Image Warping

http://www.jeffrey-martin.com

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Alexei Efros, CMU, Fall 2006

Some slides from Steve Seitz
Image Warping

image filtering: change **range** of image

\[ g(x) = T(f(x)) \]

image warping: change **domain** of image

\[ g(x) = f(T(x)) \]
Image Warping

image filtering: change **range** of image
\[ g(x) = h(T(x)) \]

image warping: change **domain** of image
\[ g(x) = f(T(x)) \]
Parametric (global) warping

Examples of parametric warps:

- translation
- rotation
- aspect
- affine
- perspective
- cylindrical
Parametric (global) warping

Transformation $T$ is a coordinate-changing machine:

$$p' = T(p)$$

What does it mean that $T$ is global?

- Is the same for any point $p$
- Can be described by just a few numbers (parameters)

Let's represent $T$ as a matrix:

$$p' = Mp$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix}$$
Scaling

*Scaling* a coordinate means multiplying each of its components by a scalar.

*Uniform scaling* means this scalar is the same for all components:
Scaling

*Non-uniform scaling:* different scalars per component:

- $X \times 2$
- $Y \times 0.5$
Scaling

Scaling operation:

\[ x' = ax \]
\[ y' = by \]

Or, in matrix form:

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix}
= 
\begin{bmatrix}
  a & 0 \\
  0 & b
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

scaling matrix \( S \)

What’s inverse of \( S \)?
2-D Rotation

\[ x' = x \cos(\theta) - y \sin(\theta) \]
\[ y' = x \sin(\theta) + y \cos(\theta) \]
2-D Rotation

x = r \cos (\phi)
y = r \sin (\phi)
x' = r \cos (\phi + \theta)
y' = r \sin (\phi + \theta)

Trig Identity…
x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta)
y' = r \sin(\phi) \sin(\theta) + r \cos(\phi) \cos(\theta)

Substitute…
x' = x \cos(\theta) - y \sin(\theta)
y' = x \sin(\theta) + y \cos(\theta)
2-D Rotation

This is easy to capture in matrix form:

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}
\]

\[
R
\]

Even though \(\sin(\theta)\) and \(\cos(\theta)\) are nonlinear functions of \(\theta\),
- \(x'\) is a linear combination of \(x\) and \(y\)
- \(y'\) is a linear combination of \(x\) and \(y\)

What is the inverse transformation?
- Rotation by \(-\theta\)
- For rotation matrices \(R^{-1} = R^T\)
2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Identity?

\[ x' = x \]
\[ y' = y \]

\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

2D Scale around (0,0)?

\[ x' = s_x \times x \]
\[ y' = s_y \times y \]

\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]
2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)?

\[
x' = \cos \Theta \cdot x - \sin \Theta \cdot y \\
y' = \sin \Theta \cdot x + \cos \Theta \cdot y
\]

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
\cos \Theta & -\sin \Theta \\
\sin \Theta & \cos \Theta
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

2D Shear?

\[
x' = x + sh_x \cdot y \\
y' = sh_y \cdot x + y
\]

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} =
\begin{bmatrix}
1 & sh_x \\
sh_y & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]
2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Mirror about Y axis?
\[
x' = -x \\
y' = y
\]
\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

2D Mirror over (0,0)?
\[
x' = -x \\
y' = -y
\]
\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]
2x2 Matrices

What types of transformations can be represented with a 2x2 matrix?

2D Translation?

\[ x' = x + t_x \]
\[ y' = y + t_y \]

NO!

Only linear 2D transformations can be represented with a 2x2 matrix
All 2D Linear Transformations

Linear transformations are combinations of …

- Scale,
- Rotation,
- Shear, and
- Mirror

Properties of linear transformations:

- Origin maps to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} =
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} =
\begin{bmatrix}
  a & b & e & f \\
  c & d & g & h \\
  i & j & k & l
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]
Linear Transformations as Change of Basis

Any linear transformation is a basis!!!

- What’s the inverse transform?
- How can we change from any basis to any basis?
- What if the basis are orthogonal?

\[ \mathbf{p} = 4\mathbf{i} + 3\mathbf{j} = (4,3) \]

\[ \mathbf{p}' = 4\mathbf{u} + 3\mathbf{v} \]

\[ \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} \mathbf{p} \]

Any linear transformation is a basis!!!

- What’s the inverse transform?
- How can we change from any basis to any basis?
- What if the basis are orthogonal?

\[ \mathbf{v} = (v_x, v_y) \]

\[ \mathbf{u} = (u_x, u_y) \]

\[ \mathbf{p}' = 4\mathbf{u} + 3\mathbf{v} \]

\[ \begin{align*}
  p_x' &= 4u_x + 3v_x \\
  p_y' &= 4u_y + 3v_y
\end{align*} \]
Homogeneous Coordinates

Q: How can we represent translation as a 3x3 matrix?

\[ x' = x + t_x \]
\[ y' = y + t_y \]
Homogeneous Coordinates

*Homogeneous coordinates*

- represent coordinates in 2 dimensions with a 3-vector

\[
\begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix}
\]
Homogeneous Coordinates

Q: How can we represent translation as a 3x3 matrix?

\[ x' = x + t_x \]
\[ y' = y + t_y \]

A: Using the rightmost column:

\[
\begin{bmatrix}
1 & 0 & t_x \\
0 & 1 & t_y \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Translation =
Translation

Example of translation

Homogeneous Coordinates

\[
\begin{bmatrix}
    x' \\
    y' \\
    1
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & t_x \\
    0 & 1 & t_y \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix} =
\begin{bmatrix}
    x + t_x \\
    y + t_y \\
    1
\end{bmatrix}
\]

\[
\begin{array}{c}
\text{tx} = 2 \\
\text{ty} = 1
\end{array}
\]
Homogeneous Coordinates

Add a 3rd coordinate to every 2D point

- \((x, y, w)\) represents a point at location \((x/w, y/w)\)
- \((x, y, 0)\) represents a point at infinity
- \((0, 0, 0)\) is not allowed

Convenient coordinate system to represent many useful transformations

\((2,1,1)\) or \((4,2,2)\) or \((6,3,3)\)
Basic 2D Transformations

Basic 2D transformations as 3x3 matrices

\[
\begin{bmatrix}
x' \\
y' \\
1
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & t_x \\
0 & 1 & t_y \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]

Translate

\[
\begin{bmatrix}
x' \\
y' \\
1
\end{bmatrix} =
\begin{bmatrix}
\cos \Theta & -\sin \Theta & 0 \\
\sin \Theta & \cos \Theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]

Rotate

\[
\begin{bmatrix}
x' \\
y' \\
1
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & s_x & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]

Scale

\[
\begin{bmatrix}
x' \\
y' \\
1
\end{bmatrix} =
\begin{bmatrix}
1 & sh_x & 0 \\
sh_y & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]

Shear
Affine Transformations

Affine transformations are combinations of …

- Linear transformations, and
- Translations

Properties of affine transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition
- Models change of basis

\[
\begin{bmatrix}
  x' \\
  y' \\
  w
\end{bmatrix} = \begin{bmatrix}
  a & b & c \\
  d & e & f \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  x \\
  y \\
  w
\end{bmatrix}
\]
Projective Transformations

Projective transformations …
• Affine transformations, and
• Projective warps

Properties of projective transformations:
• Origin does not necessarily map to origin
• Lines map to lines
• Parallel lines do not necessarily remain parallel
• Ratios are not preserved
• Closed under composition
• Models change of basis

\[
\begin{bmatrix}
  x' \\
  y' \\
  w'
\end{bmatrix} = \begin{bmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{bmatrix} \begin{bmatrix}
  x \\
  y \\
  w
\end{bmatrix}
\]
Matrix Composition

Transformations can be combined by matrix multiplication

\[
\begin{bmatrix}
x'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & tx \\
0 & 1 & ty \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\cos \Theta & -\sin \Theta & 0 \\
\sin \Theta & \cos \Theta & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
sx & 0 & 0 \\
0 & sy & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y \\
w
\end{bmatrix}
\]

\[p' = T(t_x, t_y) \quad R(\Theta) \quad S(s_x, s_y) \quad p\]
2D image transformations

These transformations are a nested set of groups
• Closed under composition and inverse is a member