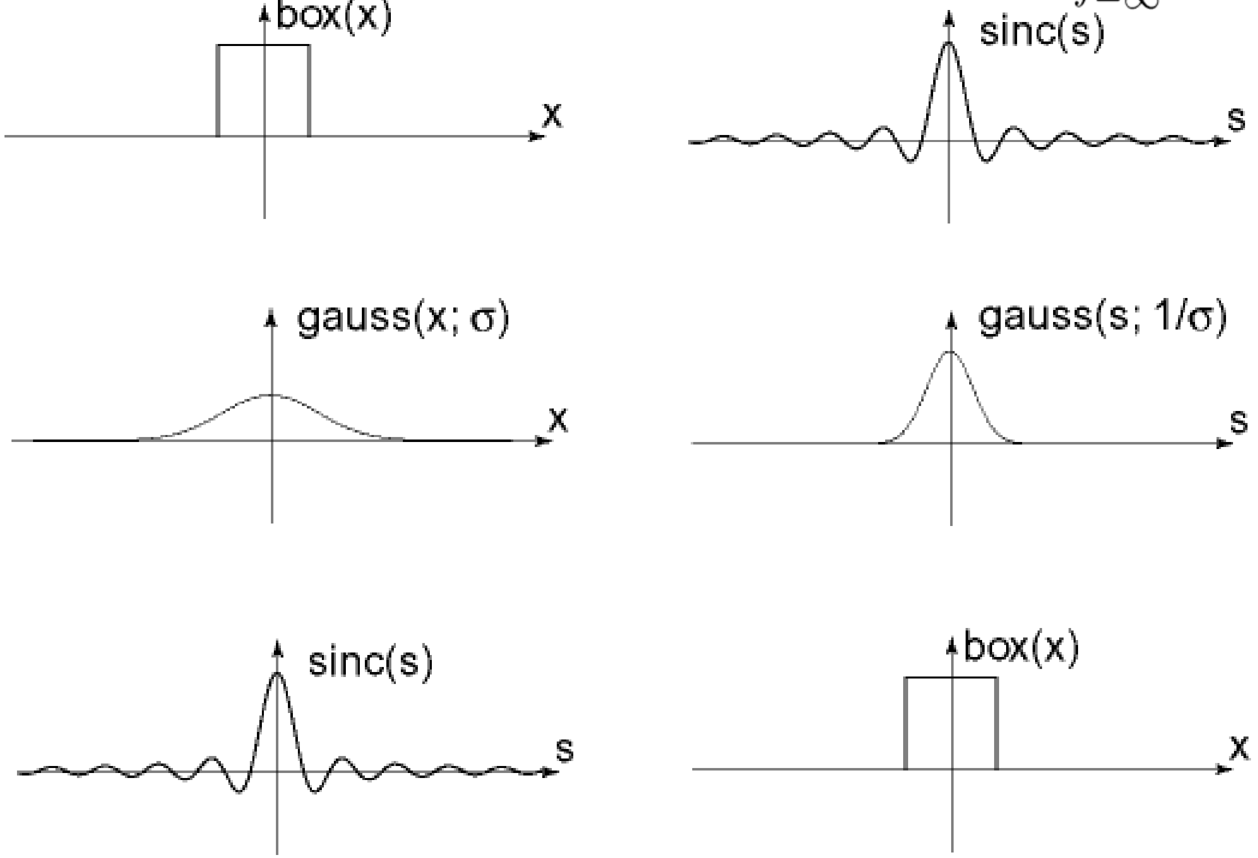
Fourier Transform pairs

Spatial domain

$$f(x)$$

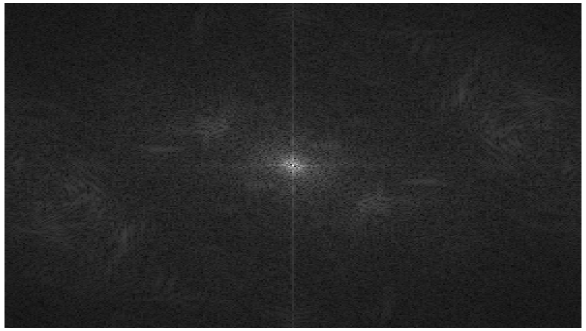
Frequency domain

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi sx} dx$$



2D convolution theorem example

$f(x,y)$

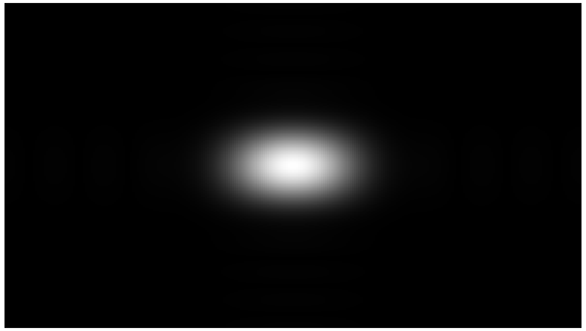


$|F(s_x,s_y)|$

*

×

$h(x,y)$

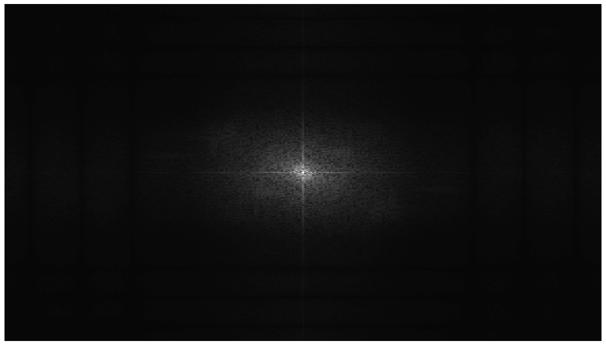


$|H(s_x,s_y)|$

$\Downarrow$

$\Downarrow$

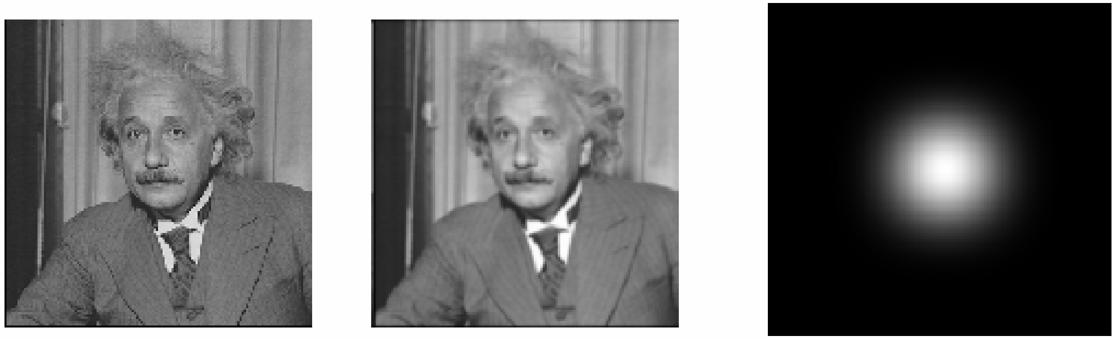
$g(x,y)$



$|G(s_x,s_y)|$

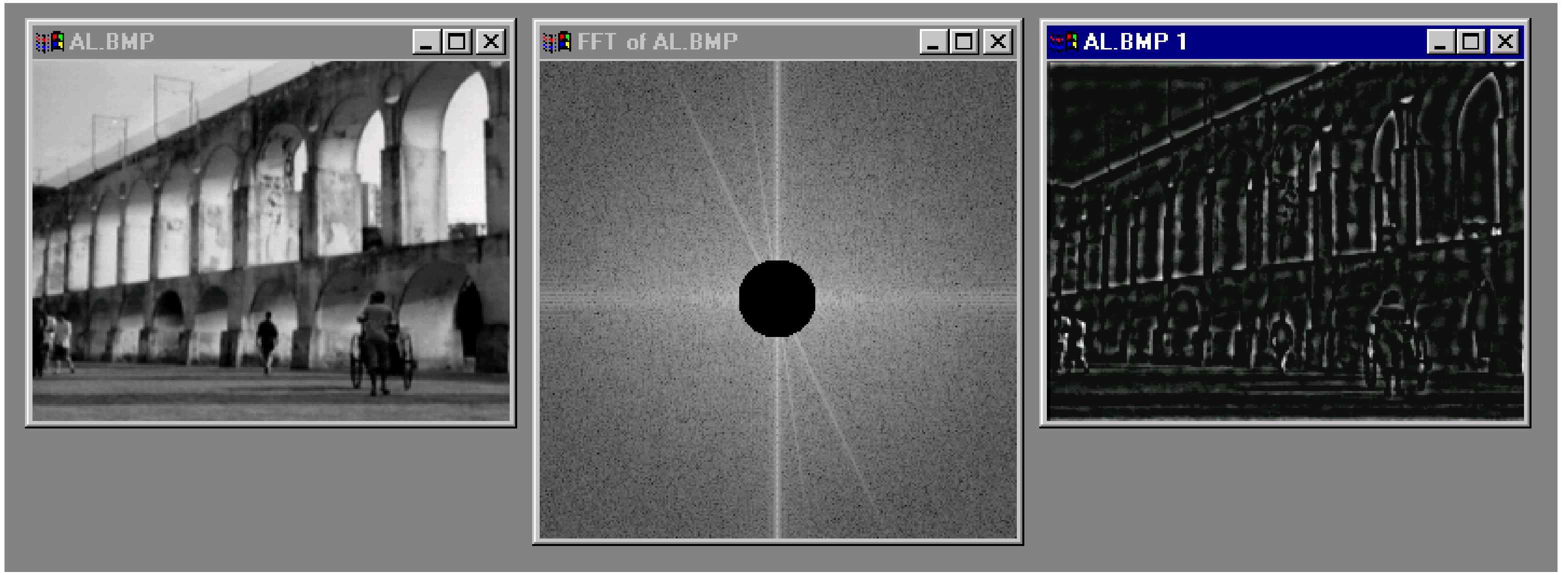
Low-pass, Band-pass, High-pass filters

low-pass:



High-pass / band-pass:

Edges in images



What does blurring take away?

original

What does blurring take away?

smoothed (5x5 Gaussian)

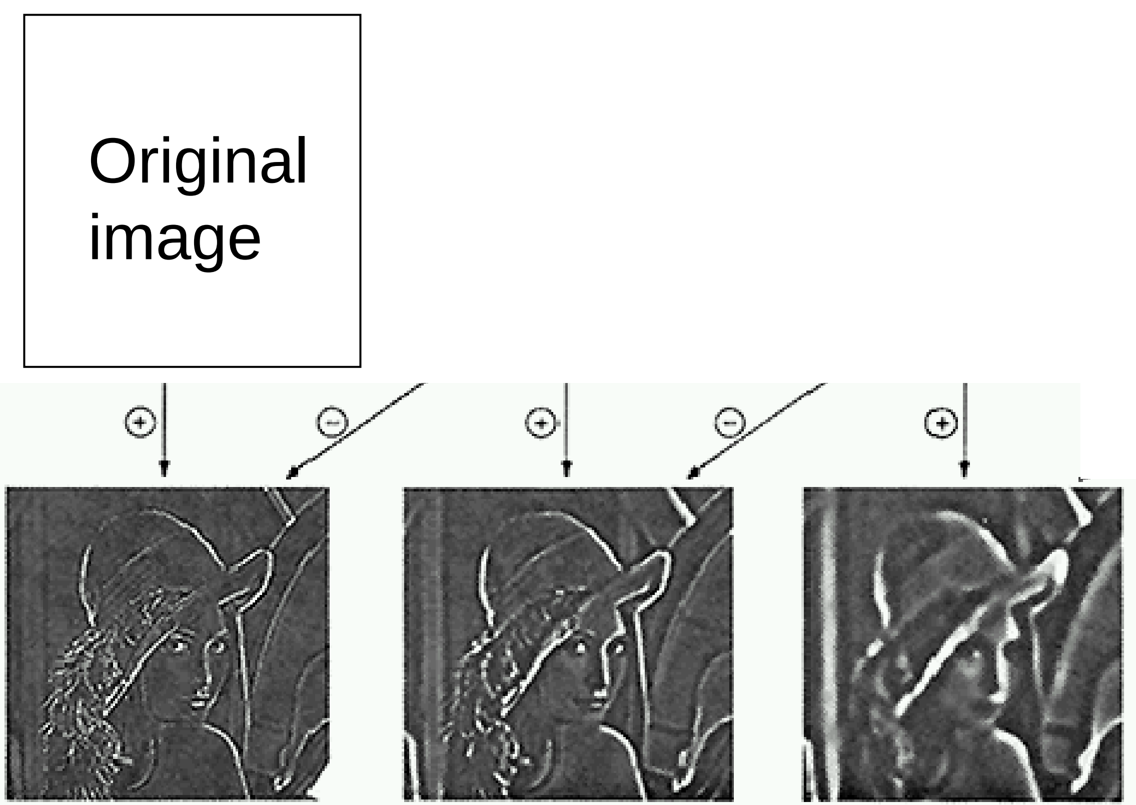
High-Pass filter

smoothed – original

Band-pass filtering

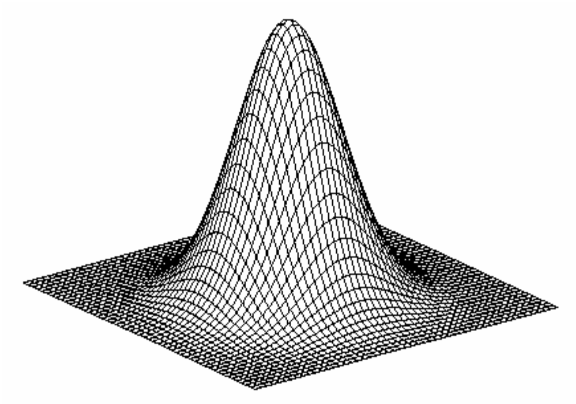
Gaussian Pyramid (low-pass images)

Laplacian Pyramid



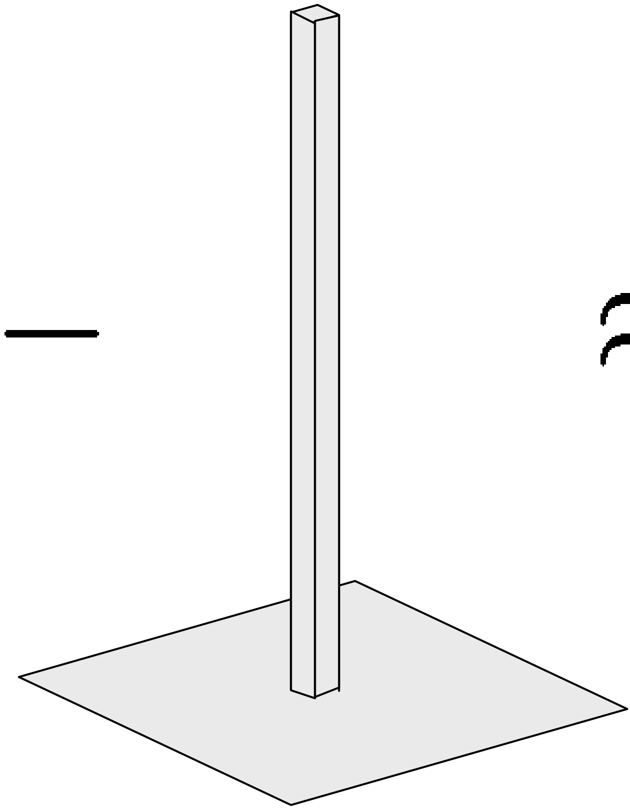
How can we reconstruct (collapse) this pyramid into the original image?

Why Laplacian?



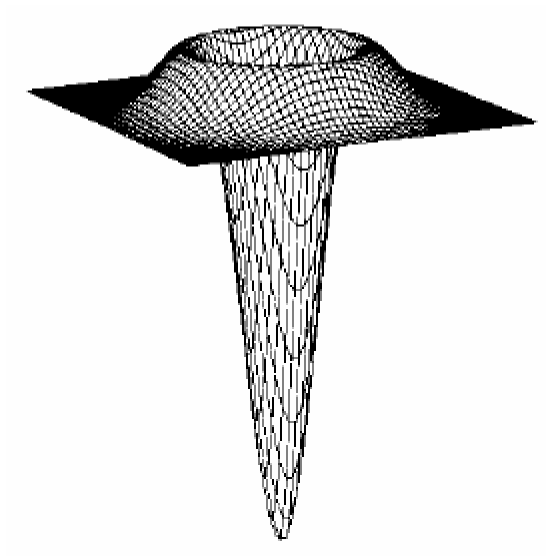
Gaussian

$-$



delta function

$\approx$



Laplacian of Gaussian

Unsharp Masking

-

=

+ α

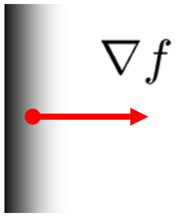
=

Image gradient

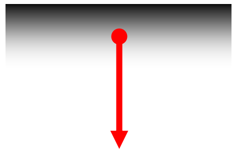
The gradient of an image:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

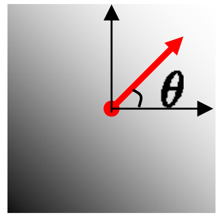
The gradient points in the direction of most rapid change in intensity



$$\nabla f = \left[\frac{\partial f}{\partial x}, 0\right]$$



$$\nabla f = \left[0, \frac{\partial f}{\partial y}\right]$$



$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

The gradient direction is given by:

$$\theta = \tan^{-1}\left(\frac{\partial f}{\partial y} / \frac{\partial f}{\partial x}\right)$$

- how does this relate to the direction of the edge?

The *edge strength* is given by the gradient magnitude

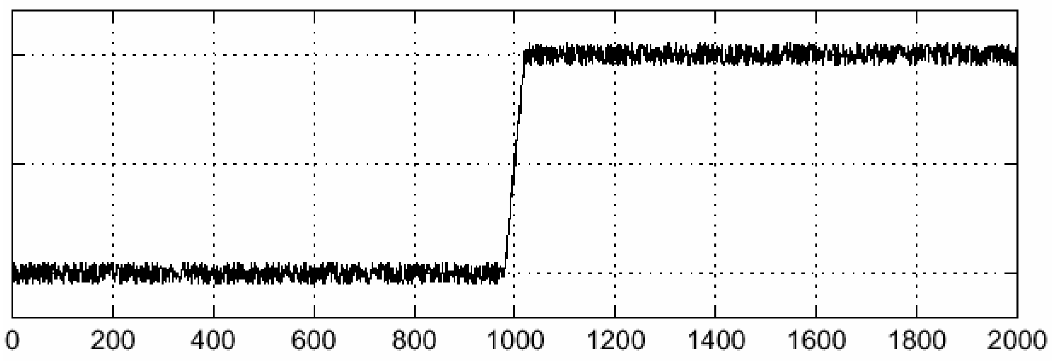
$$||\nabla f|| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

Effects of noise

Consider a single row or column of the image

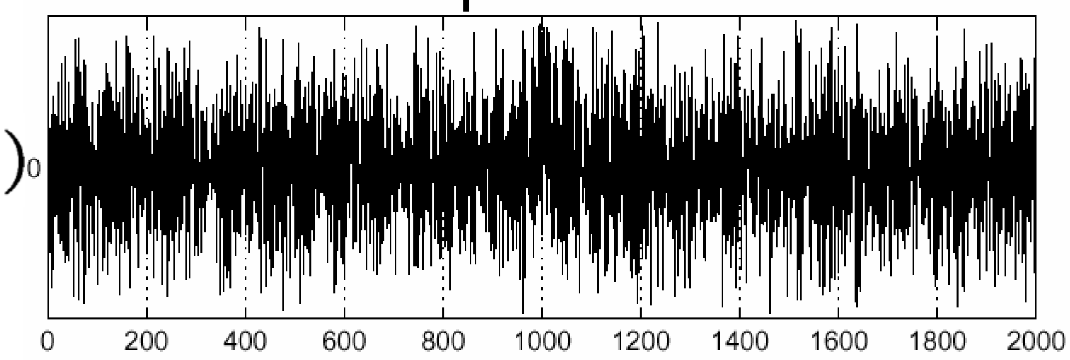
- Plotting intensity as a function of position gives a signal

$f(x)$



How to compute a derivative?

$\frac{d}{dx}f(x)$



Where is the edge?

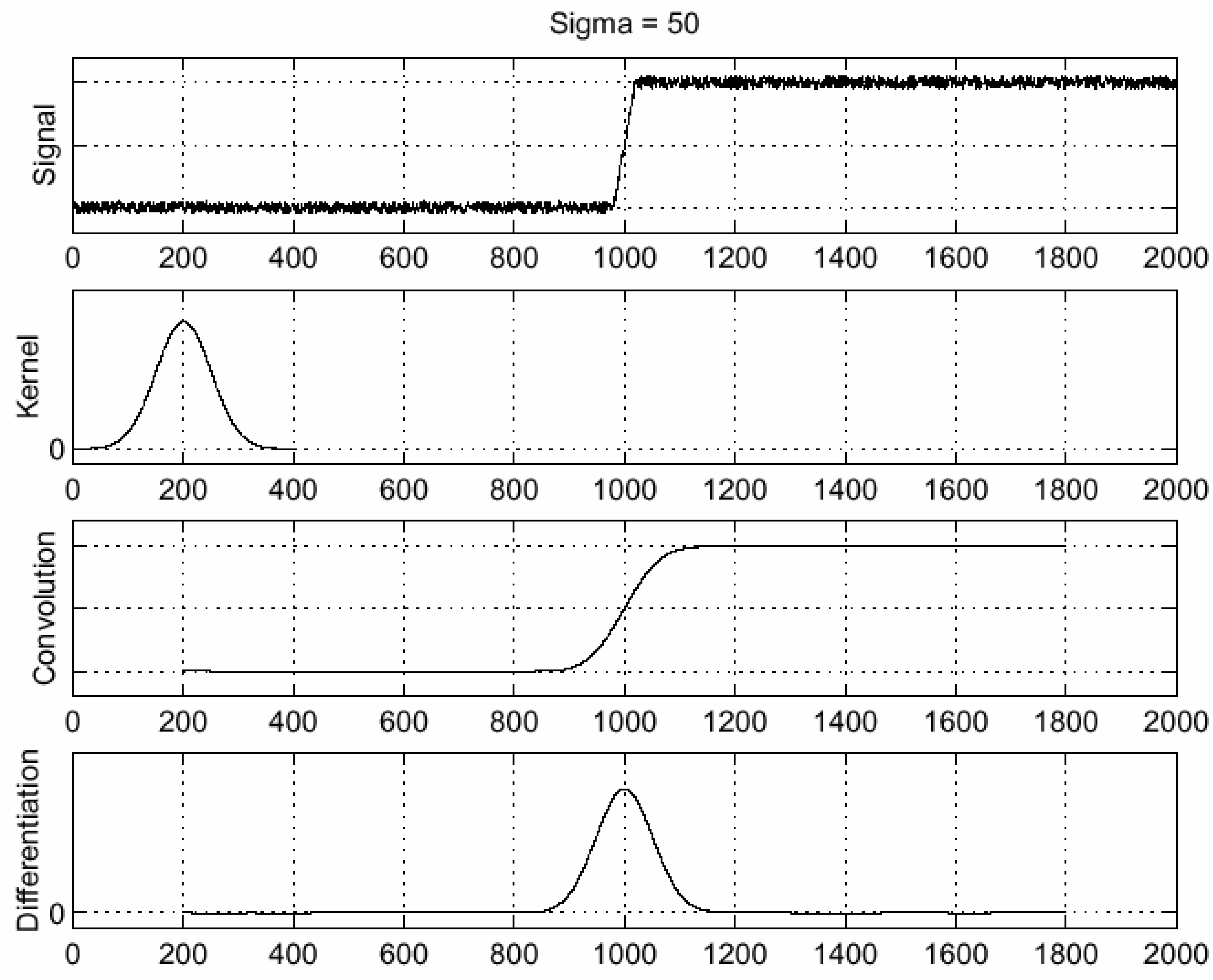
Solution: smooth first

f

h

$h \star f$

$\frac{\partial}{\partial x}(h \star f)$



Where is the edge?   Look for peaks in $\frac{\partial}{\partial x}(h \star f)$

Derivative theorem of convolution

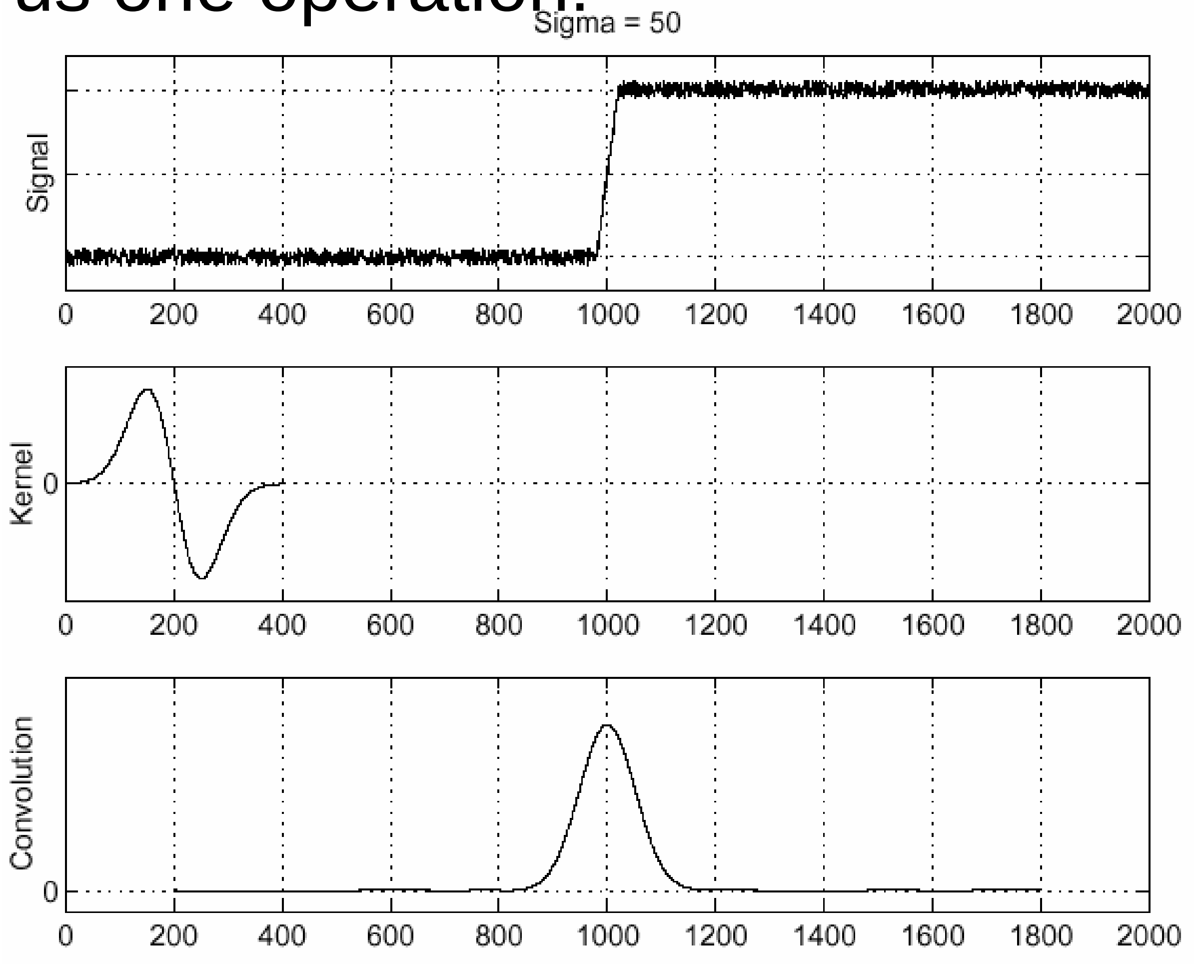
$$\frac{\partial}{\partial x}(h \star f) = (\frac{\partial}{\partial x}h) \star f$$

This saves us one operation:

f

$\frac{\partial}{\partial x}h$

$(\frac{\partial}{\partial x}h) \star f$



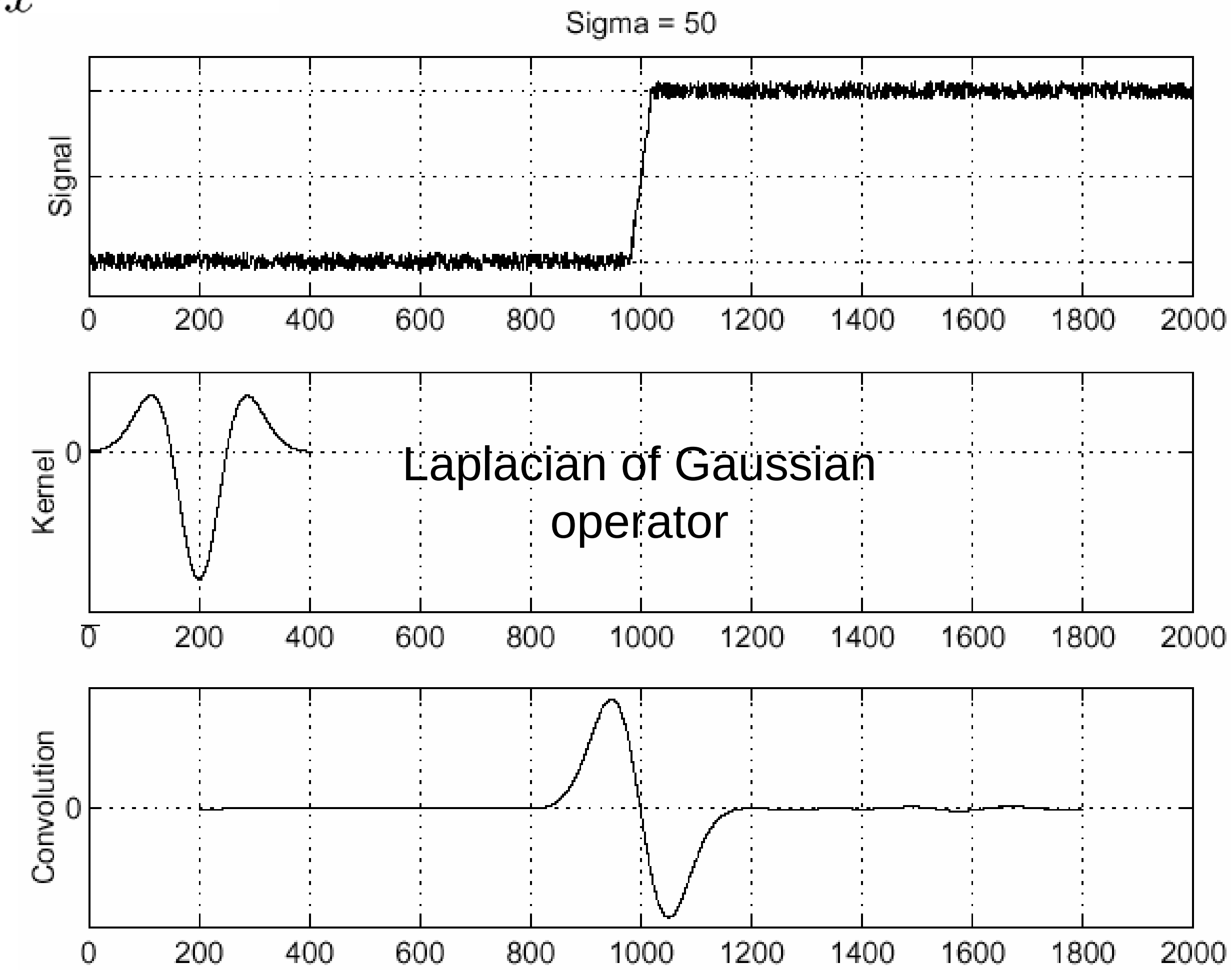
Laplacian of Gaussian

Consider $\frac{\partial^2}{\partial x^2}(h \star f)$

f

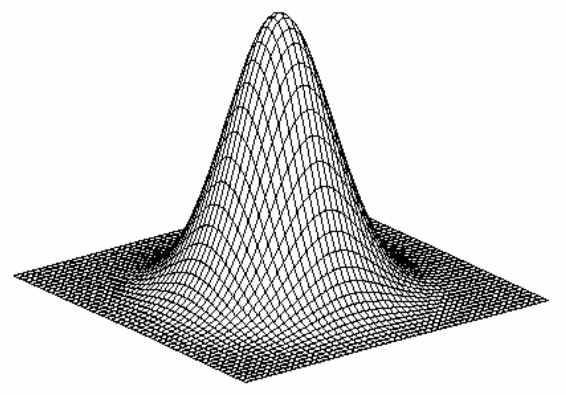
$\frac{\partial^2}{\partial x^2}h$

$(\frac{\partial^2}{\partial x^2}h) \star f$



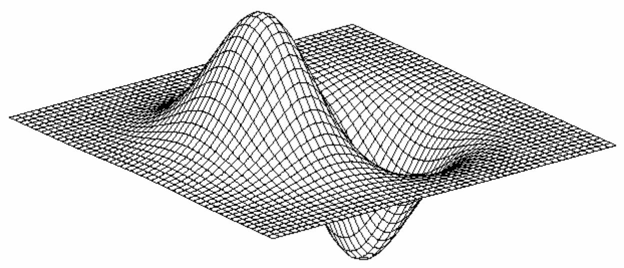
Where is the edge? Zero-crossings of bottom graph

2D edge detection filters



Gaussian

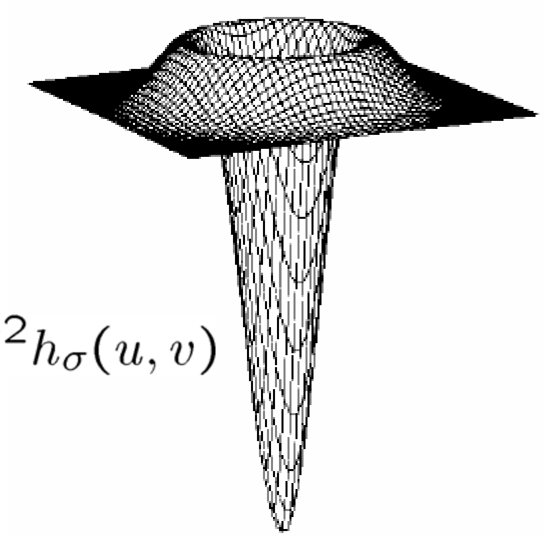
$$h_\sigma(u,v) = \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{2\sigma^2}}$$



derivative of Gaussian

$$\frac{\partial}{\partial x} h_\sigma(u,v)$$

Laplacian of Gaussian



$$\nabla^2 h_\sigma(u,v)$$

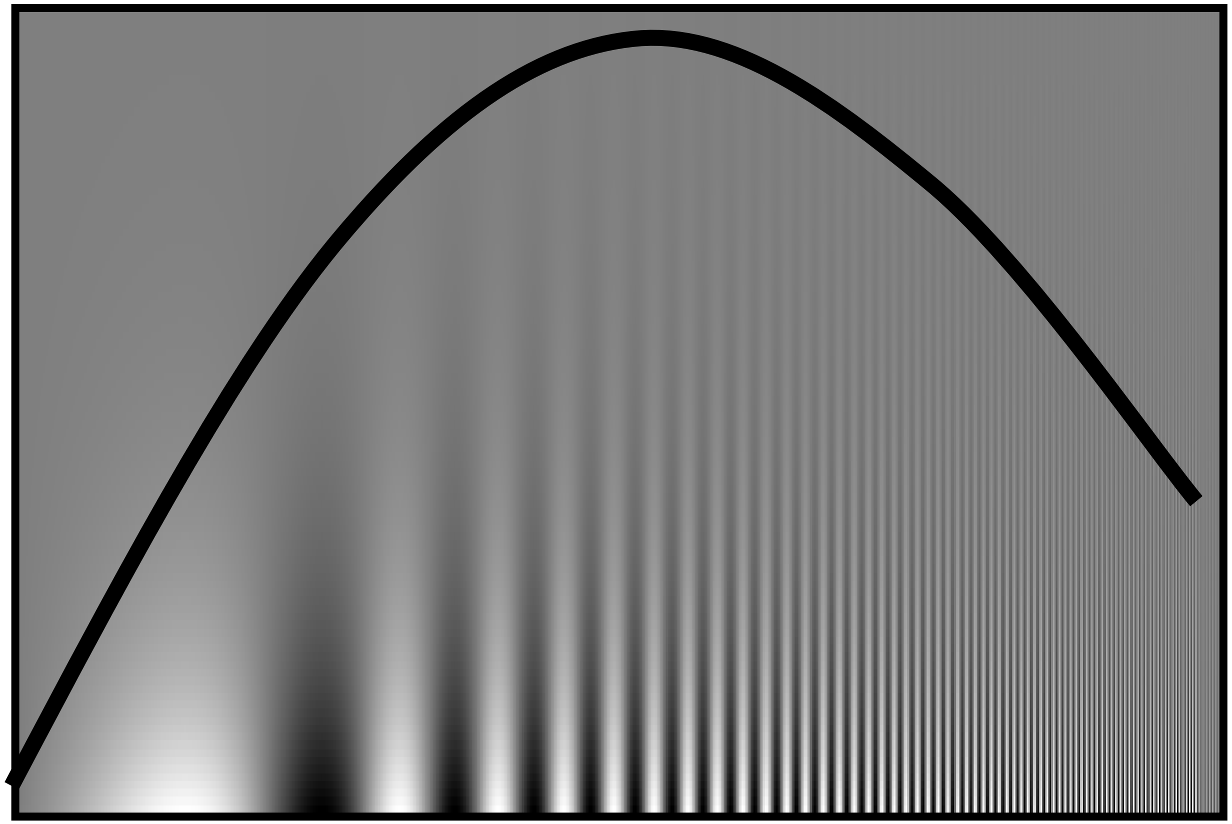
∇^2 is the **Laplacian** operator:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

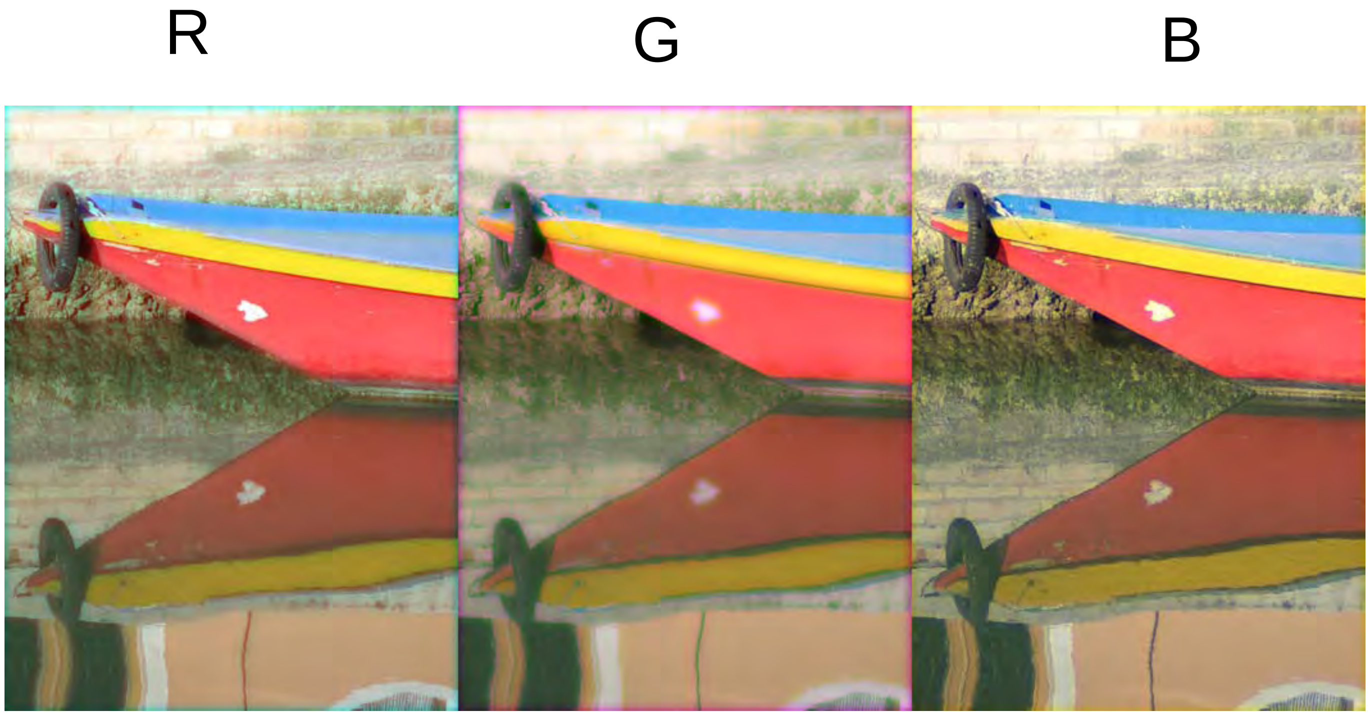
MATLAB demo

```
g = fspecial('gaussian',15,2);
imagesc(g)
surfl(g)
gclown = conv2(clown,g,'same');
imagesc(conv2(clown,[-1 1],'same'));
imagesc(conv2(gclown,[-1 1],'same'));
dx = conv2(g,[-1 1],'same');
imagesc(conv2(clown,dx,'same'));
lg = fspecial('log',15,2);
lclown = conv2(clown,lg,'same');
imagesc(lclown)
imagesc(clown + .2*lclown)
```

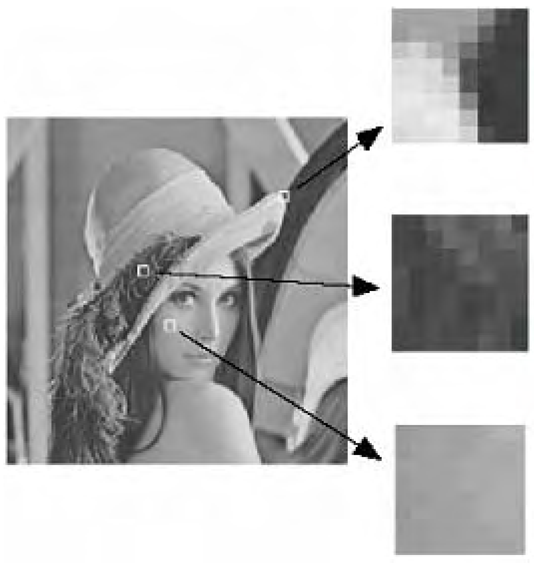
Campbell-Robson contrast sensitivity curve

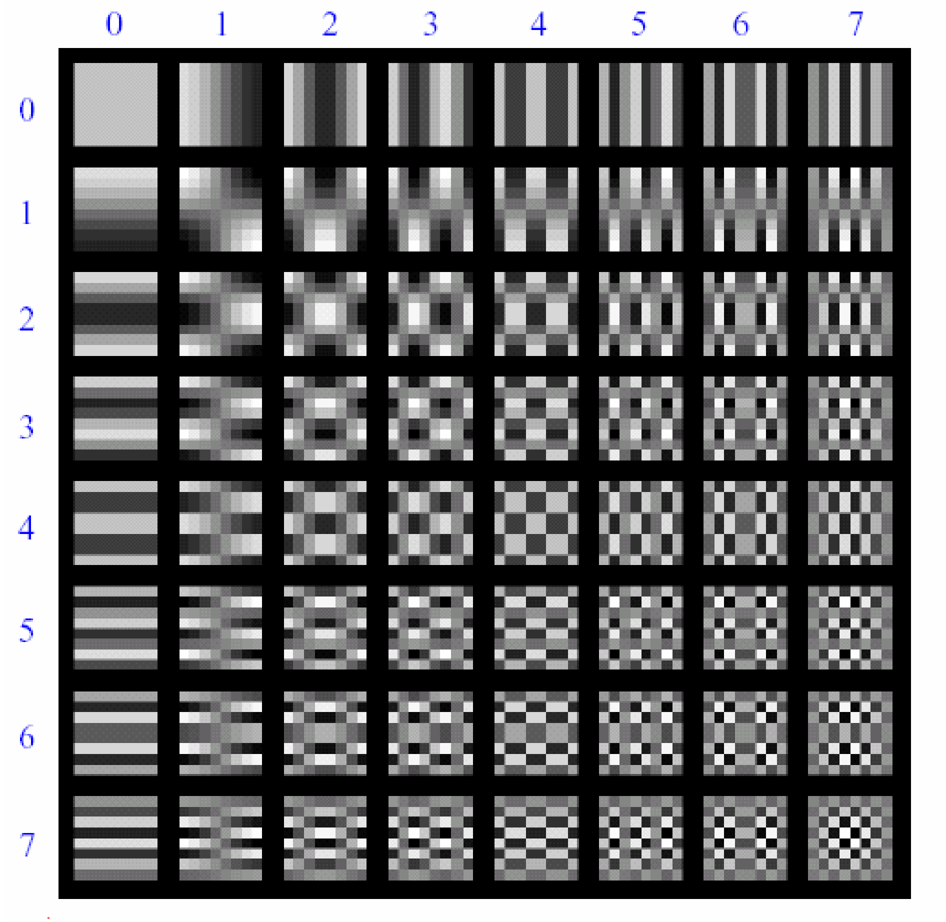


Depends on Color



Lossy Image Compression (JPEG)





Block-based Discrete Cosine Transform (DCT)

Using DCT in JPEG

A variant of discrete Fourier transform

- Real numbers
- Fast implementation

Block size

- small block
  - faster
  - correlation exists between neighboring pixels
- large block
  - better compression in smooth regions

Using DCT in JPEG

The first coefficient $B(0,0)$ is the DC component, the average intensity

The top-left coeffs represent low frequencies, the bottom right – high frequencies

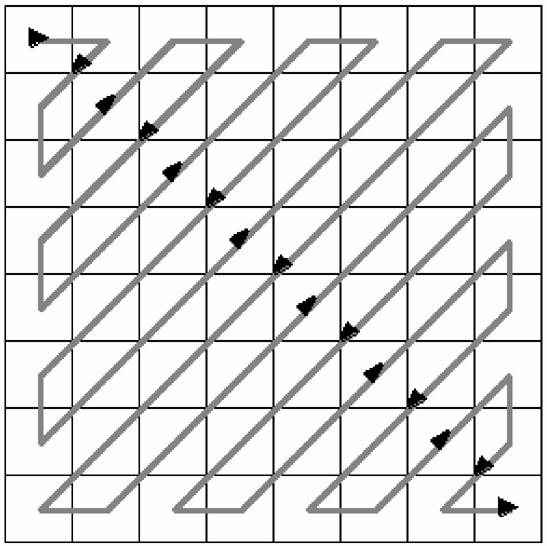


Image compression using DCT

DCT enables image compression by concentrating most image information in the low frequencies

Loose unimportant image info (high frequencies) by cutting $B(u,v)$ at bottom right

The decoder computes the inverse DCT – IDCT

- Quantization Table

3	5	7	9	11	13	15	17
5	7	9	11	13	15	17	19
7	9	11	13	15	17	19	21
9	11	13	15	17	19	21	23
11	13	15	17	19	21	23	25
13	15	17	19	21	23	25	27
15	17	19	21	23	25	27	29
17	19	21	23	25	27	29	31

JPEG compression comparison

89k

12k